Pacific Journal of Mathematics

ON $(m-n)$ PRODUCTS OF BOOLEAN ALGEBRAS

ROBERT HAMOR LA GRANGE, JR.

Vol. 31, No. 3 BadMonth 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

ON $(\mathfrak{m}-\mathfrak{n})$ PRODUCTS OF BOOLEAN ALGEBRAS

R. H. LA GRANGE

This discussion begins with the problem of whether or not all $(\mathfrak{m}-\mathfrak{n})$ products of an indexed set $\{\mathfrak{A}_t\}_{t\in T}$ of Boolean algebras can be obtained as $\mathfrak{m}$-extensions of a particular algebra $\mathscr{F}_{\mathfrak{n}}^*$. The construction of $\mathscr{F}_{\mathfrak{n}}^*$ is similar to the construction of the Boolean product of $\{\mathfrak{A}_t\}_{t\in T}$; however the $\mathscr{A}_t$ are embedded in $\mathscr{F}_{\mathfrak{n}}^*$ in such a way that their images are $\mathfrak{n}$-independent. If there is a cardinal number $\mathfrak{n}'$, satisfying $\mathfrak{n}<\mathfrak{n}'\leqq\mathfrak{m}$, then $(\mathfrak{m}-\mathfrak{n}')$ products are not obtainable in this manner. For the case $\mathfrak{n}=\mathfrak{m}$ an example shows the answer to be negative. It is explained how the class of $\mathfrak{m}$-extensions of $\mathscr{F}_{\mathfrak{n}}^*$ is situated in the class of all $(\mathfrak{m}-\mathfrak{n})$ products of $\{\mathfrak{A}_t\}_{t\in T}$. A set of $\mathfrak{m}$-representable Boolean algebras is given for which the minimal $(\mathfrak{m}-\mathfrak{n})$ product is not $\mathfrak{m}$-representable and for which there is no smallest $(\mathfrak{m}-\mathfrak{n})$ product.

These problems have been proposed by R. Sikorski (see [2]). Concerning $\{\mathfrak{A}_t\}_{t\in T}$, it is assumed throughout that each of these algebras has at least four elements. $\mathfrak{m}$ and $\mathfrak{n}$ will always denote infinite cardinals with $\mathfrak{n}\leqq\mathfrak{m}$. All definitions are taken from [2]. An $\mathfrak{m}$-homomorphism is a homomorphism that is conditionally $\mathfrak{m}$-complete. We denote the class of $(\mathfrak{m}-\mathfrak{n})$ products of $\{\mathfrak{A}_t\}_{t\in T}$ by $\boldsymbol{P}_{\mathfrak{n}}$ and the class of $(\mathfrak{m}-0)$ products by $\boldsymbol{P}$. Let $\{\{i_t\}_{t\in T},\mathscr{B}\}$ and $\{\{j_t\}_{t\in T},\mathfrak{C}\}$ be elements of $\boldsymbol{P}$. We say that

$$\{\{i_t\}_{t\in T},\mathscr{B}\}\leqq\{\{j_t\}_{t\in T},\mathfrak{C}\}$$

provided there is an $\mathfrak{m}$-homomorphism h from $\mathfrak{C}$ onto $\mathscr{B}$ such that $h\circ j_t=i_t$ for $t\in T$. The relation "$\leqq$" is a quasi-ordering of $\boldsymbol{P}$. Two $(\mathfrak{m}-0)$ products are isomorphic if each is $\leqq$ to the other.

The particular product, $\{\{g_t^*\}_{t\in T},\mathscr{F}_{\mathfrak{n}}^*\}$ of $\{\mathfrak{A}_t\}_{t\in T}$ mentioned above is defined as follows. For each $t\in T$ let X_t be the Stone space of $\mathfrak{A}_t$ and let g_t be an isomorphism from $\mathfrak{A}_t$ onto the field $\mathscr{F}_t$ of all open and closed subsets of X_t. Let X be the Cartesian product of the sets X_t, and for each $t\in T$ and each $b\in\mathfrak{A}_t$, set

$$(1)\qquad g_t^*(b)=[x\in X\colon x(t)\in g_t(b)\}\ .$$

Let $G_{\mathfrak{n}}$ be the set of all subsets a of X which satisfy the following condition:

$$a=\bigcap_{t\in S}g_t^*(b_t)\ \text{where}\ b_t\in\mathfrak{A}_t,\ S\subseteqq T\ \text{and}\ \bar{\bar{S}}\leqq\mathfrak{n}\ .$$

Finally, let $\mathscr{F}_{\mathfrak{n}}^*$ be the field of subsets of X which is generated by $G_{\mathfrak{n}}$.

