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**THE AVERAGE OF A GAUGE**

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The setting for the discussion is a real linear space  $\mathcal{L}$  with an inner product  $(x, y)$ . It is assumed that  $\mathcal{L}$  is complete with respect to the norm generated by this inner product. Clearly the gauges on  $\mathcal{L}$  and the bodies (closed, bounded, absolutely convex sets containing the origin  $\theta$  as an interior point) generate each other. The norm  $e(x) = (x, x)^{\frac{1}{2}}$  is a special gauge; it is customary to write  $\|x\|$  instead of  $e(x)$ . In general, a gauge will be denoted  $\phi(x)$ , or briefly by  $\phi$ ; the symbol  $\phi^*$  represents the conjugate of  $\phi$ .

**PROBLEM.** Consider two gauges  $\phi_1$  and  $\phi_2$  such that  $e \leq \phi_1 \leq \phi_2$ . Under what conditions may one conclude that

$$(*) \quad \frac{\phi_1 + \phi_1^*}{2} \leq \frac{\phi_2 + \phi_2^*}{2} ?$$

That such an order relation may exist is suggested by the fact that for any gauge  $\phi$ , its "average"  $(\phi + \phi^*)/2$  is well behaving with respect to  $e$ , being always  $\geq e$ . Moreover, the average of the average is a better approximation to  $e$ , and so on. Indeed, it is known that the sequence of successive averages converges decreasingly to  $e$ .

**SOLUTION.** With each gauge  $\phi$  (body  $K$ ) we associate its spread  $\mathfrak{S}(x)$ . The last concept is a very natural one and is defined as follows: For  $\theta \neq x \in \mathcal{L}$ , consider the line joining  $\theta$  and  $x$ . Let  $\omega(x)$  represent the (width) distance between the two support hyperplanes of  $K$  orthogonal to that line, while  $\delta(x)$  represents the (diameter) length of the chord of  $K$  lying on that line. We then define  $\mathfrak{S}_\phi(x) = \mathfrak{S}_K(x) = \omega(x) - \delta(x)$ . Clearly,  $\mathfrak{S}(x) = \mathfrak{S}(\lambda x)$  for  $\lambda \neq 0$ . It turns out that for any pair of gauges  $\phi_1$  and  $\phi_2$  such that  $e \leq \phi_1 \leq \phi_2$ , the relation  $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$  implies inequality (\*) and also  $\phi_1 \phi_1^* \leq \phi_2 \phi_2^*$ .

In particular, for any pair of the well-known gauges  $(\sum_{j=1}^n |x_j|^p)^{1/p}$ , the corresponding spreads satisfy the above relation. We may express, therefore, the following:

Let  $p \geq 1$ . Define  $q = p/(p-1)$  if  $p > 1$ , and  $q = \infty$  if  $p = 1$ . For any fixed point  $(x_1, x_2, \dots, x_n)$ ,

$$\frac{1}{2} \left[ \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} + \left( \sum_{j=1}^n |x_j|^q \right)^{1/q} \right]$$

and

$$\left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \left( \sum_{j=1}^n |x_j|^q \right)^{1/q}$$

are decreasing functions of  $p$  in the interval  $1 \leq p \leq 2$ .

**Preliminaries.** A body is a closed, bounded, and absolutely convex set containing  $\theta$  in its interior. The boundary of a body is called its surface. For  $\theta \neq x \in \mathcal{L}$ , the ray determined by  $x$  is  $\{\lambda x: \lambda > 0\}$ . The gauge of a body  $K$ , denoted  $\phi_K$ , is defined as follows.  $\phi_K(x) = \inf \{\lambda: \lambda > 0, x \in \lambda K\}$ . Clearly  $\phi = \phi_K$  possesses the following properties:  $\phi(x) \geq 0$ ,  $\phi(x + y) \leq \phi(x) + \phi(y)$ , and  $\phi(\alpha x) = |\alpha| \phi(x)$  for all scalars  $\alpha$ . Also,  $\phi(x)$  is necessarily continuous.

