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**ON  $(m - n)$  PRODUCTS OF BOOLEAN ALGEBRAS**

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# ON $(m - n)$ PRODUCTS OF BOOLEAN ALGEBRAS

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This discussion begins with the problem of whether or not all  $(m - n)$  products of an indexed set  $\{\mathfrak{A}_t\}_{t \in T}$  of Boolean algebras can be obtained as  $m$ -extensions of a particular algebra  $\mathcal{F}_n^*$ . The construction of  $\mathcal{F}_n^*$  is similar to the construction of the Boolean product of  $\{\mathfrak{A}_t\}_{t \in T}$ ; however the  $\mathfrak{A}_t$  are embedded in  $\mathcal{F}_n^*$  in such a way that their images are  $n$ -independent. If there is a cardinal number  $n'$ , satisfying  $n < n' \leq m$ , then  $(m - n')$  products are not obtainable in this manner. For the case  $n = m$  an example shows the answer to be negative. It is explained how the class of  $m$ -extensions of  $\mathcal{F}_n^*$  is situated in the class of all  $(m - n)$  products of  $\{\mathfrak{A}_t\}_{t \in T}$ . A set of  $m$ -representable Boolean algebras is given for which the minimal  $(m - n)$  product is not  $m$ -representable and for which there is no smallest  $(m - n)$  product.

These problems have been proposed by R. Sikorski (see [2]). Concerning  $\{\mathfrak{A}_t\}_{t \in T}$ , it is assumed throughout that each of these algebras has at least four elements.  $m$  and  $n$  will always denote infinite cardinals with  $n \leq m$ . All definitions are taken from [2]. An  $m$ -homomorphism is a homomorphism that is conditionally  $m$ -complete. We denote the class of  $(m - n)$  products of  $\{\mathfrak{A}_t\}_{t \in T}$  by  $P_n$  and the class of  $(m - 0)$  products by  $P$ . Let  $\{\{i_t\}_{t \in T}, \mathcal{B}\}$  and  $\{\{j_t\}_{t \in T}, \mathcal{C}\}$  be elements of  $P$ . We say that

$$\{\{i_t\}_{t \in T}, \mathcal{B}\} \leq \{\{j_t\}_{t \in T}, \mathcal{C}\}$$

provided there is an  $m$ -homomorphism  $h$  from  $\mathcal{C}$  onto  $\mathcal{B}$  such that  $h \circ j_t = i_t$  for  $t \in T$ . The relation " $\leq$ " is a quasi-ordering of  $P$ . Two  $(m - 0)$  products are isomorphic if each is  $\leq$  to the other.

The particular product,  $\{\{g_t^*\}_{t \in T}, \mathcal{F}_n^*\}$  of  $\{\mathfrak{A}_t\}_{t \in T}$  mentioned above is defined as follows. For each  $t \in T$  let  $X_t$  be the Stone space of  $\mathfrak{A}_t$  and let  $g_t$  be an isomorphism from  $\mathfrak{A}_t$  onto the field  $\mathcal{F}_t$  of all open and closed subsets of  $X_t$ . Let  $X$  be the Cartesian product of the sets  $X_t$ , and for each  $t \in T$  and each  $b \in \mathfrak{A}_t$ , set

$$(1) \quad g_t^*(b) = [x \in X: x(t) \in g_t(b)] .$$

Let  $G_n$  be the set of all subsets  $a$  of  $X$  which satisfy the following condition:

$$a = \bigcap_{t \in S} g_t^*(b_t) \text{ where } b_t \in \mathfrak{A}_t, S \subseteq T \text{ and } \bar{S} \leq n .$$

Finally, let  $\mathcal{F}_n^*$  be the field of subsets of  $X$  which is generated by  $G_n$ .

$\mathcal{F}_n^*$  is a base for the  $n$ -topology on  $X$ .  $g_i^*$  is a complete isomorphism from  $\mathfrak{A}_i$  into  $\mathcal{F}_n^*$ . The set  $\{g_i^*(\mathfrak{A}_i)\}$ , of subalgebras, is  $n$ -independent.

