ON (m – n) PRODUCTS OF BOOLEAN ALGEBRAS

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This discussion begins with the problem of whether or not all \((m - n)\) products of an indexed set \(\{A_t\}_{t \in T}\) of Boolean algebras can be obtained as \(m\)-extensions of a particular algebra \(\mathcal{F}_n^*\). The construction of \(\mathcal{F}_n^*\) is similar to the construction of the Boolean product of \(\{A_t\}_{t \in T}\); however the \(A_t\) are embedded in \(\mathcal{F}_n^*\) in such a way that their images are \(n\)-independent. If there is a cardinal number \(n'\), satisfying \(n < n' \leq m\), then \((m - n')\) products are not obtainable in this manner. For the case \(n = m\) an example shows the answer to be negative. It is explained how the class of \(m\)-extensions of \(\mathcal{F}_n^*\) is situated in the class of all \((m - n)\) products of \(\{A_t\}_{t \in T}\).

A set of \(m\)-representable Boolean algebras is given for which the minimal \((m - n)\) product is not \(m\)-representable and for which there is no smallest \((m - n)\) product.

These problems have been proposed by R. Sikorski (see [2]). Concerning \(\{A_t\}_{t \in T}\), it is assumed throughout that each of these algebras has at least four elements. \(m\) and \(n\) will always denote infinite cardinals with \(n \leq m\). All definitions are taken from [2]. An \(m\)-homomorphism is a homomorphism that is conditionally \(m\)-complete.

We denote the class of \((m - n)\) products of \(\{A_t\}_{t \in T}\) by \(P_n\) and the class of \((m - 0)\) products by \(P\). Let \(\{\{i_t\}_{t \in T}, \mathcal{B}\}\) and \(\{\{j_t\}_{t \in T}, \mathcal{C}\}\) be elements of \(P\). We say that

\[
\{\{i_t\}_{t \in T}, \mathcal{B}\} \preceq \{\{j_t\}_{t \in T}, \mathcal{C}\}
\]

provided there is an \(m\)-homomorphism \(h\) from \(\mathcal{C}\) onto \(\mathcal{B}\) such that \(h \circ j_t = i_t\) for \(t \in T\). The relation \(\preceq\) is a quasi-ordering of \(P\). Two \((m - 0)\) products are isomorphic if each is \(\preceq\) to the other.

The particular product, \(\{\{g_t^*\}_{t \in T}, \mathcal{F}_n^*\}\) of \(\{A_t\}_{t \in T}\) mentioned above is defined as follows. For each \(t \in T\) let \(X_t\) be the Stone space of \(A_t\) and let \(g_t\) be an isomorphism from \(\mathcal{F}_t\) onto the field \(\mathcal{F}_t\) of all open and closed subsets of \(X_t\). Let \(X\) be the Cartesian product of the sets \(X_t\), and for each \(t \in T\) and each \(b \in A_t\), set

\[
g_t^*(b) = \{x \in X: x(t) \in g_t(b)\}.
\]

Let \(G_n\) be the set of all subsets \(a\) of \(X\) which satisfy the following condition:

\[
a = \bigcap_{t \in S} g_t^*(b_t) \quad \text{where } b_t \in A_t, S \subseteq T \text{ and } \bar{S} \leq n.
\]

Finally, let \(\mathcal{F}_n^*\) be the field of subsets of \(X\) which is generated by \(G_n\).
\( \mathcal{F}_n^* \) is a base for the \( n \)-topology on \( X \). \( g_t^* \) is a complete isomorphism from \( \mathcal{A}_t \) into \( \mathcal{F}_n^* \). The set \( \{ g_t^*(\mathcal{A}_t) \} \), of subalgebras, is \( n \)-independent.

