

# Pacific Journal of Mathematics

**PARTIAL DIFFERENTIAL EQUATIONS OF  
SOBOLEV-GALPERN TYPE**

RALPH EDWIN SHOWALTER

## PARTIAL DIFFERENTIAL EQUATIONS OF SOBOLEV-GALPERN TYPE

R. E. SHOWALTER

**A mixed initial and boundary value problem is considered for a partial differential equation of the form  $Mu_t(x, t) + Lu(x, t) = 0$ , where  $M$  and  $L$  are elliptic differential operators of orders  $2m$  and  $2l$ , respectively, with  $m \leq l$ . The existence and uniqueness of a strong solution of this equation in  $H_0^l(G)$  is proved by semigroup methods.**

We are concerned here with a mixed initial boundary value problem for the equation

$$(1) \quad Mu_t + Lu = 0$$

in which  $M$  and  $L$  are elliptic differential operators. Equations of this type have been studied using various methods in [2, 3, 4, 6, 7, 10, 11, 13, 14, 15, 17, 18]. We will make use of the  $L^2$ -estimates and related results on elliptic operators to obtain a generalized solution to this problem similar to that obtained for the parabolic equation

$$u_t + Lu = 0$$

as in [7].

Let  $G$  be a bounded open domain in  $R^n$  whose boundary  $\partial G$  is an  $(n - 1)$ -dimensional manifold with  $G$  lying on one side of  $\partial G$ . By  $H^k(G) \equiv H^k$  we mean the Hilbert space (of equivalence classes) of functions in  $L^2(G)$  whose distributional derivatives through order  $k$  belong to  $L^2(G)$  with the inner product and norm given, respectively, by

$$(f, g)_k = \sum \left\{ \int_G D^\alpha f \overline{D^\alpha g} dx : |\alpha| \leq k \right\}$$

and

$$\|f\|_k = \sqrt{(f, f)_k}.$$

$H_0^k \equiv H_0^k(G)$  will denote the closure in  $H^k$  of  $C_0^\infty(G)$ , the space of infinitely differentiable functions with compact support in  $G$ .

The operators are of the form

$$M = \sum \{ (-1)^{|\rho|} D^\rho m^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leq m \}$$

and

$$L = \sum \{ (-1)^{|\rho|} D^\rho l^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leq l \},$$

and they are uniformly strongly elliptic in  $G$ . We shall investigate the existence and uniqueness of solutions to (1) which coincide with the initial function  $u_0$  in  $H_0^l$  where  $t = 0$  and vanish on  $\partial G$  together with all derivatives of order less than or equal to  $l - 1$ .

If the order of  $M$  is as high as that of  $L$  ( $2m \geq 2l$ ), then this problem can be handled as in [10] by forming the exponential of the bounded extension of  $M^{-1}L$  on  $H_0^m$  and thus obtaining a group of operators on  $H_0^m$  and a corresponding solution for all  $t$  in  $\mathbf{R}$ . The case we shall consider is that of  $m \leq l$ , and this will include the parabolic equation as a special case. We obtain a semi-group of operators on  $H_0^m$  and, hence, a solution for all  $t \geq 0$ .

2. In this section we shall formulate the problem. Assume temporarily the following.

$P'_1$ : The coefficients  $m^{\rho\sigma}$  in  $M$  belong to  $H^{|\rho|}$ , and  $D^\rho m^{\rho\sigma}$  is in  $L^\infty(G)$  whenever  $|\rho| \leq m$ . A similar statement is true for the coefficients in  $L$ . From  $P'_1$  it follows that the sesqui-linear forms defined on  $C_0^\infty(G)$  by

$$B_M(\varphi, \psi) = \sum \{(m^{\rho\sigma} D^\sigma \varphi, D^\rho \psi)_0 : |\rho|, |\sigma| \leq m\}$$

and

$$B_L(\varphi, \psi) = \sum \{(l^{\rho\sigma} D^\sigma \varphi, D^\rho \psi)_0 : |\rho|, |\sigma| \leq l\}$$