Moreover, if  $K$  is a body having  $\phi$  as its gauge, then  $K = \{x: \phi(x) \leq 1\}$ ; its surface  $S = \{x: \phi(x) = 1\}$ . A body and its gauge determine each other. Indeed,  $\phi(x) = \|x\|/\|S(x)\|$ , where  $S(x)$  is the point of the surface of  $K$  on the ray determined by  $x$ . Also,  $K_1 \subseteq K_2$  if and only if  $\phi_1(x) \geq \phi_2(x)$  for all  $x \in \mathcal{L}$ .

Let  $\phi$  be the gauge of the body  $K$ . The conjugate  $\phi^*$  of the gauge  $\phi$  is defined as  $\phi^*(x) = \text{Supremum}_{\phi(y)=1} (x, y)$ . Clearly  $\phi^*$  is the gauge of the set  $K^* = \{w: (w, v) \leq 1 \text{ for all } v \in K\}$ , commonly referred to as the conjugate or polar set of  $K$ , (Cf. Eggleston, [1]).

The average of the gauge  $\phi$  is defined as  $(\phi + \phi^*)/2$ . Similarly, the average of the body  $K$  is defined as  $(K + K^*)/2$ .

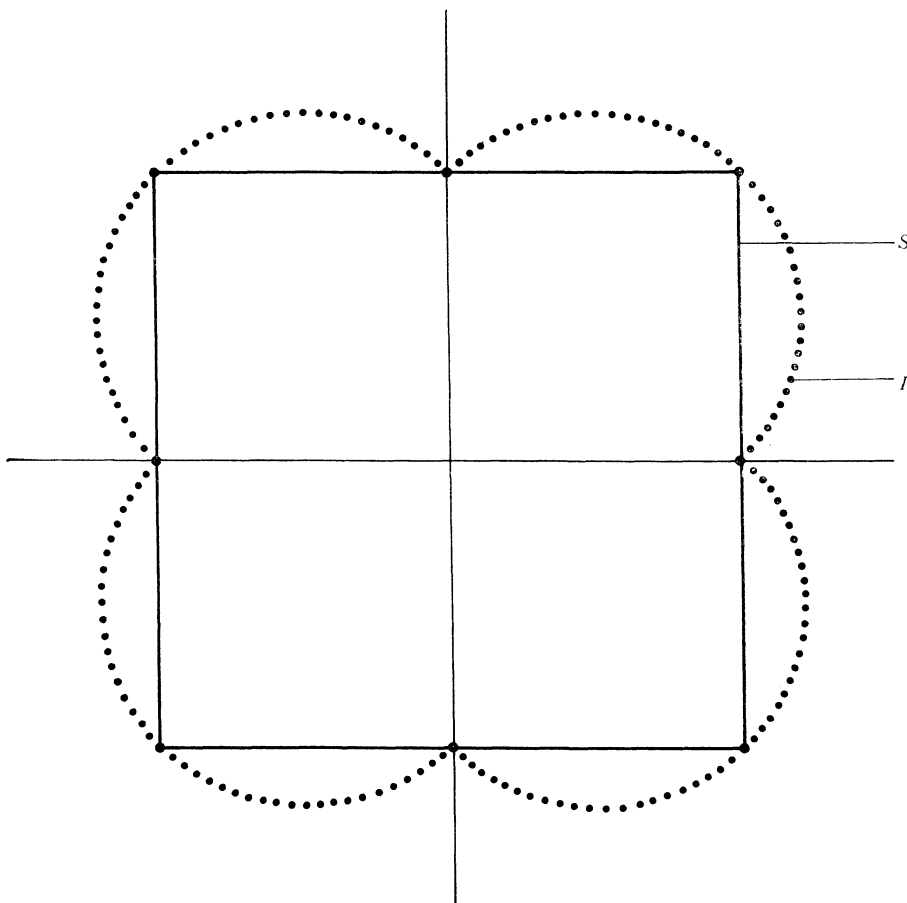
We show later that not every gauge is an average.

**The indicatrix and conjugate.** In  $\mathcal{L}$ , let a body  $K$  be given. Then  $K^*$  is necessarily a body also; we denote its surface by  $^*S$ . (Placing the asterisk on the left is unavoidable since the surface of  $K^*$  does not coincide with the polar of  $S$ .) In this section, we describe a geometric method of deriving  $^*S$  from  $S$ , and hence, of determining  $K^*$  from  $K$ .