$\mathscr{F}_{\mathfrak{n}}{}^*$ is a base for the $\mathfrak{n}$-topology on X. g_t^* is a complete isomorphism from $\mathfrak{A}_t$ into $\mathscr{F}_{\mathfrak{n}}{}^*$. The set $\{g_t^*(\mathfrak{A}_t)\}$, of subalgebras, is $\mathfrak{n}$-independent.

A Boolean $(\mathfrak{m} - \mathfrak{n})$ product $\{\{i_t\}_{t \in T}, \mathscr{B}\}$ is said to *belong to* $\boldsymbol{E}_{\mathfrak{n}}$ if and only if there is an $\mathfrak{m}$-isomorphism h (from $\mathscr{F}_{\mathfrak{n}}{}^*$ into $\mathscr{B}$) such that $\{h, \mathscr{B}\}$ is an $\mathfrak{m}$-extension of $\mathscr{F}_{\mathfrak{n}}{}^*$ and for each $t \in T$ $h \circ g_t^* = i_t$.

For every $\mathfrak{m}$-extension $\{h, \mathscr{B}\}$ of $\mathscr{F}_{\mathfrak{n}}{}^*$, $\{\{h \circ g_t^*\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$. Clearly $\boldsymbol{E}_{\mathfrak{n}} \subseteqq \boldsymbol{P}_{\mathfrak{n}}$ and $\boldsymbol{E}_{\mathfrak{n}}$ is not empty. $\mathfrak{m}$-extensions of $\mathscr{F}_{\mathfrak{n}}{}^*$ seem to provide the most natural examples of Boolean $(\mathfrak{m} - \mathfrak{n})$ products.

1. LEMMA 1.1. *Let $\{\mathscr{B}_t\}_{t \in T}$ be an $\mathfrak{n}$-independent set of subalgebras of a Boolean algebra $\mathfrak{A}$ and let S and S' be subsets of T with $\overline{\overline{S}} \leqq \mathfrak{n}$ and $\overline{\overline{S'}} \leqq \mathfrak{n}$. For each t let a_t and b_t be nonzero elements of $\mathscr{B}_t$. Then*

(i) *$\prod_{t \in S}^{\mathfrak{A}} a_t \leqq \prod_{t \in S}^{\mathfrak{A}} b_t$ if and only if $a_t \leqq b_t$ for each $t \in S$;*

(ii) *$\prod_{t \in S}^{\mathfrak{A}} a_t = \prod_{t \in S'}^{\mathfrak{A}} b_t$ implies that $a_t = b_t$ for $t \in S \cap S'$, $a_t = 1$ for $t \in S - S'$, and $b_t = 1$ for $t \in S' - S$.*

Proof. (i) Assume that for some $t_0 \in S$, $a_{t_0} \not\leqq b_{t_0}$. Define

$$C_t = \begin{cases} a_t \text{ if } t \in S \text{ and } t \neq t_0 \,, \\ a_{t_0} \cdot (-b_{t_0}) \text{ if } t = t_0 \,. \end{cases}$$

Set $c = \prod_{t \in S}^{\mathfrak{A}} c_t$, and note that $c \neq 0$, $c \leqq \prod_{t \in S}^{\mathfrak{A}} a_t$, and $c \cdot \prod_{t \in S}^{\mathfrak{A}} b_t = 0$. The converse is clear.

