ON DYADIC SUBSPACES

H. LeRoy Peterson

We prove a necessary condition for a (compact, Hausdorff) space to be dyadic (= image of product of 2-point spaces):

THEOREM. Let $Y$ be a dyadic space of weight $m$, and let $r$ be a cardinal number less than $m$. Then $X$ has a dyadic subspace of weight $r$.

It may be observed (with the aid of Corollary 2, below) that this theorem is a stronger and more general version of a result published in a previous paper by the author [this Journal, 28 (1969), 173-182; Lemma III.6.]

A dyadic space is a Hausdorff space which is a continuous image of $\{0, 1\}^I$ (with the product topology) for some set $I$. Sanin has shown (see [2], Th. 1) that, if $X$ is an infinite dyadic space, then the smallest possible cardinality for the exponent $I$ is the weight of $X$, i.e., the least cardinality for a basis for the topology of $X$, hereinafter denoted by $w(X)$. Other observations concerning the significance of $w(X)$ for an infinite dyadic space include the following: Esenin-Volpin showed (see [3], Th. 4) that $w(X)$ is the least upper bound of the characters of the points of $X$; in [6] (Th. III.3) it is shown that a dyadic space having a dense subset of cardinality $m$ must have weight no greater than $2^m$. (The converse of this last statement follows from the well-known theorem of Hewitt, et. al., in [4]).

In what follows we shall use, whenever necessary, the fact that, if $X$ and $Y$ are compact Hausdorff spaces and $X$ is a continuous image of $Y$, then $w(X) \leq w(Y)$. ([1], Appendix.) For a set $S$, $|S|$ denotes the cardinality of $S$.

2. Proof of the theorem. (1) Suppose $X$ is a dyadic space and $f$ a continuous function from $\{0, 1\}^I$ onto $X$. Define $\varepsilon \in I$ to be redundant if, whenever two points $p$ and $q$ in $\{0, 1\}^I$ differ only in the $\varepsilon$th coordinate, we have $f(p) = f(q)$. By induction, if $p$ and $q$ differ only on a finite set of redundant coordinates, then $f(p) = f(q)$. Since $f$ is continuous, we must have that $f(p) = f(q)$ whenever $p$ and $q$ differ only on an arbitrary set of redundant coordinates. Thus we may assume that all the indices in $I$ are nonredundant.

(2) Given $\varepsilon \in I$, there must exist two points $p = p'$ and $q = q'$ such that $p_\mu = q_\mu$ for all $\mu \neq \varepsilon$, $p_\varepsilon = 0$ for all but finitely many $\mu$, and $f(p) \neq f(q)$; this follows from the continuity of $f$ and the assumption that $\varepsilon$ is nonredundant.

(3) Now let $r < w(X)$; if $r$ is finite the conclusion is obvious.
Assuming \( r \) is infinite, choose a subset \( R_1 \) of \( I \) such that \(| R_1 | = r \). For each \( i \in R_1 \), choose \( p^i \) and \( q^i \) as in (2). Let

\[
E_i = \{ \mu \in R_1 : p_\mu^i = 1 \} \cup \{ i \}, \quad \text{and} \quad R = \bigcup \{ E_i : i \in R_1 \}.
\]

Let \( X_R = f(P_R) \), where \( P_R = \{ 0, 1 \}^R \times \{ 0 \} = \{ p \in \{ 0, 1 \}^R : p_\mu = 0 \text{ for } \mu \in R \} \). It is clear that \( \{ p^i : i \in R_1 \} \cup \{ q^i : i \in R_1 \} \subset P_R \), and that \(| R_1 | = r \), so that \( w(X_R) \leq r \). We wish to show that \( w(X_R) = r \); suppose \( w(X_R) < r \), and let \( B \) be a basis for the topology of \( X_R \) with \(| B | = w(X_R) \). For each \( i \in R_1 \), there exist \( U \) and \( V \), members of \( B \) with disjoint closures, such that \( f(p^i) \in U \) and \( f(q^i) \in V \). Since \( r = | R_1 | > | B \times B | \), there must exist \( U \) and \( V \) such that \( R_2 = \{ i : f(p^i) \in U, f(q^i) \in V \} \) has cardinality \( > w(X_R) \). The choice function \( i \mapsto (p^i, q^i) \) is one-to-one, thus \( \{ (p^i, q^i) : i \in R_1 \} \) has cardinality \( > w(X_R) \), and we may as well assume that \( \{ p^i : i \in R_1 \} \) is infinite. Since \( P_R \) is compact, there is an infinite net \( \{ p^i \} \) which converges to some point \( p^s \), and since each \( p^i \) differs from the corresponding \( q^i \) only in a single coordinate, we must have that \( \{ q^i \} \) converges to \( p^s \) also. But then \( f(p^s) \in \text{cl}(U) \cap \text{cl}(V) \), which we have assumed to be impossible. Thus \( w(X_R) = r \). [Note: by a slight modification of the argument in this paragraph, we could take \( R = I \) (containing only nonredundant indices) and get \(| I | = w(X) \), as in Šanin's theorem.]

**Corollary 1.** Every infinite dyadic space contains an infinite compact metric space.

**Corollary 2.** Every nonmetrizable dyadic space has a dyadic subspace of weight \( \aleph_1 \).

**Corollary 3.** Let \( X \) be a dyadic space, \( w(X) = m \). Then \( X \) contains a chain \( \{ X_n : n \leq m \} \) of dyadic subspaces with \( w(X_n) = n \) for each \( n \leq m \).

**Proof.** It is easy to see, in part (3) of the proof of the theorem, that if \( w(X_R) = r < n \), we can choose \( R' \supset R \) so that \( w(X_{R'}) = n \). Clearly \( X_{R'} \supset X_R \) if \( R' \supset R \).

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