**DEFINITION.** Let  $x \neq \theta$  be a point of  $\mathcal{L}$ . The inversion of  $x$ , denoted  $1/x$ , is that point on the ray determined by  $x$  whose norm is  $1/\|x\|$ . If  $X$  is any collection of points in  $\mathcal{L} - \{\theta\}$ , then the inversion of  $X$ , in symbol  $1/X$ , is the set of points obtained by inverting each point of  $X$ .

We are now ready to determine  $^*S$  when  $S$  is given. Consider, in  $\mathcal{L}$ , a fixed ray from the origin  $\theta$ . Let  $i$  be its intersection with the support hyperplane of  $K$  which is orthogonal to this ray. The collection of all such points—one for each ray—defines the *indicatrix* of  $K$ . It is denoted by  $I_K$  or briefly by  $I$ . We shall presently show that the inversion of  $I$  is precisely  $^*S$ .

Let  $\phi$  be the gauge of the body  $K$ . Consider a fixed ray from the origin, and let  $v \neq \theta$  be a point on this ray. Then the distance from the origin  $\theta$  to the support hyperplane of  $K$  orthogonal to this



ray is precisely  $\phi^*(v/\|v\|)$ . Letting  $i$  denote the point of  $I$  on this ray, we have  $\phi^*(v/\|v\|) = \|i\|$ . Thus,  $\phi^*(v) = \|v\| \|i\|$ . This is true for any point  $v \neq \theta$  on this ray. We, however, are interested in finding that point  $v$  for which  $\phi^*(v) = \|v\| \|i\| = 1$ . Clearly, the desired point is  $1/i$ . Thus, the inversion of  $I$  is, indeed,  $^*S$ .

The symbols  $K$  and  $K_j$  will be reserved to represent bodies. We denote by  $\hat{I}_K$  the collection of points "within and on"  $I_K$  and term it the solid indicatrix of  $K$ . Then the following statements are immediate.

$$(1) \quad K_1 \subseteq K_2 \Leftrightarrow \hat{I}_{K_1} \subseteq \hat{I}_{K_2} \Leftrightarrow K_1^* \supseteq K_2^* .$$

$$(2) \quad \phi_1 \geq \phi_2 \Leftrightarrow \phi_1^* \leq \phi_2^* .$$

**THEOREM 1.**  $\phi = e \Leftrightarrow \phi = \phi^*$ .

*Proof.* ( $\Rightarrow$ ) Consider the body  $U = \{x: \|x\| \leq 1\}$  corresponding to the gauge  $e$ . The Pythagorean Theorem implies that  $\hat{I}_U = U$ . It fol-

lows that  $U = U^*$ , and consequently,  $e = e^*$ .

( $\Leftarrow$ ) Now suppose  $\phi = \phi^*$ . Clearly, for  $x$  and  $y$  in  $\mathcal{L}$ ,  $(x, y) \leq \phi(x)\phi^*(y)$ . If also  $x = y$ , then

$$[e(x)]^2 = (x, x) \leq \phi(x)\phi^*(x) = [\phi(x)]^2.$$

Therefore,  $e \leq \phi$ . It then follows that  $e = e^* \geq \phi^* = \phi$ . Thus,  $\phi = e$ .

COROLLARY.  $K = U \Leftrightarrow \hat{I}_K = U \Leftrightarrow K = K^*$ .

THEOREM 2.  $K^{**} = K$ .

COROLLARY.  $\phi_K^{**} = \phi_K$ .

Theorem 2 and its corollary are easily verified.

**The indicatrix of a vector sum.** In expressing the theorem of this section, we require an additional piece of notation.

DEFINITION. For two subsets  $X_1$  and  $X_2$  of  $\mathcal{L}$ , their *vector sum*  $X_1 + X_2$  and *radial sum*  $X_1 \oplus X_2$  are defined as follows:

$$\begin{aligned} X_1 + X_2 &= \{x_1 + x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}, \\ X_1 \oplus X_2 &= \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2, \text{ and} \\ &\quad x_1, x_2 \text{ both on the same ray}\}. \end{aligned}$$

The vector sum and radial sum are related as described in the next theorem.

THEOREM 3. Let  $K_1$  and  $K_2$  be bodies. Then  $I_{(K_1+K_2)} = I_{K_1} \oplus I_{K_2}$ .