To prove (ii) we define

$$x_t = \begin{cases} a_t \text{ if } t \in S \,, \\ 1 \text{ if } t \in S' - S \,; \end{cases} \quad \text{and} \quad y_t = \begin{cases} b_t \text{ if } t \in S' \,, \\ 1 \text{ if } t \in S - S' \,. \end{cases}$$

Now

$$\prod_{t \in S \cup S'}^{\mathfrak{A}} x_t = \prod_{t \in S}^{\mathfrak{A}} a_t = \prod_{t \in S'}^{\mathfrak{A}} b_t = \prod_{t \in S \cup S'}^{\mathfrak{A}} y_t$$

and (ii) follows from (i).

LEMMA 1.2. *Let $\{\mathscr{B}_t\}_{t \in T}$ be an $\mathfrak{n}$-independent set of subalgebras of a Boolean algebra $\mathfrak{A}$. Let G be the set of all meets $\prod_{t \in S}^{\mathfrak{A}} a_t$ such that $S \subseteqq T$, $\overline{\overline{S}} \leqq \mathfrak{n}$, and for each $t \in S$ a_t is a nonzero element of $\mathscr{B}_t$. Assume further that G generates $\mathfrak{A}$. Then G is dense in $\mathfrak{A}$.*

Proof. First note that for $g, g' \in G$ *either* $g \cdot g' = 0$ or else $g \cdot g' \in G$. Thus every nonzero element of $\mathfrak{A}$ is a finite join of elements of the form $g \cdot \prod_{i<k}^{\mathfrak{A}} (-g_i)$ with $g, g_i \in G$ and k finite. (This notation is intended

to include the special cases g and $-g$.) Now suppose $g \cdot \prod_{i<k}^{\mathfrak{A}} (-g_i) \neq 0$, so that $g \not\leqq \sum_{i<k} g_i$. We write a common form $g = \prod_{t \in S}^{\mathfrak{A}} a_t$, and for each $i < k$ $g_i = \prod_{t \in S}^{\mathfrak{A}} a_{i,t}$ where $S \subseteqq T$, $\overline{\overline{S}} \leqq \mathfrak{n}$, and for each $t \in S$ a_t and $a_{i,t}$ are nonzero elements of $\mathscr{B}_t$. Since k is finite every Boolean algebra is $(k - \mathfrak{n})$-distributive (see [2], p. 62). We have

$$\prod_{t \in S} a_t \not\leqq \sum_{i<k} \prod_{t \in S} a_{i,t} = \prod_{\psi \in S^k} \sum_{i<k} a_{i,\psi(i)} .$$

(Here S^k denotes the set of all functions from $k = \{0, 1, \cdots, k-1\}$ into S.) Choose $\phi \in S^k$ such that $\prod_{t \in S} a_t \not\leqq \sum_{i<k} a_{i,\phi(i)}$. We have, for each $s \in \{\phi(i): i < k\}$, $a_s \not\leqq \sum_{\phi(i)=s} a_{i,\phi(i)}$. Define

$$b_t = \begin{cases} a_t \text{ if } t \in S - \{\phi(i): i < k\} \\ a_t \cdot - \sum\limits_{\phi(i)=t} a_{i,\phi(i)} \text{ if } t \in \{\phi(i): i < k\} . \end{cases}$$

Finally let $b = \prod_{t \in S}^{\mathfrak{A}} b_t$. Clearly $b \neq 0, b \in G$ and $b \leqq g$. For each $t \in \{\phi(i): i < k\}$, $b_t \cdot \sum_{\phi(i)=t} a_{i,\phi(i)} = 0$, so that $b \cdot \sum_{i<k} a_{i,\phi(i)} = 0$. It follows that $b \cdot \sum_{i<k} g_i = 0$, hence $b \leqq g \cdot \prod_{i<k} (-g_i)$.

COROLLARY 1.3. *If* $\overline{\overline{S}} > \mathfrak{n}$, *and for each* $t \in S, a_t \neq 1$, *then* $\prod_{t \in S}^{\mathscr{A}} a_t = 0$.