satisfy the identities

$$(2) \quad B_M(\varphi, \psi) = (M\varphi, \psi)_0$$

and

$$(2') \quad B_L(\varphi, \psi) = (L\varphi, \psi)_0$$

for all  $\varphi, \psi$  in  $C_0^\infty(G)$ . Since  $P'_1$  implies that

$$K_m = \sup \{\|m^{\rho\sigma}\|_\infty : |\rho|, |\sigma| \leq m\}$$

and

$$K_l = \sup \{\|l^{\rho\sigma}\|_\infty : |\rho|, |\sigma| \leq l\}$$

are finite, we see that

$$|B_M(\varphi, \psi)| \leq K_m \|\varphi\|_m \|\psi\|_m$$

and

$$|B_L(\varphi, \psi)| \leq K_l \|\varphi\|_l \|\psi\|_l$$

for all  $\varphi, \psi$  in  $C_0^\infty(G)$ . Hence these sesqui-linear forms may be extended by continuity to all of  $H_0^m$  and  $H_0^l$ , respectively.

The final properties which we shall assume are the following. For any  $\varphi, \psi$  in  $C_0^\infty(G)$  we have

$$P_2: \operatorname{Re} B_M(\varphi, \varphi) \geq k_m \|\varphi\|_m^2, k_m > 0, \\ \operatorname{Re} B_L(\varphi, \varphi) \geq k_l \|\varphi\|_l^2, k_l > 0,$$

and

$$P_3: |B_M(\varphi, \psi)|^2 \leq (\operatorname{Re} B_M(\varphi, \varphi))(\operatorname{Re} B_M(\psi, \psi)).$$

These inequalities are valid for the respective extensions to  $H_0^m$  and  $H_0^l$ . The assumptions of  $P_2$  are inequalities of the Garding type which imply that  $M$  and  $L$  are uniformly strongly elliptic. Only the first of these is essential in applications, for the usual change of dependent variable  $u = ve^{2t}$  changes our equation to one with  $L$  replaced by  $L + \lambda M$ , and the Garding inequality is true for  $B_{L+\lambda M}$  if  $\lambda$  is sufficiently large and if the coefficients  $l^{\rho\sigma}(x), |\rho| = |\sigma| = l$  are uniformly continuous in  $G$ . See [3, 8] for sufficient conditions that  $P_2$  be true.

The assumption  $P_3$  is a Cauchy-Schwarz inequality for the form  $B_M$ . In view of the positivity of  $B_M$ , a necessary and sufficient condition for  $P_3$  is that  $M$  be symmetric, that is,  $m^{\rho\sigma} = \overline{m^{\sigma\rho}}$  for all  $\rho, \sigma$ . Such is the case for the examples

- (i)  $ku_t - \Delta u = 0$  ( $m = 0$ ) and
- (ii)  $-\gamma \Delta u_t + ku_t - \Delta u = 0$ ,

where  $\Delta$  is the Laplacian and  $\gamma$  and  $k$  are positive. Example (i) is a parabolic equation, and examples like (ii) appear in various problems of fluid mechanics and soil mechanics, where a solution is sought which satisfies an initial condition  $u(x, 0) = u_0(x)$  on  $G$  and the Dirichlet condition  $u(x, t) = 0$  on the boundary of  $G$ . See [1, 11, 12, 13].

We shall not need the full strength of  $P_1'$  so we replace it with the following weaker assumption.

$P_1$ : The coefficients  $m^{\rho\sigma}$  and  $l^{\rho\sigma}$  belong to  $L^\infty(G)$  for all  $\rho, \sigma$ .

We shall proceed under the assumptions  $P_1, P_2$  and  $P_3$  and remark that  $P_1'$  is needed only when we wish to interpret our weak solutions by means of (2) and (2').