*Proof.* Consider a fixed ray from the origin. Let  $i_1$  and  $i_2$  be the points of  $I_{K_1}$  and  $I_{K_2}$ , respectively, on this ray. If  $\{x: f(x) = \alpha_1 > 0\}$  and  $\{x: f(x) = \alpha_2 > 0\}$  are the support hyperplanes of  $K_1$  and  $K_2$ , respectively, which are orthogonal to this ray, then there exist points  $v_1 \in K_1$  and  $v_2 \in K_2$  for which  $f(v_1) = \alpha_1$  and  $f(v_2) = \alpha_2$ . Consider now the point  $v_1 + v_2$ . It belongs to  $K_1 + K_2$  and  $f(v_1 + v_2) = \alpha_1 + \alpha_2$ . Moreover, the hyperplane  $\{x: f(x) = \alpha_1 + \alpha_2\}$  is orthogonal to our ray and intersects it in the point  $i_1 + i_2$ . Clearly,  $i_1 + i_2$  is the point of  $I_{K_1} \oplus I_{K_2}$  on the ray. To complete the proof, it remains to show that  $i_1 + i_2$  is a point of  $I_{(K_1+K_2)}$ .

It is sufficient to prove that  $\{x: f(x) = \alpha_1 + \alpha_2\}$  is a support hyperplane of  $K_1 + K_2$ . Otherwise, there would be a point  $y \in K_1 + K_2$  for which  $f(y) > \alpha_1 + \alpha_2$ . Let  $y = y_1 + y_2$ , where  $y_1 \in K_1$  and  $y_2 \in K_2$ . Since  $\{x: f(x) = \alpha_1\}$  and  $\{x: f(x) = \alpha_2\}$  are support hyperplanes of  $K_1$  and  $K_2$ ,

respectively, we have  $f(y_1) \leq \alpha_1$  and  $f(y_2) \leq \alpha_2$ . Therefore,

$$f(y) = f(y_1 + y_2) \leq \alpha_1 + \alpha_2 .$$

Thus, we reach a contradiction.

The preceding theorem enables one to focus attention on one fixed ray at a time.

**COROLLARY 1.** *If  $K_1$  and  $K_2$  are bodies and  $\alpha, \beta$  nonzero scalars, then  $I_{(\alpha K_1 + \beta K_2)} = \alpha I_{K_1} \oplus \beta I_{K_2}$ .*

**COROLLARY 2.** *If  $\phi_1$  and  $\phi_2$  are the gauges of  $K_1$  and  $K_2$ , respectively, then the gauge of the set  $K = \alpha K_1 + \beta K_2$  is  $\phi = (\alpha \phi_1^* + \beta \phi_2^*)^*$ .*

**COROLLARY 3.** *If  $K$  is a body having  $\phi$  as its gauge, then*

$$\frac{K + K^*}{2} \supseteq U = \{x: \|x\| \leq 1\} \quad \text{and} \quad \frac{\phi + \phi^*}{2} \geq e .$$

Corollary 3 is significant, but not surprising, in the light of the preceding exposition, since for any positive number  $\alpha$ ,  $[\alpha + (1/\alpha)]/2 \geq 1$ . Indeed, equality holds in the last if and only if  $\alpha = 1$ . Consequently, we also have the following corollary.

**COROLLARY 4.** *The average of  $K$  equals  $U$  if and only if  $K = U$ ; the average of  $\phi$  equals  $e$  if and only if  $\phi = e$ .*

The study of the average of a gauge is motivated by the following theorem, (Cf. Schatten, [3], p. 73): "Let  $\phi$  be an arbitrary gauge on  $\mathcal{L}$ . Define  $\phi_1 = (\phi + \phi^*)/2$  and  $\phi_n = (\phi_{n-1} + \phi_{n-1}^*)/2$  for  $n > 1$ . Then the sequence  $\{\phi_n\}$  converges decreasingly to  $e$ ." Since its publication, this theorem has appeared in the literature in a variety of forms, (Cf., e.g., Mityagin and Shvarts, [2], p. 116). The preceding corollaries permit us to state a different version of the above theorem, valid in Euclidean  $n$ -space  $E^n$ .

