PARTIAL DIFFERENTIAL EQUATIONS OF
SOBOLEV-GALPERN TYPE

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A mixed initial and boundary value problem is considered for a partial differential equation of the form \( Mu_t(x, t) + Lu(x, t) = 0 \), where \( M \) and \( L \) are elliptic differential operators of orders \( 2m \) and \( 2l \), respectively, with \( m \leq l \). The existence and uniqueness of a strong solution of this equation in \( H^0(G) \) is proved by semigroup methods.

We are concerned here with a mixed initial boundary value problem for the equation

\[
(1) \quad Mu_t + Lu = 0
\]

in which \( M \) and \( L \) are elliptic differential operators. Equations of this type have been studied using various methods in [2, 3, 4, 6, 7, 10, 11, 13, 14, 15, 17, 18]. We will make use of the \( L^2 \)-estimates and related results on elliptic operators to obtain a generalized solution to this problem similar to that obtained for the parabolic equation

\[
u_t + Lu = 0
\]
as in [7].

Let \( G \) be a bounded open domain in \( \mathbb{R}^n \) whose boundary \( \partial G \) is an \((n - 1)\)-dimensional manifold with \( G \) lying on one side of \( \partial G \). By \( H^k(G) \equiv H^k \) we mean the Hilbert space (of equivalence classes) of functions in \( L^2(G) \) whose distributional derivatives through order \( k \) belong to \( L^2(G) \) with the inner product and norm given, respectively, by

\[
(f, g)_k = \sum \left\{ \int_G D^\alpha f \overline{D^\alpha g} \, dx : |\alpha| \leq k \right\}
\]

and

\[
\| f \|_k = \sqrt{(f, f)_k}.
\]

\( H^k_0 \equiv H^k_0(G) \) will denote the closure in \( H^k \) of \( C^\infty_0(G) \), the space of infinitely differentiable functions with compact support in \( G \).

The operators are of the form

\[
M = \sum \{ (-1)^\rho \partial^\rho m^\sigma(x) \partial^\sigma : |\rho|, |\sigma| \leq m \}
\]

and

\[
L = \sum \{ (-1)^\rho \partial^\rho l^\sigma(x) \partial^\sigma : |\rho|, |\sigma| \leq l \}.
\]
and they are uniformly strongly elliptic in $G$. We shall investigate the existence and uniqueness of solutions to (1) which coincide with the initial function $u_0$ in $H^1_0$ where $t = 0$ and vanish on $\partial G$ together with all derivatives of order less than or equal to $l - 1$.

If the order of $M$ is as high as that of $L (2m \geq 2l)$, then this problem can be handled as in [10] by forming the exponential of the bounded extension of $M^{-1}L$ on $H^m_0$ and thus obtaining a group of operators on $H^m_0$ and a corresponding solution for all $t$ in $R$. The case we shall consider is that of $m \leq l$, and this will include the parabolic equation as a special case. We obtain a semi-group of operators on $H^m_0$ and, hence, a solution for all $t \geq 0$.

2. In this section we shall formulate the problem. Assume temporarily the following.

$P'_1$: The coefficients $m^{\rho \sigma}$ in $M$ belong to $H^{\lfloor \rho \rfloor}$, and $D^\rho m^{\rho \sigma}$ is in $L^\omega (G)$ whenever $|\rho| \leq m$. A similar statement is true for the coefficients in $L$. From $P'_1$ it follows that the sesqui-linear forms defined on $C_0^\omega (G)$ by

$$B_M (\varphi, \psi) = \sum \{ (m^{\rho \sigma} D^\rho \varphi, D^\sigma \psi)_0; \ |\rho|, |\sigma| \leq m \}$$

and

$$B_L (\varphi, \psi) = \sum \{ (l^{\rho \sigma} D^\rho \varphi, D^\sigma \psi)_0; \ |\rho|, |\sigma| \leq l \}$$