A Boolean  $(m - n)$  product  $\{\{i_t\}_{t \in T}, \mathcal{B}\}$  is said to belong to  $E_n$  if and only if there is an  $m$ -isomorphism  $h$  (from  $\mathcal{F}_n^*$  into  $\mathcal{B}$ ) such that  $\{h, \mathcal{B}\}$  is an  $m$ -extension of  $\mathcal{F}_n^*$  and for each  $t \in T$   $h \circ g_t^* = i_t$ .

For every  $m$ -extension  $\{h, \mathcal{B}\}$  of  $\mathcal{F}_n^*$ ,  $\{\{h \circ g_t^*\}_{t \in T}, \mathcal{B}\} \in E_n$ . Clearly  $E_n \subseteq P_n$  and  $E_n$  is not empty.  $m$ -extensions of  $\mathcal{F}_n^*$  seem to provide the most natural examples of Boolean  $(m - n)$  products.

1. LEMMA 1.1. Let  $\{\mathcal{B}_t\}_{t \in T}$  be an  $n$ -independent set of subalgebras of a Boolean algebra  $\mathfrak{A}$  and let  $S$  and  $S'$  be subsets of  $T$  with  $\bar{S} \leq n$  and  $\bar{S}' \leq n$ . For each  $t$  let  $a_t$  and  $b_t$  be nonzero elements of  $\mathcal{B}_t$ . Then

(i)  $\prod_{t \in S} a_t \leq \prod_{t \in S} b_t$  if and only if  $a_t \leq b_t$  for each  $t \in S$ ;

(ii)  $\prod_{t \in S} a_t = \prod_{t \in S} b_t$  implies that  $a_t = b_t$  for  $t \in S \cap S'$ ,  $a_t = 1$  for  $t \in S - S'$ , and  $b_t = 1$  for  $t \in S' - S$ .

*Proof.* (i) Assume that for some  $t_0 \in S$ ,  $a_{t_0} \not\leq b_{t_0}$ . Define

$$C_t = \begin{cases} a_t & \text{if } t \in S \text{ and } t \neq t_0, \\ a_{t_0} \cdot (-b_{t_0}) & \text{if } t = t_0. \end{cases}$$

Set  $c = \prod_{t \in S} c_t$ , and note that  $c \neq 0$ ,  $c \leq \prod_{t \in S} a_t$ , and  $c \cdot \prod_{t \in S} b_t = 0$ . The converse is clear.

To prove (ii) we define

$$x_t = \begin{cases} a_t & \text{if } t \in S, \\ 1 & \text{if } t \in S' - S; \end{cases} \quad \text{and} \quad y_t = \begin{cases} b_t & \text{if } t \in S', \\ 1 & \text{if } t \in S - S'. \end{cases}$$

Now

$$\prod_{t \in S \cup S'} x_t = \prod_{t \in S} a_t = \prod_{t \in S'} b_t = \prod_{t \in S \cup S'} y_t$$

and (ii) follows from (i).

LEMMA 1.2. Let  $\{\mathcal{B}_t\}_{t \in T}$  be an  $n$ -independent set of subalgebras of a Boolean algebra  $\mathfrak{A}$ . Let  $G$  be the set of all meets  $\prod_{t \in S} a_t$  such that  $S \subseteq T$ ,  $\bar{S} \leq n$ , and for each  $t \in S$   $a_t$  is a nonzero element of  $\mathcal{B}_t$ . Assume further that  $G$  generates  $\mathfrak{A}$ . Then  $G$  is dense in  $\mathfrak{A}$ .

*Proof.* First note that for  $g, g' \in G$  either  $g \cdot g' = 0$  or else  $g \cdot g' \in G$ . Thus every nonzero element of  $\mathfrak{A}$  is a finite join of elements of the form  $g \cdot \prod_{i < k} (-g_i)$  with  $g, g_i \in G$  and  $k$  finite. (This notation is intended