A Boolean \((m - n)\) product \( \{ \{ i_t \}_{t \in T}, \mathcal{B} \} \) is said to belong to \( E_n \) if and only if there is an \( m \)-isomorphism \( h \) (from \( \mathcal{F}_n^* \) into \( \mathcal{B} \)) such that \( \{ h, \mathcal{B} \} \) is an \( m \)-extension of \( \mathcal{F}_n^* \) and for each \( t \in T \) \( h \circ g_t^* = i_t \).

For every \( m \)-extension \( \{ h, \mathcal{B} \} \) of \( \mathcal{F}_n^* \), \( \{ h \circ g_t^* \}_{t \in T}, \mathcal{B} \) \( \in E_n \). Clearly \( E_n \subseteq P_n \) and \( E_n \) is not empty. \( m \)-extensions of \( \mathcal{F}_n^* \) seem to provide the most natural examples of Boolean \((m - n)\) products.

1. **Lemma 1.1.** Let \( \{ A_t \}_{t \in T} \) be an \( n \)-independent set of subalgebras of a Boolean algebra \( \mathcal{A} \) and let \( S \) and \( S' \) be subsets of \( T \) with \( \bar{S} \leq n \) and \( \bar{S'} \leq n \). For each \( t \) let \( a_t \) and \( b_t \) be nonzero elements of \( A_t \). Then

\[
(1) \quad \prod_{t \in S} a_t \leq \prod_{t \in S} b_t \text{ if and only if } a_t \leq b_t \text{ for each } t \in S;
\]

\[
(2) \quad \prod_{t \in S} a_t = \prod_{t \in S} b_t \text{ implies that } a_t = b_t \text{ for } t \in S \cap S', \quad a_t = 1 \text{ for } t \in S - S', \text{ and } b_t = 1 \text{ for } t \in S' - S.
\]

**Proof.** (i) Assume that for some \( t_0 \in S \), \( a_{t_0} \not\leq b_{t_0} \). Define

\[
C_t = \begin{cases} a_t & \text{if } t \in S \text{ and } t \neq t_0, \\ a_t \cdot (-b_{t_0}) & \text{if } t = t_0. \end{cases}
\]

Set \( c = \prod_{t \in S} c_t \), and note that \( c \neq 0 \), \( c \leq \prod_{t \in S} a_t \), and \( c \cdot \prod_{t \in S} b_t = 0 \). The converse is clear.

To prove (ii) we define

\[
x_t = \begin{cases} a_t & \text{if } t \in S, \\ 1 & \text{if } t \in S' - S; \end{cases} \quad \text{and} \quad y_t = \begin{cases} b_t & \text{if } t \in S', \\ 1 & \text{if } t \in S - S'. \end{cases}
\]

Now

\[
\prod_{t \in S \cup S'} x_t = \prod_{t \in S} a_t = \prod_{t \in S} b_t = \prod_{t \in S \cup S'} y_t
\]

and (ii) follows from (i).

**Lemma 1.2.** Let \( \{ A_t \}_{t \in T} \) be an \( n \)-independent set of subalgebras of a Boolean algebra \( \mathcal{A} \). Let \( G \) be the set of all meets \( \prod_{t \in S} a_t \) such that \( S \subseteq T, \bar{S} \leq n \), and for each \( t \in S \) \( a_t \) is a nonzero element of \( A_t \). Assume further that \( G \) generates \( \mathcal{A} \). Then \( G \) is dense in \( \mathcal{A} \).

**Proof.** First note that for \( g, g' \in G \) either \( g \cdot g' = 0 \) or else \( g \cdot g' \in G \). Thus every nonzero element of \( \mathcal{A} \) is a finite join of elements of the form \( g \cdot \prod_{t \leq k} (-g_t) \) with \( g, g_t \in G \) and \( k \) finite. (This notation is intended
to include the special cases $g$ and $-g$.) Now suppose $g \cdot \prod_{i<k} (-g_i) \neq 0$, so that $g \not= \sum_{i<k} g_i$. We write a common form $g = \prod_{t \in \mathcal{S}} a_t$, and for each $i < k$ $g_i = \prod_{t \in \mathcal{S}} a_{i,t}$ where $S \subseteq T$, $\tilde{S} \leq n$, and for each $t \in S a_t$ and $a_{i,t}$ are nonzero elements of $\mathbb{B}_t$. Since $k$ is finite every Boolean algebra is $(k - n)$-distributive (see [2], p. 62). We have

$$\prod_{t \in \mathcal{S}} a_t \cong \sum_{i<k} \prod_{t \in \mathcal{S}} a_{i,t} = \prod_{\phi \in \mathcal{S}^k} \sum_{i<k} a_{i,\phi(i)}.$$  