satisfy the identities

$$(2) \quad B_M (\varphi, \psi) = (M\varphi, \psi)_0$$

and

$$(2') \quad B_L (\varphi, \psi) = (L\varphi, \psi)_0$$

for all $\varphi, \psi$ in $C_0^\omega (G)$. Since $P'_1$ implies that

$$K_m = \sup \{ ||m^{\rho \sigma}||_\omega; \ |\rho|, |\sigma| \leq m \}$$

and

$$K_l = \sup \{ ||l^{\rho \sigma}||_\omega; \ |\rho|, |\sigma| \leq l \}$$

are finite, we see that

$$|B_M (\varphi, \psi)| \leq K_m ||\varphi||_m ||\psi||_m$$

and

$$|B_L (\varphi, \psi)| \leq K_l ||\varphi||_l ||\psi||_l$$

for all $\varphi, \psi$ in $C_0^\omega (G)$. Hence these sesqui-linear forms may be extended by continuity to all of $H^m_0$ and $H^l_0$, respectively.
The final properties which we shall assume are the following. For any \( \varphi, \psi \) in \( C^o(G) \) we have
\[
P_2: \Re B_M(\varphi, \varphi) \geq k_m \| \varphi \|_m^2, k_m > 0,
\]
\[
\Re B_L(\varphi, \varphi) \geq k_i \| \varphi \|_i^2, k_i > 0,
\]
and
\[
P_3: |B_M(\varphi, \psi)|^2 \leq (\Re B_M(\varphi, \varphi))(\Re B_M(\psi, \psi)) .
\]
These inequalities are valid for the respective extensions to \( H_o^m \) and \( H_i^l \). The assumptions of \( P_2 \) are inequalities of the Garding type which imply that \( M \) and \( L \) are uniformly strongly elliptic. Only the first of these is essential in applications, for the usual change of dependent variable \( u = ve^{it} \) changes our equation to one with \( L \) replaced by \( L + \lambda M \), and the Garding inequality is true for \( B_{L+\lambda M} \) if \( \lambda \) is sufficiently large and if the coefficients \( l^{\rho \sigma}(x), |\rho| = |\sigma| = l \) are uniformly continuous in \( G \). See [3, 8] for sufficient conditions that \( P_2 \) be true.

The assumption \( P_3 \) is a Cauchy-Schwarz inequality for the form \( B_M \). In view of the positivity of \( B_M \), a necessary and sufficient condition for \( P_3 \) is that \( M \) be symmetric, that is, \( m^{\rho \sigma} = \overline{m^{\rho \sigma}} \) for all \( \rho, \sigma \). Such is the case for the examples

(i) \( ku_t - \Delta u = 0 (m = 0) \) and
(ii) \( -\gamma \Delta u_t + ku_t - \Delta u = 0, \)

where \( \Delta \) is the Laplacian and \( \gamma \) and \( k \) are positive. Example (i) is a parabolic equation, and examples like (ii) appear in various problems of fluid mechanics and soil mechanics, where a solution is sought which satisfies an initial condition \( u(x, 0) = u_0(x) \) on \( G \) and the Dirichlet condition \( u(x, t) = 0 \) on the boundary of \( G \). See [1, 11, 12, 13].

We shall not need the full strength of \( P_3 \) so we replace it with the following weaker assumption.

\( P_3' \): The coefficients \( m^{\rho \sigma} \) and \( l^{\sigma} \) belong to \( L^\infty(G) \) for all \( \rho, \sigma \).

We shall proceed under the assumptions \( P_1, P_2 \) and \( P_3 \) and remark that \( P_3' \) is needed only when we wish to interpret our weak solutions by means of (2) and (2').