to include the special cases  $g$  and  $-g$ .) Now suppose  $g \cdot \prod_{i < k}^{\mathfrak{A}} (-g_i) \neq 0$ , so that  $g \not\leq \sum_{i < k} g_i$ . We write a common form  $g = \prod_{t \in S} a_t$ , and for each  $i < k$   $g_i = \prod_{t \in S} a_{i,t}$  where  $S \subseteq T$ ,  $\bar{S} \leq n$ , and for each  $t \in S$   $a_t$  and  $a_{i,t}$  are nonzero elements of  $\mathcal{B}_i$ . Since  $k$  is finite every Boolean algebra is  $(k - n)$ -distributive (see [2], p. 62). We have

$$\prod_{t \in S} a_t \not\leq \sum_{i < k} \prod_{t \in S} a_{i,t} = \prod_{\gamma \in S^k} \sum_{i < k} a_{i,\gamma(i)} .$$

(Here  $S^k$  denotes the set of all functions from  $k = \{0, 1, \dots, k - 1\}$  into  $S$ .) Choose  $\phi \in S^k$  such that  $\prod_{t \in S} a_t \not\leq \sum_{i < k} a_{i,\phi(i)}$ . We have, for each  $s \in \{\phi(i) : i < k\}$ ,  $a_s \not\leq \sum_{\phi(i)=s} a_{i,\phi(i)}$ . Define

$$b_t = \begin{cases} a_t & \text{if } t \in S - \{\phi(i) : i < k\} \\ a_t \cdot - \sum_{\phi(i)=t} a_{i,\phi(i)} & \text{if } t \in \{\phi(i) : i < k\} . \end{cases}$$

Finally let  $b = \prod_{t \in S} b_t$ . Clearly  $b \neq 0$ ,  $b \in G$  and  $b \leq g$ . For each  $t \in \{\phi(i) : i < k\}$ ,  $b_t \cdot \sum_{\phi(i)=t} a_{i,\phi(i)} = 0$ , so that  $b \cdot \sum_{i < k} a_{i,\phi(i)} = 0$ . It follows that  $b \cdot \sum_{i < k} g_i = 0$ , hence  $b \leq g \cdot \prod_{i < k} (-g_i)$ .

**COROLLARY 1.3.** *If  $\bar{S} > n$ , and for each  $t \in S$ ,  $a_t \neq 1$ , then  $\prod_{t \in S}^{\mathfrak{A}} a_t = 0$ .*

**THEOREM 1.4.** *Let  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in \mathbf{P}_n$ . There is one and only one isomorphism  $h_n$  from  $\mathcal{F}_n^*$  into  $\mathcal{B}$  which satisfies the following completeness condition:*

$$(c) \quad h_n\left(\prod_{t \in S}^{\mathfrak{A}_n} g_t^*(a_t)\right) = \prod_{t \in S} i_t(a_t) \text{ whenever } S \subseteq T, \bar{S} \leq n, \\ a_t \in \mathfrak{A}_t \text{ and } a_t \neq 0 .$$

*Proof.* Let  $G$  be the set of all meets  $\prod_{t \in S} i_t(a_t)$  such that  $S \subseteq T$ ,  $\bar{S} \leq n$ , each  $a_t \in \mathfrak{A}_t$  and  $a_t \neq 0$ . Let  $\mathfrak{A}$  be the subalgebra of  $\mathcal{B}$  which is generated by  $G$ . For  $\prod_{t \in S} i_t(a_t) \in G$  it is clear that  $\prod_{t \in S} i_t(a_t) = \prod_{t \in S}^{\mathfrak{A}} i_t(a_t)$ . By Lemma 1.2  $G$  is dense in  $\mathfrak{A}$ . Also  $G_n$  is dense in  $\mathcal{F}_n^*$ . For  $a \in G_n$  write  $a = \bigcap_{t \in S} g_t^*(a_t) = \prod_{t \in S}^{\mathfrak{A}_n} g_t^*(a_t)$ . Define  $h(a) = \prod_{t \in S}^{\mathfrak{A}} i_t(a_t)$ . It is easily seen, using Lemma 1.1, that

- (i)  $h$  is a one to one function from  $G_n$  onto  $G$ ;
- (ii) for  $a, b \in G_n$ ,  $a \leq b$  if and only if  $h(a) \leq h(b)$ .