(Here $\mathcal{S}^k$ denotes the set of all functions from $k = \{0, 1, \ldots, k - 1\}$ into $\mathcal{S}$.) Choose $\phi \in \mathcal{S}^k$ such that $\prod_{t \in \mathcal{S}} a_t \cong \sum_{i<k} a_{i,\phi(i)}$. We have, for each $s \in \{\phi(i): i < k\}$, $a_s \cong \sum_{\phi(i) = s} a_{i,\phi(i)}$. Define

$$b_t = \begin{cases} a_t & \text{if } t \in S - \{\phi(i): i < k\} \\ a_t - \sum_{\phi(i) = t} a_{i,\phi(i)} & \text{if } t \in \{\phi(i): i < k\}. \end{cases}$$

Finally let $b = \prod_{t \in \mathcal{S}} b_t$. Clearly $b \neq 0$, $b \in G$ and $b \leq g$. For each $t \in \{\phi(i): i < k\}$, $b \cdot \sum_{\phi(i) = t} a_{i,\phi(i)} = 0$, so that $b \cdot \sum_{i<k} a_{i,\phi(i)} = 0$. It follows that $b \cdot \sum_{i<k} g_i = 0$, hence $b \leq g \cdot \prod_{i<k} (-g_i)$. 

**Corollary 1.3.** If $\tilde{S} > n$, and for each $t \in S, a_t \neq 1$, then $\prod_{t \in \mathcal{S}} a_t = 0$.

**Theorem 1.4.** Let $\{(i_t)_{t \in \mathcal{T}}, \mathcal{B}\} \in P_n$. There is one and only one isomorphism $h_n$ from $\mathcal{B}_n^\approx$ into $\mathcal{B}$ which satisfies the following completeness condition:

$$h_n(\prod_{t \in \mathcal{S}} g_t^\approx(a_t)) = \prod_{t \in \mathcal{S}} i_t(a_t) \text{ whenever } S \subseteq T, \tilde{S} \leq n,$$

$$a_t \in \mathcal{A}_t \text{ and } a_t \neq 0.$$  

**Proof.** Let $G$ be the set of all meets $\prod_{t \in \mathcal{S}} i_t(a_t)$ such that $S \subseteq T$, $\tilde{S} \leq n$, each $a_t \in \mathcal{A}_t$, and $a_t \neq 0$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{B}$ which is generated by $G$. For $\prod_{t \in \mathcal{S}} i_t(a_t) \in G$ it is clear that $\prod_{t \in \mathcal{S}} i_t(a_t) = \prod_{t \in \mathcal{S}} i_t(a_t)$. By Lemma 1.2 $G$ is dense in $\mathcal{A}$. Also $G_n$ is dense in $\mathcal{B}_n^\approx$. For $a \in G_n$ write $a = \bigcap_{t \in \mathcal{S}} g_t^\approx(a_t) = \prod_{t \in \mathcal{S}} i_t(a_t)$. Define $h(a) = \prod_{t \in \mathcal{S}} i_t(a_t)$. It is easily seen, using Lemma 1.1, that

(1) $h$ is a one to one function from $G_n$ onto $G$;

(ii) for $a, b \in G_n$, $a \leq b$ if and only if $h(a) \leq h(b)$.