Under the hypotheses above there is by the theorem of Lax and Milgram [7] a closed linear operator  $M_0$  with domain  $D_m$  dense in  $H_0^m$  and range equal to  $H^0 = L^2(G)$  such that

$$(3) \quad B_M(\varphi, \psi) = (M_0\varphi, \psi)_0$$

whenever  $\varphi$  belongs to  $D_m$  and  $\psi$  to  $H_0^m$ . Furthermore,  $M_0^{-1}$  is a bounded operator from  $H^0$  into  $H_0^m$ . Similarly, there is a closed linear operator  $L_0$  with domain  $D_l$  dense in  $H_0^l$  and range equal to  $H^0$  with

$$(3') \quad B_L(\varphi, \psi) = (L_0\varphi, \psi)_0$$

whenever  $\varphi$  belongs to  $D_l$  and  $\psi$  to  $H_0^l$ . Also,  $L_0^{-1}$  is bounded from  $H^0$  into  $H_0^l$ .

Consider the bijection  $A = -M_0^{-1}L_0$  from  $D_l$  onto  $D_m$ . For any  $\varphi$  in  $D_m$  we have

$$\begin{aligned} k_l \|A^{-1}\varphi\|_l^2 &= k_l \|L_0^{-1}M_0\varphi\|_l^2 \\ &\leq \operatorname{Re} B_L(L_0^{-1}M_0\varphi, L_0^{-1}M_0\varphi) = \operatorname{Re} (M_0\varphi, L_0^{-1}M_0\varphi)_0 \\ &= \operatorname{Re} B_M(\varphi, L_0^{-1}M_0\varphi) \leq K_m \|\varphi\|_m \|A^{-1}\varphi\|_m \\ &\leq K_m \|\varphi\|_m \|A^{-1}\varphi\|_l \end{aligned}$$

which yields

$$(4) \quad \|A^{-1}\varphi\|_l \leq (K_m/k_l) \|\varphi\|_m$$

for all  $\varphi$  in  $D_m$ . But  $D_m$  is dense in  $H_0^m$  so  $A^{-1}$  has a unique extension by continuity from  $H_0^m$  onto the set  $D = A^{-1}(H_0^m)$  in  $H_0^l$ , the domain of the closed extension of  $A$ . The continuity of the injection of  $H_0^l$  into  $H_0^m$  implies that  $A^{-1}$  is a bounded operator on  $H_0^m$ , and this is the space in which we formulate the Generalized Problem:

For a given initial function  $u_0$  in  $D$ , find a differentiable map  $u(t)$  of  $R^+$  into  $H_0^m$  for which  $u(t)$  belongs to  $H_0^l$  for all  $t \geq 0$ ,  $u(0) = u_0$ , and

$$(5) \quad B_M(u'(t), \varphi) + B_L(u(t), \varphi) = 0$$

for all  $\varphi$  in  $C_0^\infty(G)$  and  $t \geq 0$ .

Sufficient conditions for a solution of this generalized problem to be a classical solution will be discussed in [9].

3. The objective of this section is to prove the following results.

**THEOREM.** *There exists a unique solution of the generalized problem. If  $u(t)$  is in  $D_l$  then  $u'(t)$  is in  $D_m$  and*

$$(6) \quad M_0 u'(t) + L_0 u(t) = 0$$

in  $H^0$ . The mapping of  $u_0$  to  $u(t)$  is continuous from  $H_0^m$  into itself for each  $t \geq 0$  and furthermore satisfies

$$(7) \quad \|u(t)\|_m \leq \sqrt{(K_m/k_m)} \|u_0\|_m \exp(-k_l t/K_m).$$

We first show that the operator  $A$  is the infinitesimal generator of a semi-group of bounded operators on  $H_0^m$ ; this semi-group will provide a means of constructing a solution to the problem. From the assumptions on  $B_M$ , it follows that the function defined by

$$|\varphi|_M = \sqrt{(\operatorname{Re} B_M(\varphi, \varphi))}$$

is a norm on  $H_0^m$  that is equivalent to the norm  $\|\cdot\|_m$ . In the following we shall use  $|\cdot|_M$  as the norm on  $H_0^m$ , noting further that

$$(8) \quad k_m^{1/2} \|\varphi\|_m \leq |\varphi|_M \leq K_m^{1/2} \|\varphi\|_m$$

for  $\varphi$  in  $H_0^m$ .