Under the hypotheses above there is by the theorem of Lax and Milgram [7] a closed linear operator \( M_0 \) with domain \( D_m \) dense in \( H_o^m \) and range equal to \( H_o^0 = L^2(G) \) such that
\[
(3) \quad B_M(\varphi, \psi) = (M_0\varphi, \psi)_0
\]
whenever \( \varphi \) belongs to \( D_m \) and \( \psi \) to \( H_o^m \). Furthermore, \( M_0^{-1} \) is a bounded operator from \( H_o^0 \) into \( H_o^m \). Similarly, there is a closed linear operator \( L_0 \) with domain \( D_l \) dense in \( H_i^l \) and range equal to \( H_i^0 \) with
\[
(3') \quad B_L(\varphi, \psi) = (L_0\varphi, \psi)_0
\]
whenever $\varphi$ belongs to $D_t$ and $\psi$ to $H^n_0$. Also, $L_0^{-1}$ is bounded from $H^0$ into $H^n_0$.

Consider the bijection $A = -M^{-1}_0L_0$ from $D_t$ onto $D_m$. For any $\varphi$ in $D_m$ we have

$$k_i \| A^{-1} \varphi \|_i^2 = k_i \| L_0^{-1} M_0 \varphi \|_i^2 \leq \text{Re} B_L(L_0^{-1} M_0 \varphi, L_0^{-1} M_0 \varphi) = \text{Re} (M_0 \varphi, L_0^{-1} M_0 \varphi)_0$$

$$= \text{Re} B_M(\varphi, L_0^{-1} M_0 \varphi) \leq K_m \| \varphi \|_m \| A^{-1} \varphi \|_m \leq K_m \| \varphi \|_m \| A^{-1} \varphi \|_i,$$

which yields

$$(4) \quad \| A^{-1} \varphi \|_i \leq (K_m/k_i) \| \varphi \|_m$$

for all $\varphi$ in $D_m$. But $D_m$ is dense in $H^0_0$ so $A^{-1}$ has a unique extension by continuity from $H^m_0$ onto the set $D = A^{-1}(H^m_0)$ in $H^n_0$, the domain of the closed extension of $A$. The continuity of the injection of $H^n_0$ into $H^m_0$ implies that $A^{-1}$ is a bounded operator on $H^m_0$, and this is the space in which we formulate the Generalized Problem:

For a given initial function $u_0$ in $D$, find a differentiable map $u(t)$ of $R^+$ into $H^m_0$ for which $u(t)$ belongs to $H^n_0$ for all $t \geq 0$, $u(0) = u_0$, and

$$(5) \quad B_M(u'(t), \varphi) + B_L(u(t), \varphi) = 0$$

for all $\varphi$ in $C_0^\infty(G)$ and $t \geq 0$.

Sufficient conditions for a solution of this generalized problem to be a classical solution will be discussed in [9].

3. The objective of this section is to prove the following results.

**Theorem.** There exists a unique solution of the generalized problem. If $u(t)$ is in $D$, then $u'(t)$ is in $D_m$ and

$$(6) \quad M_0 u'(t) + L_0 u(t) = 0$$

in $H^0$. The mapping of $u_0$ to $u(t)$ is continuous from $H^m_0$ into itself for each $t \geq 0$ and furthermore satisfies

$$(7) \quad \| u(t) \|_m \leq \sqrt{(K_m/k_i)} \| u_0 \|_m \exp(-k_i t/K_m).$$

We first show that the operator $A$ is the infinitesimal generator of a semi-group of bounded operators on $H^m_0$; this semi-group will provide a means of constructing a solution to the problem. From the assumptions on $B_M$, it follows that the function defined by

$$| \varphi |_M = \sqrt{(\text{Re} B_M(\varphi, \varphi))}$$

is a norm on $H^m_0$ that is equivalent to the norm $\| \cdot \|_m$. In the following we shall use $| \cdot |_M$ as the norm on $H^m_0$, noting further that
for \( \varphi \) in \( H^m_0 \).