It follows (see [2], p. 37) that $h$ can be extended to an isomorphism $h_n$ from $\mathcal{B}_n^\approx$ onto $\mathcal{A}$. $h_n$ is uniquely determined by condition (c) because $G_n$ generates $\mathcal{B}_n^\approx$.

**Corollary 1.5.** The product $\{(i_t)_{t \in \mathcal{T}}, \mathcal{B}\} \in E_n$ if and only if $h_n$ is $m$-complete.
Proof. Let \( \{i_t\}_{t \in T}, \mathcal{B} \} \in E_n. \) There is an \( m \)-isomorphism \( f \) from \( \mathcal{F}_n^* \) into \( \mathcal{B} \) such that for each \( t \in T, f \circ g_t^* = i_t \). \( f \) satisfies condition (c) so \( f = h_n. \)

COROLLARY 1.6. Assume \( T > n \) and that \( m \geq n' > n. \) Then \( P_{n'} \cap E_n \) is empty.

Proof. Let \( \{i_t\}_{t \in T}, \mathcal{B} \} \in P_{n'}. \) Consider the isomorphism \( h_n \) from \( \mathcal{F}_n^* \) into \( \mathcal{B}. \) Choose \( S \subseteq T, S = n^+, \) and for each \( t \in S \) choose \( a_t \in \mathcal{A}_t \) with \( a_t \neq 0, a_t \neq 1. \) By Corollary 1.3

\[
\prod_{t \in S} g_t^*(a_t) = 0.
\]

However \( 0 \neq \prod_{t \in S} i_t(a_t) = \prod_{t \in S} h_n \circ g_t^*(a_t) \) so that \( h_n \) is not \( m \)-complete.

There is an interesting contrast between \( E_n \) and \( P_n. \) (under the hypotheses of Corollary 1.6). Let \( \{i_t\}_{t \in T}, \mathcal{B} \} \) and \( \{j_t\}_{t \in T}, \mathcal{C} \) be elements of \( P_n \) with \( \{i_t\}_{t \in T}, \mathcal{B} \} \subseteq \{j_t\}_{t \in T}, \mathcal{C} \). It is known (see [2], p. 179) that if \( \{i_t\}_{t \in T}, \mathcal{B} \} \in P_n, \) then \( \{j_t\}_{t \in T}, \mathcal{C} \} \in P_{n'}. \) On the other hand if \( \{j_t\}_{t \in T}, \mathcal{C} \} \in E_n \) then we have \( \{i_t\}_{t \in T}, \mathcal{B} \} \in E_n.

COROLLARY 1.7. Assume \( T > n \) and \( m > n. \) Then \( E_n \cup P_{n'} \neq P_{n}. \)

Proof. Let \( S \subseteq T \) with \( S = n^+. \) Choose, for each \( t \in S, d_t \in \mathcal{A}_t \) with \( d_t \neq 0, d_t \neq 1. \) Let \( d = \bigcap_{t \in S} g_t^*(d_t). \) Let \( \mathcal{F} \) be the field of subsets of \( X \) which is generated by \( \mathcal{F}_n \cup \{d\}. \) Note that \( g_t^* \) is a complete isomorphism from \( \mathcal{A}_t \) into \( \mathcal{F}. \) Let \( \{f, \mathcal{C}\} \) be any \( m \)-extension of \( \mathcal{F}. \) It is easily seen that \( \{f \circ g_t^*\}_{t \in T}, \mathcal{C} \} \in P_{m}. \)

Consider the isomorphism \( h_n \) from \( \mathcal{F}_n \) into \( \mathcal{C}. \) \( h_n \circ g_t^* = f \circ g_t^* \) for every \( t \in T. \) By Corollary 1.3 \( \prod_{t \in S} g_t^*(d_t) = 0. \) However \( \prod_{t \in S} h_n \circ g_t^*(d_t) = f(d) \neq 0. \) Thus \( h_n \) is not \( m \)-complete and \( \{f \circ g_t^*\}_{t \in T}, \mathcal{C} \} \in E_n. \)