It follows (see [2], p. 37) that  $h$  can be extended to an isomorphism  $h_n$  from  $\mathcal{F}_n^*$  onto  $\mathfrak{A}$ .  $h_n$  is uniquely determined by condition (c) because  $G_n$  generates  $\mathcal{F}_n^*$ .

**COROLLARY 1.5.** *The product  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in \mathbf{E}_n$  if and only if  $h_n$  is m-complete.*

*Proof.* Let  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in E_n$ . There is an  $m$ -isomorphism  $f$  from  $\mathcal{F}_n^*$  into  $\mathcal{B}$  such that for each  $t \in T, f \circ g_t^* = i_t$ .  $f$  satisfies condition (c) so  $f = h_n$ .

**COROLLARY 1.6.** *Assume  $\bar{T} > n$  and that  $m \geq n' > n$ . Then  $P_{n'} \cap E_n$  is empty.*

*Proof.* Let  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in P_{n'}$ . Consider the isomorphism  $h_n$  from  $\mathcal{F}_n^*$  into  $\mathcal{B}$ . Choose  $S \subseteq T, \bar{S} = n^+$ , and for each  $t \in S$  choose  $a_t \in \mathfrak{A}_t$  with  $a_t \neq 0, a_t \neq 1$ . By Corollary 1.3

$$\prod_{t \in S}^{\mathcal{F}_n^*} g_t^*(a_t) = 0.$$

However  $0 \neq \prod_{t \in S} i_t(a_t) = \prod_{t \in S} h_n \circ g_t^*(a_t)$  so that  $h_n$  is not  $m$ -complete.

There is an interesting contrast between  $E_n$  and  $P_{n'}$ , (under the hypotheses of Corollary 1.6). Let  $\{\{i_t\}_{t \in T}, \mathcal{B}\}$  and  $\{\{j_t\}_{t \in T}, \mathbb{C}\}$  be elements of  $P_n$  with  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \leq \{\{j_t\}_{t \in T}, \mathbb{C}\}$ . It is known (see [2], p. 179) that if  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in P_{n'}$ , then  $\{\{j_t\}_{t \in T}, \mathbb{C}\} \in P_{n'}$ . On the other hand if  $\{\{j_t\}_{t \in T}, \mathbb{C}\} \in E_n$  then we have  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in E_n$ .

**COROLLARY 1.7.** *Assume  $\bar{T} > n$  and  $m > n$ . Then  $E_n \cup P_{n^+} \neq P_n$ .*

*Proof.* Let  $S \subseteq T$  with  $\bar{S} = n^+$ . Choose, for each  $t \in S, d_t \in \mathfrak{A}_t$  with  $d_t \neq 0, d_t \neq 1$ . Let  $d = \bigcap_{t \in S} g_t^*(d_t)$ . Let  $\mathcal{F}$  be the field of subsets of  $X$  which is generated by  $\mathcal{F}_n^* \cup \{d\}$ . Note that  $g_t^*$  is a complete isomorphism from  $\mathfrak{A}_t$  into  $\mathcal{F}$ . Let  $\{f, \mathbb{C}\}$  be any  $m$ -extension of  $\mathcal{F}$ . It is easily seen that  $\{\{f \circ g_t^*\}_{t \in T}, \mathbb{C}\} \in P_n$ .

Consider the isomorphism  $h_n$  from  $\mathcal{F}_n^*$  into  $\mathbb{C}$ .  $h_n \circ g_t^* = f \circ g_t^*$  for every  $t \in T$ . By Corollary 1.3  $\prod_{t \in S}^{\mathcal{F}_n^*} g_t(d_t) = 0$ . However  $\prod_{t \in S}^{\mathbb{C}} h_n \circ g_t^*(d_t) = f(d) \neq 0$ . Thus  $h_n$  is not  $m$ -complete and  $\{\{f \circ g_t^*\}_{t \in T}, \mathbb{C}\} \in E_n$ .

