THE CONTENT OF SOME EXTREME SIMPLEXES

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Formulae are presented that give the content of a simplex in Euclidean $n$-space: (i) in terms of the lengths of and the angles between the vectors from a fixed point to the vertices of the simplex; (ii) in terms of the lengths of and the angles between the perpendiculars from a fixed point to the bounding faces of the simplex. We then determine the largest simplex whose vertices are given distances from a fixed point and we determine the smallest simplex whose faces are given distances from a fixed point. As special cases we find that the regular simplex is the largest simplex contained in a given sphere and is also the smallest simplex containing a given sphere.

1. Introduction and results. The $n$-dimensional simplex $S_n$ in Euclidean $n$-space is the general term in the sequence of figures $S_0$, $S_1$, $S_2$, $S_3$ \ldots known respectively otherwise as point, line segment, triangle, tetrahedron, \ldots. $S_n$ is determined by $n + 1$ points, $P_1$, $P_2$, \ldots, $P_{n+1}$ — its vertices —, which we assume do not lie in any $(n - 1)$-dimensional hyperplane. Taken $n$ at a time, these vertices determine $(n - 1)$-dimensional hyperplanes $H_1$, $H_2$, \ldots, $H_{n+1}$, where $H_i$ contains all vertices except $P_i$. We choose the normal of $H_i$ so that $P_i$ lies on the negative side of $H_i$. $S_n$ can be regarded as the intersection of these $n + 1$ nonpositive half spaces; it can also be regarded as the convex hull of its vertices.

Let $Q$ be an arbitrary point. For $i = 1, 2, \ldots, n + 1$, let $d_i > 0$ be the distance from $Q$ to $P_i$ and let $e_i > 0$ be the distance from $Q$ to $H_i$. Let $a_i$ be the unit vector in the direction from $Q$ to $P_i$ and let $b_i$ be the unit vector from $Q$ along the perpendicular to $H_i$. Let $r_{ij} = a_i \cdot a_j$, $s_{ij} = b_i \cdot b_j$, $i, j = 1, 2, \ldots, n + 1$.

In this paper, we first show that the content, $V_n$, of $S_n$ is given by

\begin{align}
(1) \quad n! V_n &= \left| \sum_{i,j} R_{ij} \frac{1}{d_i} \frac{1}{d_j} \right|^{1/2} \prod_{i} d_i \\
(2) \quad &= \left| \sum_{i,j} S_{ij} e_i e_j \right|^{n/2} \prod_{i} S_{ii}^{1/2}
\end{align}

for $n = 1, 2, \ldots$, where $R_{ij}$ is the cofactor of $r_{ij}$ in the $(n + 1) \times (n + 1)$ matrix $r = (r_{ij})$ and $S_{ij}$ is the cofactor of $s_{ij}$ in $s = (s_{ij})$. Next we determine the largest simplex with given $d$ values and the smallest simplex containing $Q$ with given $e$ values. We find

\begin{align}
(3) \quad n! V_{\text{max}} &= \theta^{-1/2} \prod_{i} \left( \theta + d_i^2 \right)^{1/2}, \quad r_{ij}' = -\frac{\theta}{d_i d_j},
\end{align}
\[(4) \quad n! V_{\text{min}} = n^n \psi^{-1/2} \prod_{i=1}^{n+1} (\psi + e_i^{2})^{1/2}, \quad s'_{ij} = -\frac{\psi}{e_i e_j}, \]

\[i, j = 1, 2, \ldots, n + 1\] where \(\theta\) and \(\psi\) are respectively the unique positive roots of

\[(5) \quad \theta \prod_{i=1}^{n+1} \frac{1}{\theta + d_i^*} = 1\]

and

\[(6) \quad \psi \sum_{i=1}^{n+1} \frac{1}{\psi + e_i^*} = 1\]

and where the \(r'_{ij}\) are the maximizing values of \(r_{ij}\) and the \(s'_{ij}\) are the minimizing values of \(s_{ij}\). \(Q\) lies inside the simplex given by (3).