In order to show that \( \{f \circ g_t^*\}_{t \in T}, \mathcal{C} \} \in P_{m} \) it suffices to show that \( \prod_{t \in S} f \circ g_t^*(-d_t) = 0. \) In particular suppose \( b = \prod_{t \in S} g_t^*(-d_t) \neq 0. \) Since \( b \cdot d = 0 \) the definition of \( \mathcal{F} \) enables us to write \( b = \bigcup_{t \in S} b_t \cdot g_t^*(-d_t) \) with \( b_t \in \mathcal{F}_n. \) Choose \( t_0 \in S \) such that \( 0 \neq b_t \cdot g_t^*(-d_t) \leq b. \) By Lemma 1.2 there is a nonzero element \( a = \bigcap_{t \in S} g_t^*(a_t) \) of \( G_n \) such that \( a \subseteq b \cdot g_t^*(-d_t). \) Now \( S^t \subseteq n^+ \) and \( S = n^+ \) and it follows that \( a \not\subseteq b. \) Thus \( \prod_{t \in S} g_t^*(-d_t) = 0 \) and since \( f \) is \( m \)-complete, \( \prod_{t \in S} f \circ g_t^*(-d_t) = 0. \)

We now consider the case \( n = m. \) It is known that \( E_m \neq P_{m} \) if \( m = n^+ \) (see [2], p. 190, Example D). In this example \( T \) is the two element set \( \{1, 2\}, \mathcal{A}, \) and \( \mathcal{A}_2 \) are \( \sigma \)-complete Boolean algebras which satisfy the \( \sigma \)-chain condition. The Boolean \( \sigma \)-product \( \{i_t, i_t, \mathcal{B}\} \) is such that the subalgebra \( \mathcal{B}_t \) of \( \mathcal{B} \) which is generated by \( i_t(\mathcal{A}_t) \cup i_t(\mathcal{A}_t) \)
is not a $\sigma$-regular subalgebra of $\mathcal{B}$. Let $\{f, \mathcal{C}\}$ be any $m$-extension of $\mathcal{B}$. It follows, using the $\sigma$-chain condition on $\mathcal{A}$ and $\mathcal{A}^*$, that $\{\{f \circ i_1, f \circ i_2\}, \mathcal{C}\} \in \mathcal{P}_m$. Since $T$ is finite $\{\{g^*, g^*_2\}, \mathcal{F}_m^*\}$ is the Boolean product of $\{\mathcal{A}_1, \mathcal{A}_2\}$. Let $h$ be the homomorphism from $\mathcal{F}_m^*$ into $\mathcal{B}$ such that $h \circ g^*_1 = i_1$ and $h \circ g^*_2 = i_2$. Then $h$ is an isomorphism from $\mathcal{F}_m^*$ onto $\mathcal{B}$. Consider the isomorphism $h_m$ from $\mathcal{F}_m^*$ into $\mathcal{C}$, given by Theorem 1.4. $h_m = f \circ h$ since they agree on $g^*_1(\mathcal{A}_1) \cup g^*_2(\mathcal{A}_2)$. $h_m$ is not $m$-complete because $f(\mathcal{B})$ is not $m$-regular in $\mathcal{C}$. Thus $\{\{f \circ i_1, f \circ i_2\}, \mathcal{C}\} \in \mathcal{E}_m$. We give a simple for the case $m \geq 2^{\aleph_0}$.

