

Pacific Journal of Mathematics

ON DYADIC SUBSPACES

HAROLD L. PETERSON, JR.

ON DYADIC SUBSPACES

H. LEROY PETERSON

We prove a necessary condition for a (compact, Hausdorff) space to be dyadic (= image of product of 2-point spaces):

THEOREM. Let Y be a dyadic space of weight m , and let r be a cardinal number less than m . Then X has a dyadic subspace of weight r .

It may be observed (with the aid of Corollary 2, below) that this theorem is a stronger and more general version of a result published in a previous paper by the author [this Journal, 28 (1969), 173-182; Lemma III.6.]

A *dyadic space* is a Hausdorff space which is a continuous image of $\{0, 1\}^I$ (with the product topology) for some set I . Šanin has shown (see [2], Th. 1) that, if X is an infinite dyadic space, then the smallest possible cardinality for the exponent I is the weight of X , i.e., the least cardinality for a basis for the topology of X , hereinafter denoted by $w(X)$. Other observations concerning the significance of $w(X)$ for an infinite dyadic space include the following: Esenin-Volpin showed (see [3], Th. 4) that $w(X)$ is the least upper bound of the characters of the points of X ; in [6] (Th. III.3) it is shown that a dyadic space having a dense subset of cardinality m must have weight no greater than 2^m . (The converse of this last statement follows from the well-known theorem of Hewitt, *et. al.*, in [4]).

In what follows we shall use, whenever necessary, the fact that, if X and Y are compact Hausdorff spaces and X is a continuous image of Y , then $w(X) \leq w(Y)$. ([1], Appendix.) For a set S , $|S|$ denotes the cardinality of S .

2. **Proof of the theorem.** (1) Suppose X is a dyadic space and f a continuous function from $\{0, 1\}^I$ onto X . Define $\iota \in I$ to be *redundant* if, whenever two points p and q in $\{0, 1\}^I$ differ only in the ι th coordinate, we have $f(p) = f(q)$. By induction, if p and q differ only on a finite set of redundant coordinates, then $f(p) = f(q)$. Since f is continuous, we must have that $f(p) = f(q)$ whenever p and q differ only on an arbitrary set of redundant coordinates. Thus we may assume that all the indices in I are nonredundant.

(2) Given $\iota \in I$, there must exist two points $p = p'$ and $q = q'$ such that $p_\mu = q_\mu$ for all $\mu \neq \iota$, $p_\mu = 0$ for all but finitely many μ , and $f(p) \neq f(q)$; this follows from the continuity of f and the assumption that ι is nonredundant.

(3) Now let $r < w(X)$; if r is finite the conclusion is obvious.

Assuming r is infinite, choose a subset R_1 of I such that $|R_1| = r$. For each $\iota \in R_1$, choose p^ι and q^ι as in (2). Let

$$E_\iota = \{\mu \in R_1: p'_\mu = 1\} \cup \{\iota\}, \quad \text{and} \quad R = \bigcup \{E_\iota: \iota \in R_1\}.$$

Let $X_R = f(P_R)$, where $P_R = \{0, 1\}^R \times \{0\} = \{p \in \{0, 1\}^I: p_\mu = 0 \text{ for } \mu \in R\}$. It is clear that $\{p^\iota: \iota \in R\} \cup \{q^\iota: \iota \in R\} \subset P_R$, and that $|R| = r$, so that $w(X_R) \leq r$. We wish to show that $w(X_R) = r$; suppose $w(X_R) < r$, and let \mathbf{B} be a basis for the topology of X_R with $|\mathbf{B}| = w(X_R)$. For each $\iota \in R_1$ there exist U and V , members of \mathbf{B} with disjoint closures, such that $f(p^\iota) \in U$ and $f(q^\iota) \in V$. Since $r = |R_1| > |\mathbf{B} \times \mathbf{B}|$, there must exist U and V such that $R_2 = \{\iota: f(p^\iota) \in U, f(q^\iota) \in V\}$ has cardinality $> w(X_R)$. The choice function $\iota \rightarrow (p^\iota, q^\iota)$ is one-to-one, thus $\{(p^\iota, q^\iota): \iota \in R_2\}$ has cardinality $> w(X_R)$, and we may as well assume that $\{p^\iota: \iota \in R_2\}$ is infinite. Since P_R is compact, there is an infinite net $\{p^\iota\}$ which converges to some point p^0 , and since each p^ι differs from the corresponding q^ι only in a single coordinate, we must have that $\{q^\iota\}$ converges to p^0 also. But then $f(p^0) \in \text{cl}(U) \cap \text{cl}(V)$, which we have assumed to be impossible. Thus $w(X_R) = r$. [Note: by a slight modification of the argument in this paragraph, we could take $R = I$ (containing only nonredundant indices) and get $|I| = w(X)$, as in Šanin's theorem.]