To obtain the necessary estimates we let  $\lambda$  be a nonnegative number and consider the operator  $\lambda M_0 + L_0 = N$  from the domain  $D_m \cap D_l$  into  $H^0$ . We can define a sesqui-linear form on  $D_m \cap D_l$  by

$$B_N(\varphi, \psi) = ((\lambda M_0 + L_0)\varphi, \psi)_0 = \lambda B_M(\varphi, \psi) + B_L(\varphi, \psi)$$

and then note that  $B_N$  is bounded as well as positive-definite with respect to the norm of  $H_0^l$ . We extend  $B_N$  by continuity to all of  $H_0^l$ , and then by the theorem of Lax and Milgram there is a closed linear operator  $N_0$  from a domain  $D_n$  in  $H_0^l$  onto  $H^0$  for which

$$B_N(\varphi, \psi) = (N_0\varphi, \psi)_0$$

whenever  $\varphi$  is in  $D_n$  and  $\psi$  in  $H_0^l$ . Clearly  $N_0$  is an extension of  $N$  whose domain is  $D_m \cap D_l$ .

For all  $\varphi$  in  $D_n$  we have

$$\begin{aligned} \operatorname{Re} (N_0\varphi, \varphi)_0 &= \lambda \operatorname{Re} B_M(\varphi, \varphi) + \operatorname{Re} B_L(\varphi, \varphi) \\ &\geq (\lambda + k_l/K_m) \operatorname{Re} B_M(\varphi, \varphi) \\ &= (\lambda + k_l/K_m) |\varphi|_M^2. \end{aligned}$$

Thus, for any  $\psi$  in  $D_m$  we see that  $N_0^{-1}M_0\psi$  belongs to  $D_n$  and from above

$$\begin{aligned} (\lambda + k_l/K_m) |N_0^{-1}M_0\psi|_M^2 &\leq \operatorname{Re} (M_0\psi, N_0^{-1}M_0\psi)_0 \\ &= \operatorname{Re} B_M(\psi, N_0^{-1}M_0\psi) \leq |\psi|_M |(N_0^{-1}M_0\psi)|_M \end{aligned}$$

by  $P_3$ , so we have obtained the estimate

$$|N_0^{-1}M_0\psi|_M \leq (\lambda + k_l/K_m)^{-1} |\psi|_M$$

for all  $\psi$  in  $D_m$ .

Letting  $\varphi$  be an element of  $D_l \cap D_m$  we see

$$\begin{aligned} (N_0^{-1}M_0)(\lambda + M_0^{-1}L_0)\varphi &= N_0^{-1}(\lambda M_0\varphi + L_0\varphi) \\ &= N_0^{-1} \cdot N\varphi = \varphi, \end{aligned}$$

so  $\lambda + M_0^{-1}L_0$  is injective and satisfies

$$(\lambda + M_0^{-1}L_0)^{-1} = N_0^{-1}M_0$$

on  $D_m \cap D_l$ . Combining this with the estimate above we see that

$$|(\lambda + M_0^{-1}L_0)^{-1}\psi|_M \leq (\lambda + k_l/K_m)^{-1} |\psi|_M$$

for all  $\psi$  in  $D_l \cap D_m$ . It follows by continuity that  $\lambda - A$  is invertible on  $H_0^m$  and satisfies the estimate

$$|(\lambda - A)^{-1}|_M \leq (\lambda + k_l/K_m)^{-1}.$$

By the theorem of Hille and Yoshida [5, 16] on the characterization of the infinitesimal generators of semi-groups of class  $C_0$  we have the following results: there exists a unique family of bounded operators  $\{S(t): t \geq 0\}$  on  $H_0^m$  for which

- (i)  $S(t_1 + t_2) = S(t_1)S(t_2)$ ,
- (ii)  $S(t)x$  is strongly continuous for each  $x$  in  $H_0^m$ ,
- (iii)  $S(0) = I$  and  $\|S(t)\|_M \leq \exp(-k_1 t/K_m)$  for all  $t \geq 0$ ,
- (iv)  $\lim_{h \rightarrow 0} h^{-1}(S(h) - I)x_0 = Ax_0$  for each  $x_0$  in  $D$ , and
- (v)  $S(t)$  commutes with  $(\lambda - A)^{-1}$  for all  $\lambda \geq 0$ .