In order to show that  $\{\{f \circ g_t^*\}_{t \in T}, \mathbb{C}\} \notin P_{n^+}$  it suffices to show that  $\prod_{t \in S} f \circ g_t^*(-d_t) = 0$ . In particular suppose  $b = \prod_{t \in S}^{\mathbb{C}} g_t^*(-d_t) \neq 0$ . Since  $b \cdot d = 0$  the definition of  $\mathcal{F}$  enables us to write  $b = \bigcup_{t \in S} b_t \cdot g_t^*(-d_t)$  with  $b_t \in \mathcal{F}_n^*$ . Choose  $t_0 \in S$  such that  $0 \neq b_{t_0} \cdot g_{t_0}^*(-d_{t_0}) \leq b$ . By Lemma 1.2 there is a nonzero element  $a = \bigcap_{t \in S'} g_t^*(a_t)$  of  $G_n$  such that  $a \subseteq b_{t_0} \cdot g_{t_0}^*(-d_{t_0})$ . Now  $S^{\bar{a}} \leq n$  and  $\bar{S} = n^+$  and it follows that  $a \not\subseteq b$ . Thus  $\prod_{t \in S}^{\mathcal{F}_n^*} g_t^*(-d_t) = 0$  and since  $f$  is  $m$ -complete,  $\prod_{t \in S}^{\mathbb{C}} f \circ g_t^*(-d_t) = 0$ .

We now consider the case  $n = m$ . It is known that  $E_m \neq P_m$  if  $m = \aleph_0$  (see [2], p. 190, Example D). In this example  $T$  is the two element set  $\{1, 2\}$ ,  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are  $\sigma$ -complete Boolean algebras which satisfy the  $\sigma$ -chain condition. The Boolean  $\sigma$ -product  $\{\{i_1, i_2\}, \mathcal{B}\}$  is such that the subalgebra  $\mathcal{B}_0$  of  $\mathcal{B}$  which is generated by  $i_1(\mathfrak{A}_1) \cup i_2(\mathfrak{A}_2)$

is not a  $\sigma$ -regular subalgebra of  $\mathcal{B}$ . Let  $\{f, \mathbb{C}\}$  be any  $m$ -extension of  $\mathcal{B}$ . It follows, using the  $\sigma$ -chain condition on  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$ , that  $\{\{f \circ i_1, f \circ i_2\}, \mathbb{C}\} \in \mathbf{P}_m$ . Since  $T$  is finite  $\{\{g_1^*, g_2^*\}, \mathcal{F}_m^*\}$  is the Boolean product of  $\{\mathfrak{A}_1, \mathfrak{A}_2\}$ . Let  $h$  be the homomorphism from  $\mathcal{F}_m^*$  into  $\mathcal{B}$  such that  $h \circ g_1^* = i_1$  and  $h \circ g_2^* = i_2$ . Then  $h$  is an isomorphism from  $\mathcal{F}_m^*$  onto  $\mathcal{B}_0$ . Consider the isomorphism  $h_m$ , from  $\mathcal{F}_m^*$  into  $\mathbb{C}$ , given by Theorem 1.4.  $h_m = f \circ h$  since they agree on  $g_1^*(\mathfrak{A}_1) \cup g_2^*(\mathfrak{A}_2)$ .  $h_m$  is not  $m$ -complete because  $f(\mathcal{B}_0)$  is not  $m$ -regular in  $\mathbb{C}$ . Thus  $\{\{f \circ i_1, f \circ i_2\}, \mathbb{C}\} \notin \mathbf{E}_m$ . We give a simple for the case  $m \geq 2^{\aleph_0}$ .

EXAMPLE 1.8. Assume  $m \geq 2^{\aleph_0}$  and let  $T$  be a set of power  $\aleph_0$ . For each  $t \in T$  let  $\mathfrak{A}_t$  be a Boolean algebra having exactly four elements. Let  $\mathcal{B}$  be the free Boolean  $m$ -algebra on  $\aleph_0$   $m$ -generators,  $(D_t; t \in T)$ .  $\mathcal{B}$  is not  $m$ -representable (see [2], p. 134). For each  $t \in T$  choose  $d_t$  to be one of the atoms of  $\mathfrak{A}_t$ . Let  $i_t$  be the isomorphism from  $\mathfrak{A}_t$  into  $\mathcal{B}$  such that  $i_t(d_t) = D_t$ . Then  $\{\{i_t\}_{t \in T}, \mathcal{B}\} \in \mathbf{P}_m$ . By Lemma 1.2  $\mathcal{F}_m^*$  is atomic, the atoms being all sets of the form  $\bigcap_{t \in T} g_t^*(a_t)$ , where for each  $t \in T$   $a_t$  is an atom of  $\mathfrak{A}_t$ . Denote the set of atoms of  $\mathcal{F}_m^*$  by  $\{C_r; r \in R\}$ , then  $\bar{R} = 2^{\aleph_0}$ . We consider the isomorphism  $h_m$  from  $\mathcal{F}_m^*$  into  $\mathcal{B}$ . For each  $r \in R$ ,  $h_m(c_r)$  is an atom of  $\mathcal{B}$ . To show this we define