**Example 1.8.** Assume $m \geq 2^{\aleph_0}$ and let $T$ be a set of power $\aleph_0$. For each $t \in T$ let $\mathcal{A}_t$ be a Boolean algebra having exactly four elements. Let $\mathcal{C}$ be the free Boolean $m$-algebra on $\{\mathcal{A}_t: t \in T\}$. $\mathcal{C}$ is not $m$-representable (see [2], p. 134). For each $t \in T$ choose $d_t$ to be one of the atoms of $\mathcal{A}_t$. Let $i_t$ be the isomorphism from $\mathcal{A}_t$ into $\mathcal{B}$ such that $i_t(d_t) = D_t$. Then $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in \mathcal{P}_m$. By Lemma 1.2 $\mathcal{F}_m^*$ is atomic, the atoms being all sets of the form $\bigcap_{t \in T} g^*_t(a_t)$, where for each $t \in T a_t$ is an atom of $\mathcal{A}_t$. Denote the set of atoms of $\mathcal{F}_m^*$ by $\{C_r: r \in R\}$, then $R = 2^{\aleph_0}$. We consider the isomorphism $h_m$ from $\mathcal{F}_m^*$ into $\mathcal{B}$. For each $r \in R, h_m(c_r)$ is an atom of $\mathcal{B}$. To show this we define

$$\mathcal{B} = \{b \in \mathcal{B}: \text{for each } r \in R \text{ either } b \cdot h_m(c_r) = 0 \text{ or } h_m(c_r) \leq b\}.$$  

It is easily seen that $\mathcal{B}$ is an $m$-subalgebra of $\mathcal{B}$ which includes $\{D_t: t \in T\}$. Hence $\mathcal{B} = \mathcal{B}$. Finally, $h_m$ is not $m$-complete. For otherwise $\sum_{c_r \in R} h_m(c_r) = 1$, and $\mathcal{B}$ would be atomic and hence isomorphic to an $m$-field of sets.

2. We now consider the problem of the existence of a smallest element of $\mathcal{P}$, relative to the quasi-ordering “$\leq$”. A minimal element of $\mathcal{P}$ always exists and can be constructed as follows. Let $\{\{f_t\}_{t \in T}, \mathcal{C}\}$ be a Boolean product of $\{\mathcal{A}_t\}_{t \in T}$ and let $\{h, \mathcal{B}\}$ be an $m$-completion of $\mathcal{C}$. Then $\{\{h \circ f_t\}_{t \in T}, \mathcal{B}\}$ is a minimal element of $\mathcal{P}$. We shall show that this product need not be a smallest element of $\mathcal{P}$. Hence $\mathcal{P}$ need not have a smallest element.

**Example 2.1.** Let $m$ be any infinite cardinal. Let $\mathcal{T} = \aleph_0$ and suppose that for each $t \in T$ $\mathcal{A}_t$ is a four element Boolean algebra. For each $t \in T$ choose $a_t$ to be one of the atoms of $\mathcal{A}_t$. $\mathcal{C}$ is a free Boolean algebra of power $\aleph_0$, one set of free generators being $\{f_t(a_t): t \in T\}$. $\mathcal{B}$ has a countable dense subset, in particular $\mathcal{B}$ satisfies the countable chain condition. Thus $\mathcal{B}$ is complete. It follows that $\mathcal{B}$ is isomorphic to the quotient algebra $\mathcal{F}/\Delta$, where $\mathcal{F}$ is the $\sigma$-field...
of Borel subsets of the unit interval \( I = \{ x: 0 < x \leq 1 \} \) of real numbers and \( \Delta \) is the ideal consisting of those Borel sets which are of the first category.

To show that \( \{ (h \circ f) \}_{t \in T}, \mathcal{B} \) is not a smallest element of \( P \) we construct another \((m-0)\) product as follows. Let \( G \) be the set of all halfopen intervals of the form \( \{ x: 0 < x \leq r \} \) such that \( r \) is rational and \( 0 < r \leq 1 \). \( \mathcal{F} \) is \( \sigma \)-generated by \( G \). The subalgebra \( \mathcal{F}_0 \) of \( \mathcal{F} \) which is generated by \( G \) is denumerable and atomless. Hence \( \mathcal{F}_0 \) is isomorphic to \( \mathbb{C} \) (see [1], p. 54). Let \( g \) be an isomorphism from \( \mathbb{C} \) onto \( \mathcal{F}_0 \). Let \( \Delta \) be the ideal of \( \mathcal{F} \) consisting of those Borel sets having Lebesgue measure 0. We note that \( \mathcal{F}_0 \cap \Delta = \{ 0 \} \). Finally for each \( t \in T \) let \( h_t \) be the isomorphism from \( \mathcal{F}_0 \) onto \( \mathcal{F}_0 / \Delta \) defined by \( h_t(a) = (g \circ f_t)(a) \). It is easily seen that \( \{ (h_t)_{t \in T}, \mathcal{F}_0 / \Delta \} \in P \).