To obtain the necessary estimates we let \( \lambda \) be a nonnegative number and consider the operator \( \lambda M_0 + L_0 = N \) from the domain \( D_m \cap D_t \) into \( H^0 \). We can define a sesqui-linear form on \( D_m \cap D_t \) by

\[
B_N(\varphi, \psi) = ((\lambda M_0 + L_0)\varphi, \psi) = \lambda B_M(\varphi, \psi) + B_L(\varphi, \psi)
\]

and then note that \( B_N \) is bounded as well as positive-definite with respect to the norm of \( H^1_0 \). We extend \( B_N \) by continuity to all of \( H^1_0 \), and then by the theorem of Lax and Milgram there is a closed linear operator \( N_0 \) from a domain \( D_n \) in \( H^1_0 \) onto \( H^0 \) for which

\[
B_N(\varphi, \psi) = (N_0 \varphi, \psi)_0
\]

whenever \( \varphi \) is in \( D_n \) and \( \psi \) in \( H^1_0 \). Clearly \( N_0 \) is an extension of \( N \) whose domain is \( D_m \cap D_t \).

For all \( \varphi \) in \( D_m \) we have

\[
\text{Re} \left( N_0 \varphi, \varphi \right)_0 = \lambda \text{Re} B_M(\varphi, \varphi) + \text{Re} B_L(\varphi, \varphi)
\]

\[
\geq (\lambda + k_i/K_m) \text{Re} B_M(\varphi, \varphi)
\]

\[
= (\lambda + k_i/K_m) |\varphi|^2_M.
\]

Thus, for any \( \psi \) in \( D_m \) we see that \( N_0^{-1}M_0\psi \) belongs to \( D_n \) and from above

\[
(\lambda + k_i/K_m) \left| N_0^{-1}M_0\psi \right|_M^2 \leq \text{Re} \left( M_0\psi, N_0^{-1}M_0\psi \right)_0
\]

\[
= \text{Re} B_M(\psi, N_0^{-1}M_0\psi) \leq |\psi|^2_M \left| (N_0^{-1}M_0\psi) \right|_M
\]

by \( P_3 \), so we have obtained the estimate

\[
\left| N_0^{-1}M_0\psi \right|_M \leq (\lambda + k_i/K_m)^{-1} |\psi|^2_M
\]

for all \( \psi \) in \( D_m \).

Letting \( \varphi \) be an element of \( D_t \cap D_m \) we see

\[
(N_0^{-1}M_0)(\lambda + M_0^{-1}L_0)\varphi = N_0^{-1}(\lambda M_0\varphi + L_0\varphi)
\]

\[
= N_0^{-1} \cdot N\varphi = \varphi,
\]

so \( \lambda + M_0^{-1}L_0 \) is injective and satisfies

\[
(\lambda + M_0^{-1}L_0)^{-1} = N_0^{-1}M_0
\]

on \( D_m \cap D_t \). Combining this with the estimate above we see that

\[
\left| (\lambda + M_0^{-1}L_0)^{-1}\psi \right|_M \leq (\lambda + k_i/K_m)^{-1} |\psi|^2_M
\]

for all \( \psi \) in \( D_t \cap D_m \). It follows by continuity that \( \lambda - A \) is invertible on \( H^m_0 \) and satisfies the estimate

\[
\left| (\lambda - A)^{-1} \right|_M \leq (\lambda + k_i/K_m)^{-1}.
\]
By the theorem of Hille and Yoshida [5, 16] on the characterization of the infinitesimal generators of semi-groups of class $C_0$ we have the following results: there exists a unique family of bounded operators \[ \{S(t) : t \geq 0\} \] on $H_0^m$ for which

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$,
(ii) $S(t)x$ is strongly continuous for each $x$ in $H_0^m$,
(iii) $S(0) = I$ and $|S(t)|_M \leq \exp \left(-h_0 t/K_m\right)$ for all $t \geq 0$,
(iv) $\lim_{h \to 0} h^{-1}(S(h) - I)x_0 = Ax_0$ for each $x_0$ in $D$, and
(v) $S(t)$ commutes with $(\lambda - A)^{-1}$ for all $\lambda \geq 0$.

The statement (v) implies in particular that $D$ is invariant under each $S(t)$.