THEOREM 1.4. *Let* $\{\{i_t\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{n}}$. *There is one and only one isomorphism* $h_{\mathfrak{n}}$ *from* $\mathscr{F}_{\mathfrak{n}}^*$ *into* $\mathscr{B}$ *which satisfies the following completeness condition*:

$$\text{(c)} \qquad h_{\mathfrak{n}}\Big(\prod_{t \in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t^*(a_t)\Big) = \prod_{t \in S}^{\mathscr{B}} i_t(a_t) \textit{ whenever } S \subseteqq T, \overline{\overline{S}} \leqq \mathfrak{n} ,$$
$$a_t \in \mathfrak{A}_t \textit{ and } a_t \neq 0 .$$

Proof. Let G be the set of all meets $\prod_{t \in S}^{\mathscr{B}} i_t(a_t)$ such that $S \subseteqq T$, $\overline{\overline{S}} \leqq \mathfrak{n}$, each $a_t \in \mathfrak{A}_t$ and $a_t \neq 0$. Let $\mathfrak{A}$ be the subalgebra of $\mathscr{B}$ which is generated by G. For $\prod_{t \in S}^{\mathscr{B}} i_t(a_t) \in G$ it is clear that $\prod_{t \in S}^{\mathscr{B}} i_t(a_t) = \prod_{t \in S}^{\mathfrak{A}} i_t(a_t)$. By Lemma 1.2 G is dense in $\mathfrak{A}$. Also $G_{\mathfrak{n}}$ is dense in $\mathscr{F}_{\mathfrak{n}}^*$. For $a \in G_{\mathfrak{n}}$ write $a = \bigcap_{t \in S} g_t^*(a_t) = \prod_{t \in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t^*(a_t)$. Define $h(a) = \prod_{t \in S}^{\mathfrak{A}} i_t(a_t)$. It is easily seen, using Lemma 1.1, that

(i) h is a one to one function from $G_{\mathfrak{n}}$ onto G;

(ii) for $a, b \in G_{\mathfrak{n}}$, $a \leqq b$ if and only if $h(a) \leqq h(b)$.

It follows (see [2], p. 37) that h can be extended to an isomorphism $h_{\mathfrak{n}}$ from $\mathscr{F}_{\mathfrak{n}}^*$ onto $\mathfrak{A}$. $h_{\mathfrak{n}}$ is uniquely determined by condition (c) because $G_{\mathfrak{n}}$ generates $\mathscr{F}_{\mathfrak{n}}^*$.

COROLLARY 1.5. *The product* $\{\{i_t\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$ *if and only if* $h_{\mathfrak{n}}$ *is* $\mathfrak{m}$-*complete.*

Proof. Let $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$. There is an $\mathfrak{m}$-isomorphism f from $\mathscr{F}_{\mathfrak{n}}{}^*$ into $\mathscr{B}$ such that for each $t \in T$, $f \circ g_t^* = i_t$. f satisfies condition (c) so $f = h_{\mathfrak{n}}$.

COROLLARY 1.6. *Assume* $\bar{\bar{T}} > \mathfrak{n}$ *and that* $\mathfrak{m} \geqq \mathfrak{n}' > \mathfrak{n}$. *Then* $\boldsymbol{P}_{\mathfrak{n}'} \cap \boldsymbol{E}_{\mathfrak{n}}$ *is empty.*

Proof. Let $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{n}'}$. Consider the isomorphism $h_{\mathfrak{n}}$ from $\mathscr{F}_{\mathfrak{n}}{}^*$ *into* $\mathscr{B}$. Choose $S \subseteqq T$, $\bar{\bar{S}} = \mathfrak{n}^+$, and for each $t \in S$ choose $a_t \in \mathfrak{A}_t$ with $a_t \neq 0$, $a_t \neq 1$. By Corollary 1.3

$$\prod_{t\in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t^*(a_t) = 0 \ .$$

However $0 \neq \prod_{t\in S}^{\mathscr{B}} i_t(a_t) = \prod^{\mathscr{B}} h_{\mathfrak{n}} \circ g_t^*(a_t)$ so that $h_{\mathfrak{n}}$ is not $\mathfrak{m}$-complete.

