

# Pacific Journal of Mathematics

**ON DYADIC SUBSPACES**

HAROLD L. PETERSON, JR.

## ON DYADIC SUBSPACES

H. LEROY PETERSON

We prove a necessary condition for a (compact, Hausdorff) space to be dyadic (= image of product of 2-point spaces):

**THEOREM.** Let  $Y$  be a dyadic space of weight  $m$ , and let  $r$  be a cardinal number less than  $m$ . Then  $X$  has a dyadic subspace of weight  $r$ .

It may be observed (with the aid of Corollary 2, below) that this theorem is a stronger and more general version of a result published in a previous paper by the author [this Journal, 28 (1969), 173-182; Lemma III.6.]

A *dyadic space* is a Hausdorff space which is a continuous image of  $\{0, 1\}^I$  (with the product topology) for some set  $I$ . Šanin has shown (see [2], Th. 1) that, if  $X$  is an infinite dyadic space, then the smallest possible cardinality for the exponent  $I$  is the weight of  $X$ , i.e., the least cardinality for a basis for the topology of  $X$ , hereinafter denoted by  $w(X)$ . Other observations concerning the significance of  $w(X)$  for an infinite dyadic space include the following: Esenin-Volpin showed (see [3], Th. 4) that  $w(X)$  is the least upper bound of the characters of the points of  $X$ ; in [6] (Th. III.3) it is shown that a dyadic space having a dense subset of cardinality  $m$  must have weight no greater than  $2^m$ . (The converse of this last statement follows from the well-known theorem of Hewitt, *et. al.*, in [4]).

In what follows we shall use, whenever necessary, the fact that, if  $X$  and  $Y$  are compact Hausdorff spaces and  $X$  is a continuous image of  $Y$ , then  $w(X) \leq w(Y)$ . ([1], Appendix.) For a set  $S$ ,  $|S|$  denotes the cardinality of  $S$ .

**2. Proof of the theorem.** (1) Suppose  $X$  is a dyadic space and  $f$  a continuous function from  $\{0, 1\}^I$  onto  $X$ . Define  $\iota \in I$  to be *redundant* if, whenever two points  $p$  and  $q$  in  $\{0, 1\}^I$  differ only in the  $\iota$ th coordinate, we have  $f(p) = f(q)$ . By induction, if  $p$  and  $q$  differ only on a finite set of redundant coordinates, then  $f(p) = f(q)$ . Since  $f$  is continuous, we must have that  $f(p) = f(q)$  whenever  $p$  and  $q$  differ only on an arbitrary set of redundant coordinates. Thus we may assume that all the indices in  $I$  are nonredundant.

(2) Given  $\iota \in I$ , there must exist two points  $p = p'$  and  $q = q'$  such that  $p_\mu = q_\mu$  for all  $\mu \neq \iota$ ,  $p_\mu = 0$  for all but finitely many  $\mu$ , and  $f(p) \neq f(q)$ ; this follows from the continuity of  $f$  and the assumption that  $\iota$  is nonredundant.

(3) Now let  $r < w(X)$ ; if  $r$  is finite the conclusion is obvious.

Assuming  $r$  is infinite, choose a subset  $R_1$  of  $I$  such that  $|R_1| = r$ . For each  $\iota \in R_1$ , choose  $p^\iota$  and  $q^\iota$  as in (2). Let

$$E_\iota = \{\mu \in R_1: p^\mu = 1\} \cup \{\iota\}, \quad \text{and} \quad R = \bigcup \{E_\iota: \iota \in R_1\}.$$

Let  $X_R = f(P_R)$ , where  $P_R = \{0, 1\}^R \times \{0\} = \{p \in \{0, 1\}^R: p_\mu = 0 \text{ for } \mu \in R\}$ . It is clear that  $\{p^\iota: \iota \in R\} \cup \{q^\iota: \iota \in R\} \subset P_R$ , and that  $|R| = r$ , so that  $w(X_R) \leq r$ . We wish to show that  $w(X_R) = r$ ; suppose  $w(X_R) < r$ , and let  $\mathbf{B}$  be a basis for the topology of  $X_R$  with  $|\mathbf{B}| = w(X_R)$ . For each  $\iota \in R_1$  there exist  $U$  and  $V$ , members of  $\mathbf{B}$  with disjoint closures, such that  $f(p^\iota) \in U$  and  $f(q^\iota) \in V$ . Since  $r = |R_1| > |\mathbf{B} \times \mathbf{B}|$ , there must exist  $U$  and  $V$  such that  $R_2 = \{\iota: f(p^\iota) \in U, f(q^\iota) \in V\}$  has cardinality  $> w(X_R)$ . The choice function  $\iota \rightarrow (p^\iota, q^\iota)$  is one-to-one, thus  $\{(p^\iota, q^\iota): \iota \in R_2\}$  has cardinality  $> w(X_R)$ , and we may as well assume that  $\{p^\iota: \iota \in R_2\}$  is infinite. Since  $P_R$  is compact, there is an infinite net  $\{p^\iota\}$  which converges to some point  $p^0$ , and since each  $p^\iota$  differs from the corresponding  $q^\iota$  only in a single coordinate, we must have that  $\{q^\iota\}$  converges to  $p^0$  also. But then  $f(p^0) \in \text{cl}(U) \cap \text{cl}(V)$ , which we have assumed to be impossible. Thus  $w(X_R) = r$ . [Note: by a slight modification of the argument in this paragraph, we could take  $R = I$  (containing only nonredundant indices) and get  $|I| = w(X)$ , as in Šanin's theorem.]

**COROLLARY 1.** *Every infinite dyadic space contains an infinite compact metric space.*

**COROLLARY 2.** *Every nonmetrizable dyadic space has a dyadic subspace of weight  $\aleph_1$ .*

**COROLLARY 3.** *Let  $X$  be a dyadic space,  $w(X) = m$ . Then  $X$  contains a chain  $\{X_n: n \leq m\}$  of dyadic subspaces with  $w(X_n) = n$  for each  $n \leq m$ .*

*Proof.* It is easy to see, in part (3) of the proof of the theorem, that if  $w(X_R) = r < n$ , we can choose  $R' \supset R$  so that  $w(X_{R'}) = n$ . Clearly  $X_{R'} \supset X_R$  if  $R' \supset R$ .

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