$$\mathfrak{A} = \{b \in \mathcal{B} : \text{for each } r \in R \text{ either } b \cdot h_m(c_r) = 0 \text{ or } h_m(c_r) \leq b\}.$$

It is easily seen that  $\mathfrak{A}$  is an  $m$ -subalgebra of  $\mathcal{B}$  which includes  $\{D_t; t \in T\}$ . Hence  $\mathfrak{A} = \mathcal{B}$ . Finally,  $h_m$  is not  $m$ -complete. For otherwise  $\sum_{r \in R} h_m(c_r) = 1$ , and  $\mathcal{B}$  would be atomic and hence isomorphic to an  $m$ -field of sets.

2. We now consider the problem of the existence of a smallest element of  $\mathbf{P}$ , relative to the quasi-ordering " $\leq$ ". A minimal element of  $\mathbf{P}$  always exists and can be constructed as follows. Let  $\{\{f_t\}_{t \in T}, \mathbb{C}\}$  be a Boolean product of  $\{\mathfrak{A}_t\}_{t \in T}$  and let  $\{h, \mathcal{B}\}$  be an  $m$ -completion of  $\mathbb{C}$ . Then  $\{\{h \circ f_t\}_{t \in T}, \mathcal{B}\}$  is a minimal element of  $\mathbf{P}$ . We shall show that this product need not be a smallest element of  $\mathbf{P}$ . Hence  $\mathbf{P}$  need not have a smallest element.

EXAMPLE 2.1. Let  $m$  be any infinite cardinal. Let  $\bar{T} = \aleph_0$  and suppose that for each  $t \in T$   $\mathfrak{A}_t$  is a four element Boolean algebra. For each  $t \in T$  choose  $a_t$  to be one of the atoms of  $\mathfrak{A}_t$ .  $\mathbb{C}$  is a free Boolean algebra of power  $\aleph_0$ , one set of free generators being  $\{f_t(a_t); t \in T\}$ .  $\mathcal{B}$  has a countable dense subset, in particular  $\mathcal{B}$  satisfies the countable chain condition. Thus  $\mathcal{B}$  is complete. It follows that  $\mathcal{B}$  is isomorphic to the quotient algebra  $\mathcal{F}/\Delta_0$  where  $\mathcal{F}$  is the  $\sigma$ -field

of Borel subsets of the unit interval  $I = \{x: 0 < x \leq 1\}$  of real numbers and  $\mathcal{A}_0$  is the ideal consisting of those Borel sets which are of the first category.

To show that  $\{\{h \circ f_i\}_{i \in T}, \mathcal{B}\}$  is not a smallest element of  $\mathbf{P}$  we construct another (m-0) product as follows. Let  $G$  be the set of all halfopen intervals of the form  $\{x: 0 < x \leq r\}$  such that  $r$  is rational and  $0 < r \leq 1$ .  $\mathcal{F}$  is  $\sigma$ -generated by  $G$ . The subalgebra  $\mathcal{F}_0$  of  $\mathcal{F}$  which is generated by  $G$  is denumerable and atomless. Hence  $\mathcal{F}_0$  is isomorphic to  $\mathbb{C}$  (see [1], p. 54). Let  $g$  be an isomorphism from  $\mathbb{C}$  onto  $\mathcal{F}_0$ . Let  $\mathcal{A}_1$  be the ideal of  $\mathcal{F}$  consisting of those Borel sets having Lebesgue measure 0. We note that  $\mathcal{F}_0 \cap \mathcal{A}_1 = \{0\}$ . Finally for each  $t \in T$  let  $h_t$  be the isomorphism from  $\mathfrak{A}_t$  into  $\mathcal{F}/\mathcal{A}_1$  defined by  $h_t(a_i) = [g \circ f_i(a_i)]\mathcal{A}_1$ . It is easily seen that  $\{\{h_t\}_{t \in T}, \mathcal{F}/\mathcal{A}_1\} \in \mathbf{P}$ .