There is an interesting contrast between $\boldsymbol{E}_{\mathfrak{n}}$ and $\boldsymbol{P}_{\mathfrak{n}'}$, (under the hypotheses of Corollary 1.6). Let $\{\{i_t\}_{t\in T}, \mathscr{B}\}$ and $\{\{j_t\}_{t\in T}, \mathfrak{C}\}$ be elements of $\boldsymbol{P}_{\mathfrak{n}}$ with $\{\{i_t\}_{t\in T}, \mathscr{B}\} \leqq \{\{j_t\}_{t\in T}, \mathfrak{C}\}$. It is known (see [2], p. 179) that if $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{n}'}$, then $\{\{j_t\}_{t\in T}, \mathfrak{C}\} \in \boldsymbol{P}_{\mathfrak{n}'}$. On the other hand if $\{\{j_t\}_{t\in T}, \mathfrak{C}\} \in \boldsymbol{E}_{\mathfrak{n}}$ then we have $\{\{i_t\}_{t\in T}, \mathscr{B}\} \in \boldsymbol{E}_{\mathfrak{n}}$.

COROLLARY 1.7. *Assume* $\bar{\bar{T}} > \mathfrak{n}$ *and* $\mathfrak{m} > \mathfrak{n}$. *Then* $\boldsymbol{E}_{\mathfrak{n}} \cup \boldsymbol{P}_{\mathfrak{n}+} \neq \boldsymbol{P}_{\mathfrak{n}}$.

Proof. Let $S \subseteqq T$ with $\bar{\bar{S}} = \mathfrak{n}^+$. Choose, for each $t \in S$, $d_t \in \mathfrak{A}_t$ with $d_t \neq 0$, $d_t \neq 1$. Let $d = \bigcap_{t\in S} g_t^*(d_t)$. Let $\mathscr{F}$ be the field of subsets of X which is generated by $\mathscr{F}_{\mathfrak{n}}{}^* \cup \{d\}$. Note that g_t^* is a complete isomorphism from $\mathfrak{A}_t$ into $\mathscr{F}$. Let $\{f, \mathfrak{C}\}$ be any $\mathfrak{m}$-extension of $\mathscr{F}$. It is easily seen that $\{\{f\circ g_t^*\}_{t\in T}, \mathfrak{C}\} \in \boldsymbol{P}_{\mathfrak{n}}$.

Consider the isomorphism $h_{\mathfrak{n}}$ from $\mathscr{F}_{\mathfrak{n}}{}^*$ into $\mathfrak{C}$. $h_{\mathfrak{n}} \circ g_t^* = f \circ g_t^*$ for every $t \in T$. By Corollary 1.3 $\prod_{t\in S}^{\mathscr{F}_{\mathfrak{n}}^*} g_t(d_t) = 0$. However $\prod_{t\in S}^{\mathfrak{C}} h_{\mathfrak{n}} \circ g_t^*(d_t) = f(d) \neq 0$. Thus $h_{\mathfrak{n}}$ is not $\mathfrak{m}$-complete and $\{\{f\circ g_t^*\}_{t\in T}, \mathfrak{C}\} \notin \boldsymbol{E}_{\mathfrak{n}}$.

In order to show that $\{\{f\circ g_t^*\}_{t\in T}, \mathfrak{C}\} \notin \boldsymbol{P}_{+\mathfrak{n}}$ it suffices to show that $\prod_{t\in S}^{\mathfrak{C}} f\circ g_t^*(-d_t) = 0$. In particular suppose $b = \prod_{t\in S}^{\mathscr{F}} g_t^*(-d_t) \neq 0$. Since $b\cdot d = 0$ the definition of $\mathscr{F}$ enables us to write $b = \bigcup_{t\in S} b_1 \cdot g_t^*(-d_t)$ with $b_1 \in \mathscr{F}_{\mathfrak{n}}{}^*$. Choose $t_0 \in S$ such that $0 \neq b_1\cdot g_{t_0}^*(-d_{t_0}) \leqq b$. By Lemma 1.2 there is a nonzero element $a = \bigcap_{t\in S'} g_t^*(a_t)$ of $G_{\mathfrak{n}}$ such that $a \subseteqq b_1 \cdot g_{t_0}^*(-d_{t_0})$. Now $\bar{\bar{S'}} \leqq \mathfrak{n}$ and $\bar{\bar{S}} = \mathfrak{n}^+$ and it follows that $a \nleqq b$. Thus $\prod_{t\in S}^{\mathscr{F}} g_t^*(-d_t) = 0$ and since f is $\mathfrak{m}$-complete, $\prod_{t\in S}^{\mathfrak{C}} f\circ g_t^*(-d_t) = 0$.