**THEOREM 4.** *Let  $K$  be a body in  $E^n$ . Define*

$$K_1 = \frac{K + K^*}{2} \quad \text{and} \quad K_n = \frac{K_{n-1} + K_{n-1}^*}{2}$$

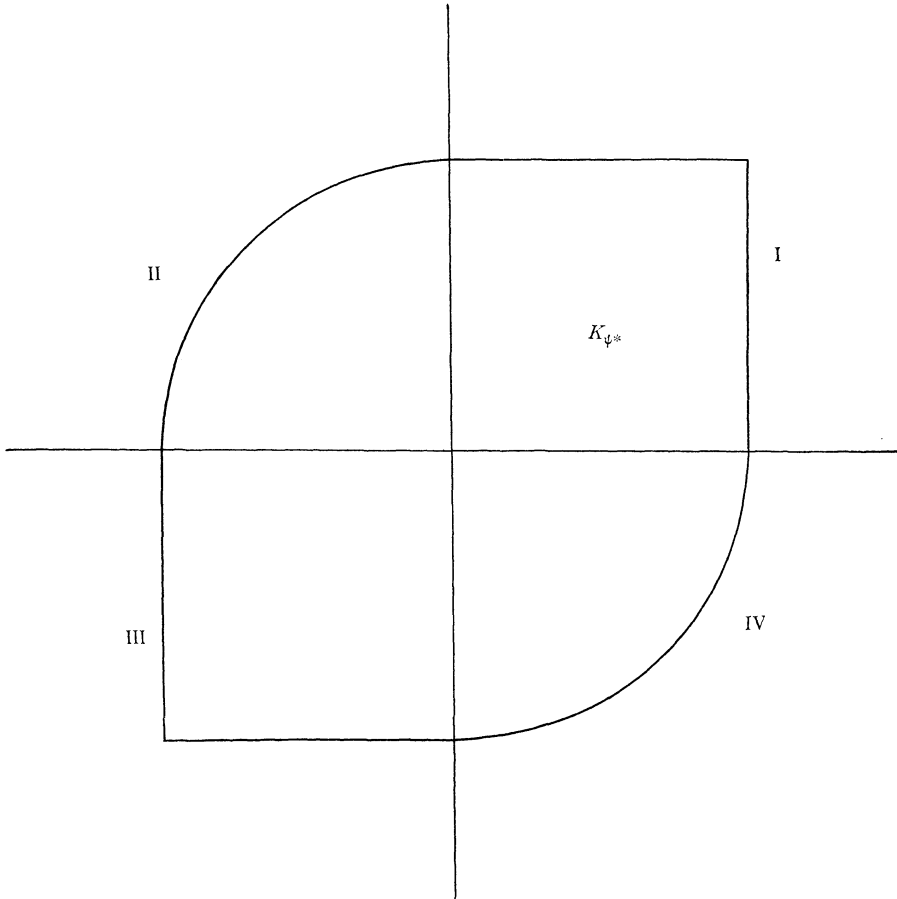
*for  $n > 1$ . Then  $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$  and the sequence  $\{K_n\}$  converges in the Blaschke sense to  $U$ .*

We conclude this section by showing that, even in the Euclidean plane, not every gauge  $\psi$  is an average, that is, of the form  $(\phi + \phi^*)/2$

for some gauge  $\phi$ . To define such a  $\psi$ , put

$$\psi(x, y) = \begin{cases} |x| + |y| & \text{if } xy > 0 \\ (x^2 + y^2)^{1/2} & \text{if } xy \leq 0. \end{cases}$$

To prove this, assume to the contrary that  $\psi = (\phi + \phi^*)/2$ . Let  $K_\psi$  and  $K_\phi$  be the convex sets corresponding to  $\psi$  and  $\phi$ , respectively. Corollary 2 of Theorem 3 implies that  $K_{\psi^*} = (K_\phi + K_{\phi^*})/2$ , (see diagram).



Let  $v \neq \theta$  be a fixed point in either quadrant II or IV. Denoting by  $I_K(v)$  and  $S_K(v)$  the points of the indicatrix and surface of  $K$ , respectively, which lie on the ray determined by  $v$ , we obtain:

$$\begin{aligned} 1 &= \|I_{K_{\psi^*}}(v)\| = \frac{1}{2}(\|I_{K_\phi}(v)\| + \|I_{K_{\phi^*}}(v)\|) \\ &\geq \frac{1}{2}(\|I_{K_\phi}(v)\| + \|S_{K_{\phi^*}}(v)\|) \geq 1. \end{aligned}$$

Therefore,  $\|I_{K_\phi}(v)\| = \|I_{K_{\phi^*}}(v)\| = 1$ . It follows that  $K_\phi = K_{\phi^*} = U$  in quadrants II and IV. The last implies that  $K_\phi \subseteq K_{\psi^*}$  and  $K_{\phi^*} \subseteq K_{\psi^*}$ .