If not all the \(d_i\) are the same, (5) has a negative real root of smallest absolute value. The simplex (3) corresponding to this root is the largest simplex with given \(d\) values having \(Q\) on the negative side of exactly one bounding face. Similarly if not all the \(e_i\) are the same, (6) has a negative real root of smallest absolute value. The simplex given by (4) corresponding to this root is the smallest simplex with given \(e\) values having \(Q\) on the positive side of exactly one bounding face.

A special case of these results states that: (a) the largest simplex contained in a given sphere is a regular simplex; (b) the smallest simplex containing a given sphere is a regular simplex.

2. Derivation of volume formula (1). Let \((x_1, x_2, \ldots, x_n)\) be the coordinates of a general point in Euclidean \(n\)-space referred to rectangular coordinate axes. We denote by \(r\) the vector from the origin to this general point. Consider the simplex whose vertices are the origin and the termini of the \(n\) vectors \(y_1, y_2, \ldots, y_n\) from the origin. The simplex is described by

\[(7) \quad r = \sum_{i=1}^{n} \xi_i y_i\]

\[\sum_{i=1}^{n} \xi_i \leq 1\]

\[\xi_1 \geq 0, \xi_2 \geq 0, \ldots, \xi_n \geq 0. \]

The volume of the simplex is given by

\[(9) \quad V = \int_{S_n} dx_1 \cdots dx_n = \int_R d\xi_1 \cdots d\xi_n |J|\]

where \(R\) is the \(\xi\)-region defined by (8) and \(J\) is the Jacobian of the
transformation (7). If \( y_i = (y_{i1}, y_{i2}, \ldots, y_{in}), i = 1, 2, \ldots, n, \) then (7) is explicitly \( x_i = \sum \xi_j y_{ji}, \) whence

\[
J = \begin{vmatrix}
  y_{11} & \cdots & y_{1n} \\
  \vdots & \ddots & \vdots \\
  y_{n1} & \cdots & y_{nn}
\end{vmatrix}
\]

which is independent of the \( \xi \)'s. The integral in (9) is readily evaluated to give the formula

\[ n! \ V = |J|. \]

To obtain the content of a simplex not located at the origin, we translate the coordinates along the vector \( x_{n+1}. \) Set \( y_i = x_i - x_{n+1}, i = 1, 2, \ldots, n. \) Then the content of a simplex with vertices given by the termini of \( x_i, i = 1, \ldots, n + 1, \) is

\[
\begin{vmatrix}
  x_{11} - x_{n+1} & \cdots & x_{1n} - x_{n+1} \\
  \vdots & \ddots & \vdots \\
  x_{n1} - x_{n+1} & \cdots & x_{nn} - x_{n+1}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
  x_{11} & \cdots & x_{1n} & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{n1} & \cdots & x_{nn} & 1
\end{vmatrix}
\]

(10)

a well-known formula [1, p. 124]. Here the double line denotes absolute value of a determinant. The equality shown in (10) can easily be established by subtracting the last row of the second determinant shown from each of the first \( n \) rows and evaluating the result by the cofactor expansion of the last column.

Squaring (10) we find \([n! \ V]^2 = ||x_1 \cdot x_j + 1||\) where the determinant is obtained by multiplying the last matrix of (10) by its transpose and we exhibit the element in the \( i \) th row and \( j \) th column of the result. Introducing the notation of § 1, we set \( x_1 \cdot x_j = d_i d_j r_{ij} \) with \( Q \) located at the origin. We have then

\[
n! \ V = ||d_i d_j r_{ij} + 1||^{1/2} = \left| r_{ij} + \frac{1}{d_i d_j} \right|^{1/2} \prod_{i=1}^{n+1} d_i.
\]

(11)

Now the \( i \) th row of \( ||r_{ij} + 1/d_i d_j|| \) is the sum of the two rows \((r_{i1}, \ldots, r_{in+1})\) and \(1/d_i(1/d_1, 1/d_2, \ldots, 1/d_{n+1}).\) The determinant can thus be written as the sum of the \( 2^{n+1} \) determinants obtained by taking for each row either a row of the matrix \( (r_{ij}) \) or a row of the matrix \((1/d_i d_j).\) But any determinant having two or more rows taken from \((1/d_i d_j)\) vanishes, and \( |r_{ij}| \) also vanishes since the \( n + 1 \) \( n \)-vectors \( a_i \)
are linearly dependent. The determinant $|r_{ij} + 1/d_1d_j|$ can therefore be expressed as the sum of $n + 1$ determinants, the $k$ th term being $|r_{ij}|$ with row $k$ replaced by $1/d_k(1/d_1, 1/d_2, \cdots, 1/d_{n+1})$. Expanding this determinant by the $k$ th row gives $\sum_j R_{k,j}(1/d_k)(1/d_j)$ and formula (1) then follows directly.