We now consider the case $\mathfrak{n} = \mathfrak{m}$. It is known that $\boldsymbol{E}_{\mathfrak{m}} \neq \boldsymbol{P}_{\mathfrak{m}}$ if $\mathfrak{m} = \aleph_0$ (see [2], p. 190, Example D). In this example T is the two element set $\{1, 2\}$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are σ-complete Boolean algebras which satisfy the σ-chain condition. The Boolean σ-product $\{\{i_1, i_2\}, \mathscr{B}\}$ is such that the subalgebra $\mathscr{B}_0$ of $\mathscr{B}$ which is generated by $i_1(\mathfrak{A}_1) \cup i_2(\mathfrak{A}_2)$

is not a σ-regular subalgebra of $\mathscr{B}$. Let $\{f, \mathfrak{C}\}$ be any $\mathfrak{m}$-extension of $\mathscr{B}$. It follows, using the σ-chain condition on $\mathfrak{A}_1$ and $\mathfrak{A}_2$, that $\{\{f \circ i_1, f \circ i_2\}, \mathfrak{C}\} \in \boldsymbol{P}_{\mathfrak{m}}$. Since T is finite $\{\{g_1^*, g_2^*\}, \mathscr{F}_m{}^*\}$ is the Boolean product of $\{\mathfrak{A}_1, \mathfrak{A}_2\}$. Let h be the homomorphism from $\mathscr{F}_{\mathfrak{m}}{}^*$ into $\mathscr{B}$ such that $h \circ g_1^* = i_1$ and $h \circ g_2^* = i_2$. Then h is an isomorphism from $\mathscr{F}_{\mathfrak{m}}{}^*$ onto $\mathscr{B}_0$. Consider the isomorphism $h_{\mathfrak{m}}$, from $\mathscr{F}_{\mathfrak{m}}{}^*$ into $\mathfrak{C}$, given by Theorem 1.4. $h_{\mathfrak{m}} = f \circ h$ since they agree on $g_1^*(\mathfrak{A}_1) \cup g_2^*(\mathfrak{A}_2)$. $h_{\mathfrak{m}}$ is not $\mathfrak{m}$-complete because $f(\mathscr{B}_0)$ is not $\mathfrak{m}$-regular in $\mathfrak{C}$. Thus $\{\{f \circ i_1, f \circ i_2\}, \mathfrak{C}\} \notin \boldsymbol{E}_{\mathfrak{m}}$. We give a simple for the case $\mathfrak{m} \geqq 2^{\aleph_0}$.

EXAMPLE 1.8. Assume $\mathfrak{m} \geqq 2^{\aleph}0$ and let T be a set of power $\aleph_0$. For each $t \in T$ let $\mathfrak{A}_t$ be a Boolean algebra having exactly four elements. Let $\mathscr{B}$ be the free Boolean $\mathfrak{m}$-algebra on $\aleph_0$ $\mathfrak{m}$-generators, $(D_t: t \in T\}$. $\mathscr{B}$ is not $\mathfrak{m}$-representable (see [2], p. 134). For each $t \in T$ choose d_t to be one of the atoms of $\mathfrak{A}_t$. Let i_t be the isomorphism from $\mathfrak{A}_t$ into $\mathscr{B}$ such that $i_t(d_t) = D_t$. Then $\{\{i_t\}_{t \in T}, \mathscr{B}\} \in \boldsymbol{P}_{\mathfrak{m}}$. By Lemma 1.2 $\mathscr{F}_{\mathfrak{m}}{}^*$ is atomic, the atoms being all sets of the form $\bigcap_{t \in T} g_t^*(a_t)$, where for each $t \in T$ a_t is an atom of $\mathfrak{A}_t$. Denote the set of atoms of $\mathscr{F}_{\mathfrak{m}}{}^*$ by $\{C_r: r \in R\}$, then $\overline{\overline{R}} = 2^{\aleph_0}$. We consider the isomorphism $h_{\mathfrak{m}}$ from $\mathscr{F}_{\mathfrak{m}}{}^*$ into $\mathscr{B}$. For each $r \in R$, $h_{\mathfrak{m}}(c_r)$ is an atom of $\mathscr{B}$. To show this we define

$$\mathfrak{A} = \{b \in \mathscr{B}\colon \text{ for each } r \in R \text{ either } b \cdot h_{\mathfrak{m}}(c_r) = 0 \text{ or } h_{\mathfrak{m}}(c_r) \leqq b\} .$$