Moreover, since  $(K_\phi + K_{\phi^*})/2 = K_{\psi^*}$ , we have  $K_\phi = K_{\phi^*} = K_{\psi^*}$ . This of course is a contradiction.

The “spread”.

DEFINITION. Let a body  $K$  be given. Then the *width* of  $K$  in the direction of  $x$ , denoted  $\omega_K(x)$ , is the distance between the two support hyperplanes of  $K$  which are orthogonal to the line determined by the origin and  $x \neq \theta$ . The *diameter* of  $K$  in the direction of  $x$ , in symbol  $\delta_K(x)$ , is the length of the chord of  $K$  lying on the line determined by  $\theta$  and  $x$ . The function  $\mathfrak{S}_K(x) = \omega_K(x) - \delta_K(x)$  is defined as the *spread* of  $K$ .

Clearly,  $\mathfrak{S}_K$  is defined on the whole space except at the origin. Being constant on each ray, it may also be considered as a function of direction.  $\mathfrak{S}_K$  is completely determined by its values on the surface  $\{u: \|u\| = 1\}$  of the unit ball; the letter “ $u$ ” is reserved to represent an arbitrary point of this surface.

Let  $\phi$  represent the gauge of the body  $K$ . Then

$$\omega_K(u) - \delta_K(u) = 2(\|I_K(u)\| - \|S_K(u)\|),$$

due to the fact that  $K$  is balanced. Since  $\|u\| = 1$ ,

$$\phi^*(u) = \frac{\|u\|}{\|S_{K^*}(u)\|} = \frac{1}{\|S_{K^*}(u)\|} = \|I_K(u)\|.$$

Moreover,  $\phi(u) = \|u\|/\|S_K(u)\| = 1/\|S_K(u)\|$ . Therefore, the *spread* of the gauge  $\phi$  is given by

$$\mathfrak{S}_\phi(u) = \mathfrak{S}_K(u) = 2\left(\phi^*(u) - \frac{1}{\phi(u)}\right).$$

We stress that the last equality holds when  $u$  is any vector of norm one. For a vector  $x$  such that  $0 < \|x\| \neq 1$ , we define

$$\mathfrak{S}_\phi(x) = \mathfrak{S}_\phi\left(\frac{x}{\|x\|}\right).$$

The following relations are immediate.

- (1)  $\mathfrak{S}_\phi \geq 0$ .
- (2)  $\mathfrak{S}_e = 0$ .
- (3)  $e \leq \phi \Rightarrow \mathfrak{S}_\phi \leq \mathfrak{S}_{\phi^*}$ .
- (4) If  $e \leq \phi_1 \leq \phi_2$ , then  $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2} \Rightarrow \mathfrak{S}_{\phi_1^*} \leq \mathfrak{S}_{\phi_2^*}$ .

**THEOREM 5.** *If  $\phi_1$  and  $\phi_2$  are gauges such that  $\phi_1 \leq \phi_2$  and  $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$ , then  $\phi_1\phi_1^* \leq \phi_2\phi_2^*$ .*



*Proof.* It is sufficient to prove the theorem for a vector  $u$  of norm one. Clearly,  $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$  amounts to

$$\phi_1^*(u) - \frac{1}{\phi_1(u)} \leq \phi_2^*(u) - \frac{1}{\phi_2(u)}.$$

Since also  $\phi_1 \leq \phi_2$ , we have  $\phi_1(u)\phi_1^*(u) \leq \phi_2(u)\phi_2^*(u)$ .