3. Derivation of volume formula (2). Consider the volume, $V_n$, of the region $S_n$ defined by

$$r \cdot b_i = \sum_{j=1}^n b_{ij}x_j \leq e_i, \ i = 1, 2, \cdots, n + 1. \tag{12}$$

Let

$$b = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \tag{13}$$

and let $B_{ij}$ be the cofactor of $b_{ij}$ in $b$. Set $b^{-1}_{ij} = B_{ij}/|b|$, $i, j = 1, 2, \cdots, n$. Define new variables $y_1, \cdots, y_n$ by

$$y_i = \sum_{j=1}^n b_{ij}x_j, \ i = 1, \cdots, n$$

so that

$$x_i = \sum_{j=1}^n b^{-1}_{ij}y_j, \ i = 1, \cdots, n. \tag{14}$$

In the new variables, the inequalities (12) are

$$y_i \leq e_i, \ i = 1, 2, \cdots, n \tag{14}$$

$$\sum_k \left( \sum_j b_{n+1,j}b^{-1}_{jk} \right)y_k \leq e_{n+1}. \tag{15}$$

If we now regard the $y$'s as rectangular coordinates, we see that (14) defines a simplex $S'_n$ in this new space. If $S'_n$ has $y$-volume $V'_n$, then

$$V_n = V'_n/|b| \tag{15}$$

since $dx_1 \cdots dx_n = dy_1 \cdots dy_n/|b|$. We proceed by finding $V'_n$.

The bounding hyperplanes of $S'_n$ are

$$H_i: \ y_i = e_i \tag{16}$$

$$H_n: \ y_n = e_n \tag{17}$$

$$H_{n+1}: \sum_k \left( \sum_{i=1} b_{n+1,i}b^{-1}_{ik} \right)y_k = e_{n+1}. \tag{17}$$
The vertex $P_{n+1}$ of this simplex, given by $H_1 \cap H_2 \cap \cdots \cap H_n$, has coordinates

$$P_{n+1} : (e_1, e_2, \ldots, e_n).$$

Consider the vertex $P_i$ given by $H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_n$, $i = 1, 2, \cdots, n$. For the $j$th coordinate of $P_i$ we find

$$y_{ij} = e_j, j \neq i, i, j = 1, 2, \cdots, n$$

from (16). The $i$th coordinate $y_{ii}$ is found from (17) as the solution of

$$y_{ii} \sum_l b_{n+1,i} b_{li}^{-1} + \sum_{k \neq i} b_{n+1,i} b_{ki} e_k = e_{n+1}$$

or

$$y_{ii} \sum_l b_{n+1,i} B_{li} + \sum_{k \neq i} b_{n+1,i} B_{ki} e_k = |b| e_{n+1}.$$ 

Now let

$$c = \begin{pmatrix} b_{11} & b_{1n} & e_1 \\ \vdots & \vdots & \vdots \\ b_{n+1,1} & \cdots & b_{n+1,n} e_{n+1} \end{pmatrix} = (c_{ij})$$

and write $C_{ij}$ for the cofactor of $c_{ij}$ in $c$. Equation (20) now becomes

$$-y_{ii} C_{i \cdot n+1} - \sum_{k \neq i} C_{k \cdot n+1} e_k = C_{i \cdot n+1} e_{n+1}$$

or

$$-y_{ii} C_{i \cdot n+1} = \sum_{k \neq i} C_{k \cdot n+1} e_k = |c| - C_{i \cdot n+1} e_i.$$