COROLLARY 1. *Every infinite dyadic space contains an infinite compact metric space.*

COROLLARY 2. *Every nonmetrizable dyadic space has a dyadic subspace of weight \aleph_1 .*

COROLLARY 3. *Let X be a dyadic space, $w(X) = m$. Then X contains a chain $\{X_n: n \leq m\}$ of dyadic subspaces with $w(X_n) = n$ for each $n \leq m$.*

Proof. It is easy to see, in part (3) of the proof of the theorem, that if $w(X_R) = r < n$, we can choose $R' \supset R$ so that $w(X_{R'}) = n$. Clearly $X_{R'} \supset X_R$ if $R' \supset R$.

3. Acknowledgment. The author wishes to express his appreciation to the referee for his helpful comments and, in particular, for suggesting a considerable simplification of the author's proof of the theorem.

REFERENCES

1. P. Alexandroff, *On bicomact extensions of topological spaces*, Mat. Sb. **5** (1939), 403-423 (Russian).
2. B. Efimov and R. Engelking, *Remarks on dyadic spaces II*, Colloq. Math. **13** (1965), 181-197.
3. R. Engelking and A. Pelczynski, *Remarks on dyadic spaces*, Colloq. Math. **11** (1963), 55-63.
4. E. Hewitt, *A remark on density characters*, Bull. Amer. Math. Soc. **52** (1946), 641-643.
5. J. L. Kelley, *General topology*, Van Nostrand, Princeton, 1955.
6. H. L. Peterson, *Regular and irregular measures on groups and dyadic spaces*, Pacific J. Math. **28** (1969), 173-182.

Received April 15, 1969.

THE UNIVERSITY OF CONNECTICUT

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD PIERCE
University of Washington
Seattle, Washington 98105

BASIL GORDON
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 31, No. 3

BadMonth, 1969

George E. Andrews, <i>On a calculus of partition functions</i>	555
Silvio Aurora, <i>A representation theorem for certain connected rings</i>	563
Lawrence Wasson Baggett, <i>A note on groups with finite dual spaces</i>	569
Steven Barry Bank, <i>On majorants for solutions of algebraic differential equations in regions of the complex plane</i>	573
Klaus R. Bichteler, <i>Locally compact topologies on a group and the corresponding continuous irreducible representations</i>	583
Mario Borelli, <i>Affine complements of divisors</i>	595
Carlos Jorge Do Rego Borges, <i>A study of absolute extensor spaces</i>	609
Bruce Langworthy Chalmers, <i>Subspace kernels and minimum problems in Hilbert spaces with kernel function</i>	619
John Dauns, <i>Representation of L-groups and F-rings</i>	629
Spencer Ernest Dickson and Kent Ralph Fuller, <i>Algebras for which every indecomposable right module is invariant in its injective envelope</i>	655
Robert Fraser and Sam Bernard Nadler, Jr., <i>Sequences of contractive maps and fixed points</i>	659
Judith Lee Gersting, <i>A rate of growth criterion for universality of regressive isols</i>	669
Robert Fred Gordon, <i>Rings in which minimal left ideals are projective</i>	679
Fred Gross, <i>Entire functions of several variables with algebraic derivatives at certain algebraic points</i>	693
W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary, <i>Wreath products of ordered permutation groups</i>	703
W. J. Kim, <i>The Schwarzian derivative and multivalence</i>	717
Robert Hamor La Grange, Jr., <i>On $(m - n)$ products of Boolean algebras</i>	725
Charles D. Masiello, <i>The average of a gauge</i>	733
Stephen H. McCleary, <i>The closed prime subgroups of certain ordered permutation groups</i>	745
Richard Roy Miller, <i>Gleason parts and Choquet boundary points in convolution measure algebras</i>	755
Harold L. Peterson, Jr., <i>On dyadic subspaces</i>	773
Derek J. S. Robinson, <i>Groups which are minimal with respect to normality being intransitive</i>	777
Ralph Edwin Showalter, <i>Partial differential equations of Sobolev-Galpern type</i>	787
David Slepian, <i>The content of some extreme simplexes</i>	795
Joseph L. Taylor, <i>Noncommutative convolution measure algebras</i>	809
B. S. Yadav, <i>Contractions of functions and their Fourier series</i>	827
Lindsay Nathan Childs and Frank Rimi DeMeyer, <i>Correction to: "On automorphisms of separable algebras"</i>	833
Moses Glasner and Richard Emanuel Katz, <i>Correction to: "Function-theoretic degeneracy criteria for Riemannian manifolds"</i>	834
Satish Shirali, <i>Correction to: "On the Jordan structure of complex Banach *algebras"</i>	834
Benjamin Rigler Halpern, <i>Addendum to: "Fixed points for iterates"</i>	834