**THEOREM 6.** *Let  $\phi_1$  and  $\phi_2$  be gauges such that  $e \leq \phi_1 \leq \phi_2$ . Moreover, suppose that  $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$ . Then  $(\phi_1 + \phi_1^*)/2 \leq (\phi_2 + \phi_2^*)/2$ .*

*Proof.* If  $K_j$  is the body of  $\phi_j$  and  $S_j$  is its surface ( $j = 1, 2$ ), then  $\mathfrak{S}_{K_1} \leq \mathfrak{S}_{K_2}$ . Therefore,  $\omega_{K_1} - \delta_{K_1} \leq \omega_{K_2} - \delta_{K_2}$ . We shall denote the indicatrix of  $K_j$  by  $I_j$  and the indicatrix of  $K_j^*$  by  $*I_j$ . (The asterisk is placed on the left here in order to distinguish  $*I_j$  from the polar of  $I_j$ .) Then, since the  $K_j$  are balanced, it follows that

$$\|I_1(x)\| - \|S_1(x)\| \leq \|I_2(x)\| - \|S_2(x)\|$$

for all  $x \neq \theta$ . Observe the following.

$$\begin{aligned} \|*I_2(x)\| - \|*I_1(x)\| &= \left\| \frac{1}{S_2(x)} \right\| - \left\| \frac{1}{S_1(x)} \right\| \\ &= \frac{1}{\|S_2(x)\|} - \frac{1}{\|S_1(x)\|} \\ &= \frac{\|S_1(x)\| - \|S_2(x)\|}{\|S_2(x)\| \|S_1(x)\|}. \end{aligned}$$

Since  $e \leq \phi_1 \leq \phi_2$ , we have  $K_2 \subseteq K_1 \subseteq U$ . The last implies that  $\|S_2(x)\| \|S_1(x)\| \leq 1$ . Therefore,

$$\begin{aligned} \|I_1(x)\| - \|I_2(x)\| &\leq \|S_1(x)\| - \|S_2(x)\| \leq \frac{\|S_1(x)\| - \|S_2(x)\|}{\|S_2(x)\| \|S_1(x)\|} \\ &= \|*I_2(x)\| - \|*I_1(x)\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|I_1(x) + *I_1(x)\| &= \|I_1(x)\| + \|*I_1(x)\| \\ &\leq \|I_2(x)\| + \|*I_2(x)\| = \|I_2(x) + *I_2(x)\|. \end{aligned}$$

It follows that the solid indicatrix of the average of  $K_1$  is a subset of the solid indicatrix of the average of  $K_2$ . Therefore,

$$\left( \frac{K_1 + K_1^*}{2} \right)^* \subseteq \left( \frac{K_2 + K_2^*}{2} \right)^*$$

and thus,  $(\phi_1 + \phi_1^*)/2 \leq (\phi_2 + \phi_2^*)/2$ .

**THEOREM 7.** *Let  $\phi_1$  and  $\phi_2$  be gauges satisfying  $\phi_2 \leq \phi_1 \leq e$  and*

suppose  $\mathfrak{S}_{\phi_1} \leq \mathfrak{S}_{\phi_2}$ . Then  $(\phi_1 + \phi_1^*)/2 \leq (\phi_2 + \phi_2^*)/2$ .

*Proof.* The argument is similar to that of Theorem 6.

REMARK. The preceding arguments will remain valid also for bounded convex sets (not necessarily balanced) containing the origin as an interior point. This necessitates only minor changes in the text. For example, one would then define the spread as

$$\mathfrak{S}_K(x) = \|I_K(x)\| - \|S_K(x)\| \quad \text{and} \quad \mathfrak{S}_\phi(u) = \phi^*(u) - \frac{1}{\phi(u)}.$$

The class of gauges  $(\sum_{j=1}^n |x_j|^p)^{1/p}$ . In this section,  $\mathcal{L}$  stands for the  $n$ -dimensional Euclidean space. There we define a class of gauges  $\{\phi_p\}$  for  $1 \leq p \leq \infty$  as follows.

$$\begin{aligned} \phi_p(x_1, \dots, x_n) &= \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \quad \text{for } 1 \leq p < \infty \\ \phi_\infty(x_1, \dots, x_n) &= \text{Maximum}_{1 \leq j \leq n} |x_j|. \end{aligned}$$

One readily verifies that for each  $x = (x_1, \dots, x_n)$ ,  $\phi_p(x)$  is a decreasing function of  $p$ . Moreover, the conjugate of  $\phi_p$  is precisely  $\phi_q$ , where  $1/p + 1/q = 1$  if  $p > 1$ , and  $q = \infty$  if  $p = 1$ . To simplify notation, we shall write  $\mathfrak{S}_p(x)$  instead of  $\mathfrak{S}_{\phi_p}(x)$  for the spread of  $\phi_p$ .