Thus

$$y_{ii} = e_i - \frac{|c|}{C_{i \cdot n+1}}, i = 1, 2, \cdots, n.$$

Formulas (18), (19), and (22) provide us with the coordinates of the vertices of $S_n'$. Using the first equality of (10) with the substitution $x_{ij} = y_{ij}$, $i, j = 1, 2, \cdots, n$, $x_{n+1,j} = e_j, j = 1, 2, \cdots, n$, we find

$$n! V_n' = \left| \text{diag} \left( -\frac{|c|}{C_{1 \cdot n+1}}, -\frac{|c|}{C_{2 \cdot n+1}}, \ldots, -\frac{|c|}{C_{n \cdot n+1}} \right) \right| = \frac{||c||^n}{\prod_{i=1}^{n} C_{i \cdot n+1}}.$$

From (15), then

$$n! V_n = \frac{||c||^n}{\prod_{i=1}^{n} C_{j \cdot n+1}}.$$
Next we note from (21), by multiplying $c$ by its transpose, that $|c|^2 = |s_{ij} + e_i e_j|$ where as before $s_{ij} = b_i \cdot b_j$. An argument analogous to that given after equation (11) then shows that $|c|^2 = \sum S_{ij} e_i e_j$ with $S_{ij}$ the cofactor of $s_{ij}$ in $(s_{ij})$. Finally, we see from (21) that $|C_{j+n+1}|$ is (apart from sign) the determinant of the $n \times n$ matrix whose rows are the $b$ vectors, $b_j$ being omitted. Multiplying this matrix by its transpose gives $|C_{j+n+1}|^2 = S_{j:j}$. This quantity is positive since we assume every $n$ of the $b$'s are independent and hence, as a matrix $S_{j:j}$ is positive definite. Formula (2) then follows by substitution in (23).

4. The largest simplex whose $i$th vertex is distant $d_i$ from a given point. We choose the origin as the special point $Q$ and denote by $a_i d_i$ the vector from $Q$ to the vertex $P_i$. Here $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ is a unit vector. Equation (10) then gives

$$n! \, V = \prod_{i=1}^{n+1} d_i \begin{vmatrix} a_{i1} & \cdots & a_{in} & \frac{1}{d_i} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+1} & a_{n+1} & \vdots & \vdots \\
 & & 1 & \frac{a_{n+1}}{d_{n+1}} \end{vmatrix}.$$ (24)

The vectors $a_i$ are linearly dependent. We write

$$a_{n+1} = \sum_{i=1}^{n} \alpha_j a_j.$$ (25)

The determinant $D$ displayed in (24) can now be expressed easily in other terms. Multiply the $j$th row of $D$ by $\alpha_j$ and subtract from the last row, $j = 1, 2, \ldots, n$. Because of (25), all elements of the last row except the diagonal entry are zero. On expanding by this last row, we then find

$$D = |a| \left( \frac{1}{d_{n+1}} - \sum_{i=1}^{n} \frac{\alpha_j}{d_j} \right)$$ (26)

where

$$a = \begin{pmatrix} a_{i1} & \cdots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}. $$

We have also $|a|^2 = |\rho|$ where

$$\rho = (\rho_{ij}) = \begin{pmatrix} r_{i1} & \cdots & r_{in} \\ \vdots & \vdots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}.$$ (27)
and as before $\mathbf{a}_i \cdot \mathbf{a}_j = r_{ij}$. Finally, defining

$$x_i = d_{n+1}/d_i, \ i = 1, 2, \ldots, n$$

equation (24) becomes

$$n! \frac{V_n}{\prod d_j} = |\rho|^{1/2} \left[ 1 - \sum_{i=1}^{n} \alpha_j x_j \right].$$

The condition that $a_{n+1}$ is a unit vector becomes from (25)

$$\prod_{i=1}^{n} \rho_{ij} \alpha_i \alpha_j = 1.$$ (30)

We now seek to maximize (29), subject to (30), over all values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ and over all symmetric $n \times n$ nonsingular matrices $\rho$ having

$$\rho_{ii} = 1, \ i = 1, 2, \ldots, n.$$ (31)

Introducing the Lagrange multiplier $\lambda$, we seek the stationary values of

$$J = |\rho|^{1/2} \left[ 1 - \sum_{i=1}^{n} \alpha_j x_j \right] - \lambda \sum_{i,j} \rho_{ij} \alpha_i \alpha_j .$$