Having been given the initial function $u_0$ in $D$, we define

\[ u(t) = S(t)u_0, \quad t \geq 0 \]

and show that $u(t)$ is a solution of the generalized problem. Clearly we see $u(t)$ belongs to $H^m_0$ and $u(0) = u_0$. Furthermore, since $S(t)$ leaves $D$ invariant and $u_0$ is in $D$, it follows that $u(t)$ belongs to $D$ and thus to $H^1$. The function $u(t)$ is differentiable with

\[ u'(t) = Au(t) \]

for all $t \geq 0$ by (i) and (iv), and hence $u'(t)$ is in $H^m_0$.

We shall verify that $u(t)$ satisfies the equation (5). Since $D_m$ is dense in $H^m_0$ there is a sequence $\{\varphi_n\}$ in $D_m$ for which $||\varphi_n - u'(t)||_m \to 0$ as $n \to \infty$. Now $\{\varphi_n\}$ is a Cauchy sequence in $H^m_0$ and it follows by (4) that $\psi_n = A^{-1}\varphi_n$ is a Cauchy sequence in the complete space $H^1$, so there is a $\psi$ in $H^1$ such that $||\psi_n - \psi||_1 \to 0$ as $n \to \infty$. Since $A^{-1}$ is continuous we have $\psi = u(t)$. Each $\psi_n$ belongs to $D_1$, since $\varphi_n$ is in $D_m$, and furthermore $M_0\varphi_n + L_0\psi_n = 0$. Now for each $\varphi$ in $C^0_c(G)$ we have by the continuity of $B_M$ and $B_L$

\[
B_M(u'(t), \varphi) + B_L(u(t), \varphi)
= \lim_{n \to \infty} B_M(\varphi_n, \varphi) + B_L(\psi_n, \varphi)
= \lim_{n \to \infty} \{B_M(\varphi_n, \varphi) + B_L(\psi_n, \varphi)\} = \lim_{n \to \infty} \{(M_0\varphi_n, \varphi)_0 + (L_0\psi_n, \varphi)_0\} = 0,
\]

so the generalized problem does have a solution.

If $u(t)$ is in $D_1$ then by (9) $u'(t)$ is in $D_m$. It follows from (5) that for every $\varphi$ in $C^0_c(G)$

\[ (M_0u'(t) + L_0u(t), \varphi)_0 = 0, \]

and this implies (6). The estimate (7) is a consequence of (iii) and (8).

To show that the generalized problem has at most one solution, we let $u(t)$ be a solution of the problem with $u_0 = 0$. By linearity it suffices to show that $u(t) \equiv 0$. The differentiability of $u(t)$ in $H^m_0$
implies that the real valued function
\[ \alpha(t) = \text{Re} \, B(u(t), u(t)) \]
is differentiable and
\[ \alpha'(t) = 2 \text{Re} \, B(u'(t), u(t)) . \]
Since (5) is true also for all \( \varphi \) in \( H_0^1 \) by continuity, we have from \( P_2 \)
\[ \alpha'(t) = -2 \text{Re} \, B_L(u(t), u(t)) \leq 0 . \]
But \( \alpha(0) = \text{Re} \, B(u(0), u(0)) = 0 \), so \( \alpha(t) = 0 \) for all \( t \geq 0 \). From \( P_2 \),
it follows that \( u(t) = 0 \) for \( t \geq 0 \).