Now assume  $\{\{h \circ f_i\}_{i \in T}, \mathcal{B}\} \leq \{\{h_t\}_{t \in T}, \mathcal{F}/\mathcal{A}_1\}$ . Then there is an m-homomorphism  $p$  from  $\mathcal{F}/\mathcal{A}_1$  onto  $\mathcal{F}/\mathcal{A}_0$ . Since  $\mathcal{F}/\mathcal{A}_1$  satisfies the countable chain condition the kernel of  $p$  is a principal ideal.  $\mathcal{F}/\mathcal{A}_0$  is isomorphic to a principal ideal of  $\mathcal{F}/\mathcal{A}_1$ . However  $\mathcal{F}/\mathcal{A}_1$  is homogeneous (see [2], p. 105). Thus  $\mathcal{F}/\mathcal{A}_0$  is isomorphic to  $\mathcal{F}/\mathcal{A}_1$ , which is a contradiction.

Next we consider the problem of the existence of a smallest element of  $\mathbf{P}_n$ . Let  $\{g, \mathcal{B}\}$  be an m-completion of  $\mathcal{F}_n^*$ . Then  $\{\{g \circ g_i^*\}_{i \in T}, \mathcal{B}\}$  is a minimal element of  $\mathbf{P}_n$ . Also it is known (see [2], p. 183) that if all the  $\mathfrak{A}_t$  are m-representable then there is an (m-n) product  $\{\{i_t\}_{t \in T}, \mathbb{C}\}$  for which  $\mathbb{C}$  is m-representable. We give an example of  $\{\mathfrak{A}_t\}_{t \in T}$  for which  $\mathcal{B}$  is not m-representable and  $\{\{g \circ g_i^*\}_{i \in T}, \mathcal{B}\}$  is not a smallest element of  $\mathbf{P}_n$ .

EXAMPLE 2.2. Assume that  $m \geq 2^{(n^+)}$ . Let  $T = n^+$  and for each  $t \in T$  let  $\mathfrak{A}_t$  be a four element Boolean algebra. We show that  $\mathcal{B}$  is not  $n^+$ -distributive. Choose, for each  $t \in T$ ,  $a_t$  to be one of the atoms of  $\mathfrak{A}_t$ . Then

$$\prod_{i \in T}^{\mathcal{B}} (g \circ g_i^*(a_i) + - g \circ g_i^*(a_i)) = 1 .$$

However for each function  $\eta \in H^T$  (here  $H = \{+1, -1\}$ ) we have

$$\prod_{i \in T}^{\mathcal{F}_n^*} \eta(t) \cdot g_i^*(a_i) = 0 .$$

This follows from Corollary 1.3. Thus  $\prod_{i \in T}^{\mathcal{B}} \eta(t) \cdot g \circ g_i^*(a_i) = 0$ . This proves  $\mathcal{B}$  is not  $n^+$ -distributive and hence not m-representable.

To show that  $\{\{g \circ g_i^*\}_{i \in T}, \mathcal{B}\}$  is not a smallest element of  $\mathbf{P}_n$ , let  $\{\{i_t\}_{t \in T}, \mathbb{C}\}$  be any (m-n) product of  $\{\mathfrak{A}_t\}_{t \in T}$  such that  $\mathbb{C}$  is m-representable.  $\mathcal{B}$  is not an m-homomorphic image of  $\mathbb{C}$ . Thus the inequality

$$\{\{g \circ g_i^*\}_{t \in T}, \mathcal{B}\} \leq \{\{i_t\}_{t \in T}, \mathbb{C}\}$$

does not hold.

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