We have

$$\frac{\partial J}{\partial \alpha_i} = -|\rho|^{1/2} x_i - 2\lambda \sum_j \rho_{ij} \alpha_j = 0, \ i = 1, 2, \ldots, n$$

(32)

$$\frac{\partial J}{\partial \rho_{ij}} = \frac{1}{2} \rho_{ji}^{-1} |\rho|^{1/2} \left[ 1 - \sum \alpha_j x_j \right] - \lambda \alpha_i \alpha_j = 0,$$

$$i \neq j, \ i, j = 1, 2, \ldots, n.$$ (33)

Multiply (32) by $\alpha_i$ and sum. By (30) one finds

$$2\lambda = +|\rho|^{1/2}/u$$

(34)

where we have written

$$u = -\frac{1}{\sum_{j=1}^{n} \alpha_j x_j}.$$ (35)

Equations (32) and (33) then become

$$\sum_{j=1}^{n} \rho_{ij} \alpha_j = -u x_i, \ i = 1, 2, \ldots, n$$

(36)

$$\rho_{ij}^{-1} = \frac{1}{1 + u} \alpha_i \alpha_j, \ i \neq j, \ i, j = 1, 2, \ldots, n .$$ (37)
Our task now is to solve the non-linear system (31), (35), (36), (37) for the α's and ρ_{ij}.

Multiply (36) by α_i to obtain

\[-uα_i x_i = \sum_j ρ_{ij} α_i α_j \]

= α_i^2 + \sum_{j \neq i} ρ_{ij} α_i α_j

= α_i^2 + (1 + u) \sum_{j \neq i} ρ_{ij} \rho_{ji} \]

= α_i^2 + (1 + u)[1 - ρ_{ii}].

Here (31) was used to obtain the second line and (37) was used to obtain the third line. We have then

(38)

\[ρ_{ii}^{-1} = 1 + \frac{1}{1 + u}[α_i^2 + uα_i x_i].\]

From (36) we also have

\[α_i = -u \sum_j ρ_{ij} x_j, \quad i = 1, 2, \cdots, n.\]

We now use (37) and (38) to replace ρ_{ii}^{-1} in this sum. There results

\[-α_i/u = ρ_{ii}^{-1} x_i + \sum_{j \neq i} ρ_{ij} x_j \]

= x_i + \frac{x_i}{1 + u} [α_i^2 + uα_i x_i] + \frac{α_i}{1 + u} \sum_{j \neq i} α_j x_j

= x_i + \frac{x_i}{1 + u} [α_i^2 + uα_i x_i] + \frac{α_i}{1 + u} \left[-\frac{1}{u} - α_i x_i\right].

To obtain the last line we have employed (35). The quadratic terms in α_i cancel in (39) and the equation yields

\[α_i = -\frac{(1 + u)x_i}{1 + ux_i^2}, \quad i = 1, 2, \cdots, n.\]

Therefore

\[\sum i α_i x_i = -(1 + u) \sum i \frac{x_i^2}{1 + ux_i^2} \]

= -\frac{1}{u}

by (35). The parameter u must therefore satisfy

(41)

\[\sum i \frac{x_i^2}{1 + ux_i^2} = \frac{1}{u(1 + u)}.\]

We now write (38) in the form

\[\sum \]
\[
\rho^{-1}_{ii} = \frac{1}{1 + u} [\alpha_i^2 + q_i], \quad i = 1, 2, \ldots, n
\]

where

\[
q_i = 1 + u + u\alpha_i x_i, \quad i = 1, 2, \ldots, n.
\]

It is easy to invert the matrix \( \rho^{-1} \) whose elements are given by (37) and (42). One finds

\[
|\rho^{-1}| = \frac{1}{(1 + u)^n} [1 + \sum \alpha_i^2/q_i] \prod q_i
\]

\[
\rho_{ii} = \frac{(1 + u) \left[ 1 + \sum_{j \neq i} \alpha_j^2/q_j \right]}{q_i [1 + \sum \alpha_j^2/q_j]}, \quad i = 1, 2, \ldots, n
\]

\[
\rho_{ij} = -\frac{(1 + u)}{1 + \sum \alpha_j^2/q_j} \cdot \frac{\alpha_i \alpha_j}{q_i q_j}, \quad i \neq j, \quad i, j = 1, 2, \ldots, n.
\]