**THEOREM 8.** Let  $x = (x_1, \dots, x_n) \neq \theta$ . Then  $\mathfrak{S}_p(x)$  is a decreasing function of  $p$  for  $1 \leq p \leq 2$ .

*Proof.* The proof is essentially based on an argument of S. Chowla. Let  $u = (u_1, \dots, u_n)$  be the vector of norm one on the ray determined by  $x$ . Since the spread is constant on each ray, we may replace  $x$  by  $u$ . We may also assume that  $u_j > 0$  for  $j = 1, 2, \dots, n$ .

To prove the theorem in case  $1 < p \leq 2$ , we define an auxiliary function  $\gamma(t) = \log(\sum_{j=1}^n u_j^t)$ . Elementary computations show that:

$$\gamma'(t) = \frac{d}{dt} [\gamma(t)] = \frac{\sum_{j=1}^n u_j^t (\log u_j)}{\sum_{j=1}^n u_j^t}$$

and

$$\gamma''(t) = \frac{\left(\sum_{j=1}^n u_j^t\right) \left(\sum_{j=1}^n u_j^t (\log u_j)^2\right) - \left(\sum_{j=1}^n u_j^t (\log u_j)\right)^2}{\left(\sum_{j=1}^n u_j^t\right)^2}.$$

Moreover, Cauchy's Inequality implies:

$$\begin{aligned} \left( \sum_{j=1}^n u_j^t (\log u_j) \right)^2 &= \left( \sum_{j=1}^n (u_j^{t/2}) (u_j^{t/2} \log u_j) \right)^2 \\ &\leq \left( \sum_{j=1}^n u_j^t \right) \left( \sum_{j=1}^n u_j^t (\log u_j)^2 \right). \end{aligned}$$

Consequently,  $\gamma''(t) \geq 0$  and therefore,  $t\gamma''(t) \geq 0$  for all  $t \geq 1$ . The last being the derivative of  $\xi(t) = t\gamma'(t) - \gamma(t)$  implies that  $\xi(t)$  is an increasing function of  $t$ . Thus,  $p \leq 2 \leq q$  implies that  $\xi(q) \geq \xi(p)$ . Also,  $(\sum_{j=1}^n u_j^q)^{1/q} \geq (\sum_{j=1}^n u_j^p)^{-1/p}$ , because

$$\left( \sum_{j=1}^n u_j^q \right)^{1/q} - \left( \sum_{j=1}^n u_j^p \right)^{-1/p} = \frac{1}{2} \mathfrak{S}_p(u) \geq 0.$$

Therefore,

$$\frac{1}{2} p^2 \cdot \frac{d}{dp} [\mathfrak{S}_p(u)] = \left( \sum_{j=1}^n u_j^p \right)^{-1/p} \cdot \xi(p) - \left( \sum_{j=1}^n u_j^q \right)^{1/q} \cdot \xi(q) \leq 0.$$

Thus,  $d/dp[\mathfrak{S}_p(u)] \leq 0$  and  $\mathfrak{S}_p(u)$  is a decreasing function of  $p$  for  $1 < p \leq 2$ .

The case  $p = 1$  is settled by observing that for any  $p$  ( $1 < p \leq 2$ ), we have  $\mathfrak{S}_1(u) \geq \mathfrak{S}_p(u)$ .

**COROLLARY.** *If  $2 \leq q \leq q'$ , then  $\mathfrak{S}_q(x) \leq \mathfrak{S}_{q'}(x)$  for each  $x \neq \theta$ .*

As a consequence of the preceding theorem, one obtains the following result.

**THEOREM 9.** *If  $1 \leq p \leq 2$  and  $1/p + 1/q = 1$ , then for each point  $(x_1, x_2, \dots, x_n)$ , both*

$$\frac{1}{2} \left[ \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} + \left( \sum_{j=1}^n |x_j|^q \right)^{1/q} \right]$$

and

$$\left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \left( \sum_{j=1}^n |x_j|^q \right)^{1/q}$$

are decreasing functions of  $p$ .

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