It is easily seen that $\mathfrak{A}$ is an $\mathfrak{m}$-subalgebra of $\mathscr{B}$ which includes $\{D_t: t \in T\}$. Hence $\mathfrak{A} = \mathscr{B}$. Finally, $h_{\mathfrak{m}}$ is not $\mathfrak{m}$-complete. For otherwise $\sum_{r \in R}^{\mathscr{B}} h_{\mathfrak{m}}(c_r) = 1$, and $\mathscr{B}$ would be atomic and hence isomorphic to an $\mathfrak{m}$-field of sets.

2. We now consider the problem of the existence of a smallest element of $\boldsymbol{P}$, relative to the quasi-ordering "$\leqq$". A minimal element of $\boldsymbol{P}$ always exists and can be constructed as follows. Let $\{\{f_t\}_{t \in T}, \mathfrak{C}\}$ be a Boolean product of $\{\mathfrak{A}_t\}_{t \in T}$ and let $\{h, \mathscr{B}\}$ be an $\mathfrak{m}$-completion of $\mathfrak{C}$. Then $\{\{h \circ f_t\}_{t \in T}, \mathscr{B}\}$ is a minimal element of $\boldsymbol{P}$. We shall show that this product need not be a smallest element of $\boldsymbol{P}$. Hence $\boldsymbol{P}$ need not have a smallest element.

EXAMPLE 2.1. Let $\mathfrak{m}$ be any infinite cardinal. Let $\overline{\overline{T}} = \aleph_0$ and suppose that for each $t \in T$ $\mathfrak{A}_t$ is a four element Boolean algebra. For each $t \in T$ choose a_t to be one of the atoms of $\mathfrak{A}_t$. $\mathfrak{C}$ is a free Boolean algebra of power $\aleph_0$, one set of free generators being $\{f_t(a_t): t \in T\}$. $\mathscr{B}$ has a countable dense subset, in particular $\mathscr{B}$ satisfies the countable chain condition. Thus $\mathscr{B}$ is complete. It follows that $\mathscr{B}$ is isomorphic to the quotient algebra $\mathscr{F}/\Delta_0$ where $\mathscr{F}$ is the σ-field of Borel subsets of the unit interval $I = \{x: 0 < x \leqq 1\}$ of real numbers and Δ_0 is the ideal consisting of those Borel sets which are of the first category.

To show that $\{\{h \circ f_t\}_{t\in T}, \mathscr{B}\}$ is not a smallest element of $\boldsymbol{P}$ we construct another ($\mathfrak{m}$-0) product as follows. Let G be the set of all halfopen intervals of the form $\{x: 0 < x \leqq r\}$ such that r is rational and $0 < r \leqq 1$. $\mathscr{F}$ is σ-generated by G. The subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ which is generated by G is denumerable and atomless. Hence $\mathscr{F}_0$ is isomorphic to $\mathfrak{C}$ (see [1], p. 54). Let g be an isomorphism from $\mathfrak{C}$ onto $\mathscr{F}_0$. Let Δ_1 be the ideal of $\mathscr{F}$ consisting of those Borel sets having Lebesgue measure 0. We note that $\mathscr{F}_0 \cap \Delta_1 = \{0\}$. Finally for each $t \in T$ let h_t be the isomorphism from $\mathfrak{A}_t$ into $\mathscr{F}/\Delta_1$ defined by $h_t(a_t) = [g \circ f_t(a_t)]\Delta_1$. It is easily seen that $\{\{h_t\}_{t\in T}, \mathscr{F}/\Delta_1\} \in \boldsymbol{P}$.