Using (40), (41) and (43) in these expressions, one verifies that \( \rho_{ii} = 1 \) and finds

\[
\rho_{ij} = -ux_i x_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n.
\]

We note that from (25)

\[
r'_{n+1;i} = a_{n+1} \cdot \alpha_i = \sum_{j=1}^{n} \alpha_j \rho_{ij}
\]

\[
= \alpha_i - \sum_{j \neq i} \alpha_j (-ux_i x_j)
\]

\[
= \alpha_i - ux_i \left( -\frac{1}{u} - \alpha_i x_i \right)
\]

\[
= -ux_i.
\]

Here we have used (47) to obtain the second line, (35) to obtain the third line and (40) to obtain the final line. From (44), using (40), (41) and (43), one finds

\[
|\rho| = \frac{u}{1 + u} \prod_{i=1}^{n} (1 + ux_i^2).
\]

We now symmetrize the formulae thus far obtained by introducing

\[
\theta = ud_{n+1}^2.
\]

With the help of (28), (41) becomes

\[
\theta \sum_{i=1}^{n+1} \frac{1}{\theta + d_j^2} = 1.
\]
Equations (47) and (48) can be written jointly as

\[ r_{ij}' = -\frac{\theta}{d_i d_j}, \quad i \neq j, \quad i, j = 1, 2, \ldots, n + 1. \]

Finally, (29), (35) and (49) give us

\[ n! V = |\theta|^{-|\Omega|} \prod_1^{n+1} |\theta + d_i^2|^{1/2}. \]

To complete our demonstration of (3) and (5), we must show that \( \theta \) must be chosen as the unique positive root of (51).

Let us suppose that the distances \( d_i \) are all distinct and that \( 0 < d_1 < d_2 < \cdots < d_{n+1} \). The modifications of our argument necessary when several \( d \)'s are identical are easily made. It is readily seen from (51) that \( \theta \) is the root of a polynomial of degree \( n + 1 \) whose \( n + 1 \) roots are real and can be labelled so that

\[ \theta_1 > 0 > -d_1^2 > \theta_2 > -d_2^2 > \cdots > \theta_{n+1} > -d_{n+1}^2. \]

We shall show that the roots \( \theta_3, \theta_4, \ldots, \theta_{n+1} \) do not correspond to a realizable simplex. Let \( H(\theta) = \theta^{-1} \prod_{i=1}^{n+1} (\theta + d_i^2) \) so that \( n! V = |H(\theta)|^{1/2} \). We shall also show that \( H(\theta_3) > H(\theta_2) > 0 \) which will then complete the proof.

Consider the \( (n + 1) \times (n + 1) \) matrix \( r \) whose elements are given by (52) and \( r_{ii} = 1, \quad i = 1, 2, \ldots, n + 1 \). The elements \( r_{ij} = a_i \cdot a_j \) of this matrix are scalar products of the optimal \( a \)'s and since for arbitrary real numbers \( \gamma_i \),

\[ |\sum \gamma_i a_i|^2 = \sum \gamma_i a_i \cdot \sum \gamma_j a_j = \sum r_{ij} \gamma_i \gamma_j \geq 0 \]

it follows that \( r \) must be nonnegative definite. The determinant of \( r \) and all the principal minors of \( r \) must then also be nonnegative. One readily finds

\[ |r| = \left[1 - \theta \sum_1^{n+1} \frac{1}{\theta + d_j^2}\right]^{n+1} \prod_1^{n+1} \frac{\theta + d_j^2}{\theta + d_i^2}. \]

An expression for the principal minor of \( r \) obtained by deleting rows and columns \( j, j_2, \ldots, j_l \) is given by (53) by omitting the terms and factors involving \( d_{j_1}, d_{j_2}, \ldots, d_{j_l} \).

Suppose now \( \theta = \theta_3 \). Since \( \theta_3 \) is a root of (51),

\[ 0 = 1 - \theta_3 \sum_1^{n+1} \frac{1}{\theta_3 + d_j^2} < 1 - \theta_3 \sum_2^{n+1} \frac{1}{\theta_3 + d_j^2} \]

since \( \theta_3/(\theta_3 + d_j^2) > 0 \). The principal minor of \( r \) obtained by deleting the first row and column has value
We have seen that the bracketed expression is positive. Of the factors, \( \theta_3 + d_2^2 \) is negative, and all others positive. \( R_{11} \) is therefore negative and we must reject the root \( \theta_3 \).