REFERENCES


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<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>George E. Andrews</td>
<td>On a calculus of partition functions</td>
<td>555</td>
</tr>
<tr>
<td>Silvio Aurora</td>
<td>A representation theorem for certain connected rings</td>
<td>563</td>
</tr>
<tr>
<td>Lawrence Wasson Baggett</td>
<td>A note on groups with finite dual spaces</td>
<td>569</td>
</tr>
<tr>
<td>Steven Barry Bank</td>
<td>On majorants for solutions of algebraic differential equations in regions of the complex plane</td>
<td>573</td>
</tr>
<tr>
<td>Klaus R. Bichteler</td>
<td>Locally compact topologies on a group and the corresponding continuous irreducible representations</td>
<td>583</td>
</tr>
<tr>
<td>Mario Borelli</td>
<td>Affine complements of divisors</td>
<td>595</td>
</tr>
<tr>
<td>Carlos Jorge Do Rego Borges</td>
<td>A study of absolute extensor spaces</td>
<td>609</td>
</tr>
<tr>
<td>Bruce Langworthy Chalmers</td>
<td>Subspace kernels and minimum problems in Hilbert spaces with kernel function</td>
<td>619</td>
</tr>
<tr>
<td>John Dauns</td>
<td>Representation of L-groups and F-rings</td>
<td>629</td>
</tr>
<tr>
<td>Spencer Ernest Dickson and Kent Ralph Fuller</td>
<td>Algebras for which every indecomposable right module is invariant in its injective envelope</td>
<td>655</td>
</tr>
<tr>
<td>Robert Fraser and Sam Bernard Nadler, Jr.</td>
<td>Sequences of contractive maps and fixed points</td>
<td>659</td>
</tr>
<tr>
<td>Judith Lee Gersting</td>
<td>A rate of growth criterion for universality of regressive isols</td>
<td>669</td>
</tr>
<tr>
<td>Robert Fred Gordon</td>
<td>Rings in which minimal left ideals are projective</td>
<td>679</td>
</tr>
<tr>
<td>Fred Gross</td>
<td>Entire functions of several variables with algebraic derivatives at certain algebraic points</td>
<td>693</td>
</tr>
<tr>
<td>W. Charles (Wilbur) Holland Jr. and Stephen H. McCleary</td>
<td>Wreath products of ordered permutation groups</td>
<td>703</td>
</tr>
<tr>
<td>W. J. Kim</td>
<td>The Schwarzian derivative and multivalence</td>
<td>717</td>
</tr>
<tr>
<td>Robert Hamor La Grange, Jr.</td>
<td>(m − n) products of Boolean algebras</td>
<td>725</td>
</tr>
<tr>
<td>Charles D. Masiello</td>
<td>The average of a gauge</td>
<td>733</td>
</tr>
<tr>
<td>Stephen H. McCleary</td>
<td>The closed prime subgroups of certain ordered permutation groups</td>
<td>745</td>
</tr>
<tr>
<td>Richard Roy Miller</td>
<td>Gleason parts and Choquet boundary points in convolution measure algebras</td>
<td>755</td>
</tr>
<tr>
<td>Harold L. Peterson, Jr.</td>
<td>On dyadic subspaces</td>
<td>773</td>
</tr>
<tr>
<td>Derek J. S. Robinson</td>
<td>Groups which are minimal with respect to normality being intransitive</td>
<td>777</td>
</tr>
<tr>
<td>Ralph Edwin Showalter</td>
<td>Partial differential equations of Sobolev-Galpern type</td>
<td>787</td>
</tr>
<tr>
<td>David Slepian</td>
<td>The content of some extreme simplexes</td>
<td>795</td>
</tr>
<tr>
<td>Joseph L. Taylor</td>
<td>Noncommutative convolution measure algebras</td>
<td>809</td>
</tr>
<tr>
<td>B. S. Yadav</td>
<td>Contractions of functions and their Fourier series</td>
<td>827</td>
</tr>
<tr>
<td>Lindsay Nathan Childs and Frank Rimi DeMeyer</td>
<td>Correction to: “On automorphisms of separable algebras”</td>
<td>833</td>
</tr>
<tr>
<td>Moses Glasner and Richard Emanuel Katz</td>
<td>Correction to: “Function-theoretic degeneracy criteria for Riemannian manifolds”</td>
<td>834</td>
</tr>
<tr>
<td>Satish Shirali</td>
<td>Correction to: “On the Jordan structure of complex Banach *algebras”</td>
<td>834</td>
</tr>
<tr>
<td>Benjamin Rigler Halpern</td>
<td>Addendum to: “Fixed points for iterates”</td>
<td>834</td>
</tr>
</tbody>
</table>