Now assume $\{\{h \circ f_t\}_{t\in T}, \mathscr{B}\} \leqq \{\{h_t\}_{t\in T}, \mathscr{F}/\Delta_1$. Then there is an $\mathfrak{m}$-homomorphism p from $\mathscr{F}/\Delta_1$ onto $\mathscr{F}/\Delta_0$. Since $\mathscr{F}/\Delta_1$ satisfies the countable chain condition the kernel of p is a principal ideal. $\mathscr{F}/\Delta_0$ is isomorphic to a principal ideal of $\mathscr{F}/\Delta_1$. However $\mathscr{F}/\Delta_1$ is homogeneous (see [2], p. 105). Thus $\mathscr{F}/\Delta_0$ is isomorphic to $\mathscr{F}/\Delta_1$, which is a contradiction.

Next we consider the problem of the existence of a smallest element of $\boldsymbol{P}_\mathfrak{n}$. Let $\{g, \mathscr{B}\}$ be an $\mathfrak{m}$-completion of $\mathscr{F}_\mathfrak{n}^*$. Then $\{\{g \circ g_t^*\}_{t\in T}, \mathscr{B}\}$ is a minimal element of $\boldsymbol{P}_\mathfrak{n}$. Also it is known (see [2], p. 183) that if all the $\mathfrak{A}_t$ are $\mathfrak{m}$-representable then there is an ($\mathfrak{m}$-$\mathfrak{n}$) product $\{\{i_t\}_{t\in T}, \mathfrak{C}\}$ for which $\mathfrak{C}$ is $\mathfrak{m}$-representable. We give an example of $\{\mathfrak{A}_t\}_{t\in T}$ for which $\mathscr{B}$ is not $\mathfrak{m}$-representable and $\{\{g \circ g_t^*\}_{t\in T}, \mathscr{B}\}$ is not a smallest element of $\boldsymbol{P}_\mathfrak{n}$.

EXAMPLE 2.2. Assume that $\mathfrak{m} \geqq 2^{(\mathfrak{n}^+)}$. Let $\overline{\overline{T}} = \mathfrak{n}^+$ and for each $t \in T$ let $\mathfrak{A}_t$ be a four element Boolean algebra. We show that $\mathscr{B}$ is not $\mathfrak{n}^+$-distributive. Choose, for each $t \in T$, a_t to be one of the atoms of $\mathfrak{A}_t$. Then

$$\prod_{t\in T}^{\mathscr{B}} (g \circ g_t^*(a_t) + - g \circ g_t^*(a_t)) = 1 .$$

However for each function $\eta \in H^T$ (here $H = \{+1, -1\}$) we have

$$\prod_{t\in T}^{\mathscr{F}_\mathfrak{n}^*} \eta(t) \cdot g_t^*(a_t) = 0 .$$

This follows from Corollary 1.3. Thus $\prod_{t\in T}^{\mathscr{B}} \eta(t) \cdot g \circ g_t^*(a_t) = 0$. This proves $\mathscr{B}$ is not $\mathfrak{n}^+$-distributive and hence not $\mathfrak{m}$-representable.

To show that $\{\{g \circ g_t^*\}_{t\in T}, \mathscr{B}\}$ is not a smallest element of $\boldsymbol{P}_\mathfrak{n}$, let $\{\{i_t\}_{t\in T}, \mathfrak{C}\}$ be any ($\mathfrak{m}$-$\mathfrak{n}$) product of $\{\mathfrak{A}_t\}_{t\in T}$ such that $\mathfrak{C}$ is $\mathfrak{m}$-representable. $\mathscr{B}$ is not an $\mathfrak{m}$-homomorphic image of $\mathfrak{C}$. Thus the inequality

$$\{\{g \circ g_t^*\}_{t \in T}, \mathscr{B}\} \leqq \{\{i_t\}_{t \in T}, \mathfrak{C}\}$$

does not hold.

REFERENCES

1. P. H. Dwinger, *Introduction to Boolean algebras,* Wurzburg, 1961.
2. R. Sikorski, *Boolean algebras,* Second Edition, Springer Verlag, 1964.

Received July 19, 1968.
UNIVERSITY OF WYOMING

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 31, No. 3 BadMonth, 1969