In a similar manner one sees that for \( \theta = \theta_k, k > 2 \) the principal minor obtained by deleting rows and columns \( 1, 2, \cdots, k - 2 \) is negative. We complete the proof by showing \( H(\theta_1) > H(\theta_2) > 0 \). Since \( \theta_1 > 0 \) while \( 0 > -d_1^2 > \theta_2 > -d_2^2 \cdots \)

\[
\frac{d_1^2}{\theta_1} > \frac{d_2^2}{\theta_2}
\]

so

\[
1 + \frac{d_2^2}{\theta_1} > 1 + \frac{d_2^2}{\theta_2}
\]

or

\[
\frac{\theta_1 + d_1^2}{\theta_1} > \frac{\theta_2 + d_2^2}{\theta_2} > 0.
\]

Now

\[
\theta_1 + d_1^2 > \theta_2 + d_2^2 > 0 \quad \text{for} \quad j \geq 2
\]

so that

\[
H(\theta_i) = \frac{\theta_1 + d_1^2}{\theta_1} \prod_{j=2}^{n-1} (\theta_1 + d_j^2) > \frac{\theta_2 + d_2^2}{\theta_2} \prod_{j=2}^{n-1} (\theta_2 + d_j^2) = H(\theta_2) > 0.
\]

We close this section with the remark that the origin and \( P_j \) lie on the same side of \( H_i \) if and only if \( (\theta_1 + d_1^2)/\theta_1 \) is positive. We omit the direct demonstration of this fact here. Corresponding to the root \( \theta_1 > 0 \) of (51) we obtain a simplex containing the special point \( Q \). For the root \( \theta_2 \), satisfying \( -d_1^2 > \theta_2 > -d_2^2 \), we see that \( Q \) lies outside the simplex, since \( (\theta_2 + d_2^2)/\theta_2 < 0 \) for example.

5. The smallest simplex whose \( i \)th bounding plane is distant \( e_i \) from a given interior point. We choose the origin as the given interior point. Let \( b_i \) be the unit vector from the origin along the perpendicular to boundary \( H_i, i = 1, 2, \cdots, n + 1 \). The volume of the simplex is given by (23) with \( c \) defined in (21). Now the vectors \( b_i \) are linearly dependent. We write

\[
b_{n+1} = \sum_{j=1}^{n} \beta_j b_j
\]
in analogy with (25). Making an obvious association between $|c|$ and the determinant in (24), we find from (26) that

$$|c| = |b| \left| e_{n+1} - \sum_{i}^{n} \beta_{i} e_{j} \right|$$

where $b$ is the $n \times n$ matrix given in (13). We note that $|C_{j,n+1}| = |\alpha_{j}||b||, \ j = 1, \ldots, n$ while $C_{n+1,n+1} = |b|$. Equation (23) then gives us

$$n! \ V_{n} = \left| \frac{|b|^{n} \left| e_{n+1} - \sum_{i}^{n} \beta_{i} e_{j} \right|^n}{|b|^{n+1} \prod_{i}^{n} \beta_{j}} \right| = \left| \frac{e_{n+1} - \sum_{i}^{n} \beta_{i} e_{j}}{|\sigma|^{1/2} \prod_{i}^{n} \beta_{j}} \right|^{n}$$

where

$$\sigma = (\sigma_{ij}) = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}$$

where as before $s_{ij} = b_{i} \cdot b_{j}$. Finally, defining

$$y_{i} = e_{i}/e_{n+1}, \ i = 1, \ldots, n$$

(55) becomes

$$n! \ V_{n} = \left| \frac{1 - \sum \beta_{j} y_{j}}{|\sigma|^{1/2} \prod_{i}^{n} \beta_{j}} \right|^n$$