Now assume \( \{ (h \circ f) \}_{t \in T}, \mathcal{B} \} \leq \{ (h_t)_{t \in T}, \mathcal{F}_0 / \Delta \} \). Then there is an \( m \)-homomorphism \( p \) from \( \mathcal{F}_0 / \Delta \) onto \( \mathcal{F}_0 / \Delta_0 \). Since \( \mathcal{F}_0 / \Delta_0 \) satisfies the countable chain condition the kernel of \( p \) is a principal ideal. \( \mathcal{F}_0 / \Delta_0 \) is isomorphic to a principal ideal of \( \mathcal{F}_0 / \Delta_0 \). However \( \mathcal{F}_0 / \Delta_0 \) is homogeneous (see [2], p. 105). Thus \( \mathcal{F}_0 / \Delta_0 \) is isomorphic to \( \mathcal{F}_0 / \Delta_1 \), which is a contradiction.

Next we consider the problem of the existence of a smallest element of \( P_n \). Let \( \{ g, \mathcal{B} \} \) be an \( m \)-completion of \( \mathcal{F}_n^* \). Then \( \{ (g \circ g^*_t) \}_{t \in T}, \mathcal{B} \} \) is a minimal element of \( P_n \). Also it is known (see [2], p. 183) that if all the \( \mathcal{A}_t \) are \( m \)-representable then there is an \((m-n)\) product \( \{ (i_t)_{t \in T}, \mathbb{C} \} \) for which \( \mathbb{C} \) is \( m \)-representable. We give an example of \( \{ \mathcal{A}_t \}_{t \in T} \) for which \( \mathcal{B} \) is not \( m \)-representable and \( \{ (g \circ g^*_t) \}_{t \in T}, \mathcal{B} \} \) is not a smallest element of \( P_n \).

**Example 2.2.** Assume that \( m \geq 2^{n+1} \). Let \( T = n+1 \) and for each \( t \in T \) let \( \mathcal{A}_t \) be a four element Boolean algebra. We show that \( \mathcal{B} \) is not \( n^t \)-distributive. Choose, for each \( t \in T, a_t \) to be one of the atoms of \( \mathcal{A}_t \). Then

\[
\prod_{t \in T} (g \circ g^*_t(a_t) + g \circ g^*_t(a_t)) = 1. 
\]

However for each function \( \eta \in H^T \) (here \( H = \{ +1, -1 \} \)) we have

\[
\prod_{t \in T} \eta(t) \cdot g \circ g^*_t(a_t) = 0. 
\]

This follows from Corollary 1.3. Thus \( \prod_{t \in T} \eta(t) \cdot g \circ g^*_t(a_t) = 0 \). This proves \( \mathcal{B} \) is not \( n^t \)-distributive and hence not \( m \)-representable.

To show that \( \{ (g \circ g^*_t) \}_{t \in T}, \mathcal{B} \} \) is not a smallest element of \( P_n \), let \( \{ (i_t)_{t \in T}, \mathbb{C} \} \) be any \( (m-n) \) product of \( \{ \mathcal{A}_t \}_{t \in T} \) such that \( \mathbb{C} \) is \( m \)-representable. \( \mathcal{B} \) is not an \( m \)-homomorphic image of \( \mathbb{C} \). Thus the inequality
\[
\{g \circ g_i^*\}_{i \in T}, \mathcal{B} \subseteq \{i_i\}_{i \in T}, \mathcal{C} \]
does not hold.

**References**


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