The statement (v) implies in particular that  $D$  is invariant under each  $S(t)$ .

Having been given the initial function  $u_0$  in  $D$ , we define

$$u(t) = S(t)u_0, t \geq 0$$

and show that  $u(t)$  is a solution of the generalized problem. Clearly we see  $u(t)$  belongs to  $H_0^m$  and  $u(0) = u_0$ . Furthermore, since  $S(t)$  leaves  $D$  invariant and  $u_0$  is in  $D$ , it follows that  $u(t)$  belongs to  $D$  and thus to  $H_0^l$ . The function  $u(t)$  is differentiable with

$$(9) \quad u'(t) = Au(t)$$

for all  $t \geq 0$  by (i) and (iv), and hence  $u'(t)$  is in  $H_0^m$ .

We shall verify that  $u(t)$  satisfies the equation (5). Since  $D_m$  is dense in  $H_0^m$  there is a sequence  $\{\varphi_n\}$  in  $D_m$  for which  $\|\varphi_n - u'(t)\|_m \rightarrow 0$  as  $n \rightarrow \infty$ . Now  $\{\varphi_n\}$  is a Cauchy sequence in  $H_0^m$  and it follows by (4) that  $\psi_n = A^{-1}\varphi_n$  is a Cauchy sequence in the complete space  $H_0^l$ , so there is a  $\psi$  in  $H_0^l$  such that  $\|\psi_n - \psi\|_l \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $A^{-1}$  is continuous we have  $\psi = u(t)$ . Each  $\psi_n$  belongs to  $D_l$ , since  $\varphi_n$  is in  $D_m$ , and furthermore  $M_0\varphi_n + L_0\psi_n = 0$ . Now for each  $\varphi$  in  $C_0^\infty(G)$  we have by the continuity of  $B_M$  and  $B_L$

$$\begin{aligned} & B_M(u'(t), \varphi) + B_L(u(t), \varphi) \\ &= \lim_{n \rightarrow \infty} B_M(\varphi_n, \varphi) + \lim_{n \rightarrow \infty} B_L(\psi_n, \varphi) \\ &= \lim_{n \rightarrow \infty} \{B_M(\varphi_n, \varphi) + B_L(\psi_n, \varphi)\} = \lim_{n \rightarrow \infty} \{(M_0\varphi_n, \varphi)_0 + (L_0\psi_n, \varphi)_0\} \equiv 0, \end{aligned}$$

so the generalized problem does have a solution.

If  $u(t)$  is in  $D_l$  then by (9)  $u'(t)$  is in  $D_m$ . It follows from (5) that for every  $\varphi$  in  $C_0^\infty(G)$

$$(M_0u'(t) + L_0u(t), \varphi)_0 = 0,$$

and this implies (6). The estimate (7) is a consequence of (iii) and (8).

To show that the generalized problem has at most one solution, we let  $u(t)$  be a solution of the problem with  $u_0 = 0$ . By linearity it suffices to show that  $u(t) \equiv 0$ . The differentiability of  $u(t)$  in  $H_0^m$

implies that the real valued function

$$\alpha(t) = \operatorname{Re} B_M(u(t), u(t))$$

is differentiable and

$$\alpha'(t) = 2 \operatorname{Re} B_M(u'(t), u(t)) .$$

Since (5) is true also for all  $\varphi$  in  $H_0^l$  by continuity, we have from  $P_2$

$$\alpha'(t) = -2 \operatorname{Re} B_L(u(t), u(t)) \leq 0 .$$

But  $\alpha(0) = \operatorname{Re} B_M(u(0), u(0)) = 0$ , so  $\alpha(t) = 0$  for all  $t \geq 0$ . From  $P_2$  it follows that  $u(t) = 0$  for  $t \geq 0$ .

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