The condition that $b_{n+1}$ is a unit vector becomes from (54)

$$\sum_{i=1}^{n} \sigma_{ij} \beta_{i} \beta_{j} = 1 .$$

We now seek to minimize (57), subject to (58), over all values of $\beta_{i}, \cdots, \beta_{n}$ and all symmetric $n \times n$ nonsingular matrices $\sigma$ having

$$\sigma_{ii} = 1, \ i = 1, 2, \ldots, n .$$

Introducing the Lagrange multiplier $\mu$, we seek the stationary values of

$$K = n \log \left[ 1 - \sum \beta_{j} y_{j} \right] - \frac{1}{2} \log |\sigma| - \sum \log \beta_{j} - \mu \sum \sigma_{ij} \beta_{i} \beta_{j} .$$

We have

$$\frac{\partial K}{\partial \beta_{i}} = \frac{-ny_{i}}{1 - \sum \beta_{j} y_{j}} - \frac{1}{\beta_{i}} - 2\mu \sum \sigma_{ij} \beta_{j} = 0, \ i = 1, 2, \ldots, n ,$$

(60)

$$\frac{\partial K}{\partial \sigma_{ij}} = -\frac{1}{2} \sigma_{ii}^{-1} - \mu \beta_{i} \beta_{j} = 0, \ i \neq j, \ i, j = 1, 2, \ldots, n .$$

(61)
Multiply (60) by \( \beta_i \) and sum. By (58) one finds

\[
2\mu = -\frac{n}{1 - \sum \beta_j y_j} = -\frac{1}{1 + v}
\]

where we have set

\[
v = -\frac{1}{n} \left[ n - 1 + \sum \beta_j y_j \right].
\]

Equations (60) and (61) then become

\[
\sum_j \sigma_{ij} \beta_j = y_i + \frac{1 + v}{\beta_i}, \quad i = 1, 2, \ldots, n
\]

\[
\sigma^\dagger_{ij} = \frac{1}{1 + v} \beta_i \beta_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n.
\]

Our task now is to solve the nonlinear system (59), (63), (64), (65) for the \( \beta \)'s and \( \sigma_{ij} \).

Multiply (64) by \( \beta_i \) to obtain

\[
\beta_i y_i + 1 + v = \beta_i^2 + \sum_{j \neq i} \sigma_{ij} \beta_i \beta_j
\]

\[
= \beta_i^2 + (1 + v) \sum_{j \neq i} \sigma_{ij} \sigma^\dagger_{ji}
\]

\[
= \beta_i^2 + (1 + v)(1 - \sigma^\dagger_{ii})
\]

whence

\[
\sigma^\dagger_{ii} = \frac{1}{1 + v} [\beta_i^2 - \beta_i y_i].
\]

From (64)

\[
\beta_i = \sum_{j=1}^n \sigma_{ij} \left[ y_j + \frac{1 + v}{\beta_j} \right].
\]

Replace \( \sigma^\dagger_{ij} \) by values given in (65) and (66). Use (63). There results

\[
\beta_i = -\frac{(1 + v)y_i}{v + y_i^2}, \quad i = 1, 2, \ldots, n.
\]

Multiply by \( y_i \) and sum. Insert the result in (63). One finds that \( v \) must satisfy

\[
\sum \frac{1}{v + y_j^2} = \frac{1}{v(1 + v)}.
\]

The analogy between (37) and (65) and between (42) and (66) permits us to use (44), (45) and (46) directly to obtain
The substitution

\[
v = \frac{\psi}{e_{n+1}^2}
\]

now yields (4) and (6). We omit the details.

In analogy with (5), the roots of (6) are all real and can be labelled so that \( \psi_1 > 0 > -e_1 > \psi_2 > -e_2 > \cdots > \psi_{n+1} > -e_{n+1}^2 \), if \( e_1 < e_2 \cdots < e_{n+1} \). Only \( \psi_1 \) and \( \psi_2 \) correspond to realizable simplexes and the content corresponding to \( \psi_1 \) is greater than the content of the simplex corresponding to the root \( \psi_2 \). It is not difficult to show that \( P_j \) and the origin lie on the same side of \( H_j \) if and only if \( \psi + e_j^2 > 0 \). For the solution corresponding to \( \psi_1 \), then, \( Q \) lies within the simplex; for the solution corresponding to \( \psi_2 \), \( Q \) and the simplex lie on opposite sides of \( H_1 \).

**Reference**


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