Pacific Journal of Mathematics

IN THIS ISSUE—

Vol. 32, No. 1 **January, 1970**

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

COMPACT SOBOLEV IMBEDDINGS FOR UNBOUNDED DOMAINS

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A condition on an open set $G \subset E_n$ which is both necessary and sufficient for the compactness of the (Sobolev) imbedding $H_0^{u+1}(G) \rightarrow H_0^m(G)$ is not yet known. C. Clark has given a necessary condition (quasiboundedness) and a much stronger sufficient condition. We show here that (unless $n=1$) quasiboundedness is not sufficient, and answer in the negative a question raised by Clark on whether the imbedding can be compact if ∂G consists of isolated points. We also substantially weaken Clark's sufficient condition so as to include a wide class of domains with null exterior. The gap between necessary and sufficient conditions is thus considerably narrowed.

Let G be an open set in Euclidean n-space, E_n. Let $H_0^m(G)$ for each nonnegative integer m denote the Sobolev space obtained by completing with respect to the norm

$$\| u \|_{m,G} = \left\{ \sum_{|\alpha| \leqq m} \int_G | D^\alpha u(x)|^2 \, dx \right\}^{1/2}$$

the space $C_0^\infty(G)$ of all infinitely differentiable complex valued functions having compact support in G. Here, as usual, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers; $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = \partial / \partial x_j$, $j = 1, \cdots, n$.

We shall say that G has the Rellich property if for each integer $m \geqslant 0$ the imbedding mapping $H_0^{m+1}(G) \rightarrow H_0^m(G)$ is compact. It is well known that any bounded G has this property. An unbounded domain G is called quasibounded if dist $(x, \partial G) \rightarrow 0$ whenever x tends to infinity in G. If G is unbounded and not quasibounded then it contains an infinite number of mutually disjoint, congruent balls. If φ is infinitely differentiable, has support in one of these balls, and has nonzero $L^2(G)$ norm then the set of its translates with supports in the other balls provides a counterexample showing the imbedding $H_0^1(G) \rightarrow H_0^0(G) \equiv L^2(G)$ is not compact. Thus for an unbounded domain quasiboundedness is necessary for the Rellich property.

In [2] Clark showed that the following Condition 1 is sufficient to guarantee that G has the Rellich property.

CONDITION 1. To each $R \geqq 0$ there correspond positive numbers $d(R)$ and $\delta(R)$ satisfying

(a) $d(R) + \delta(R) \rightarrow 0$ as $R \rightarrow \infty$,

(b) $d(R)/\delta(R) \leqq M < \infty$ for all R,

(c) for each $x \in G$ with $|x| > R$ there exists y such that $|x-y| < d(R)$ and $G \cap \{z: |z-y| < \delta(R)\} = \varnothing$.

This condition is considerably stronger than quasiboundedness. It implies, for example, that G has nonnull exterior. In [3] Clark gave an example of an unbounded domain having the Rellich property but not satisfying Condition 1. His example was the "spiny urchin," an open connected set in E_2 obtained by removing from the plane all points whose polar coordinates (r, θ) satisfy for any $k = 1, 2, \cdots$ the two restrictions $r \geqq k$ and $\theta = 2^{-k}m\pi$, $m = 1, 2, \cdots, 2^{k+1}$.

In this paper the gap between quasiboundedness as a necessary condition and Condition 1 as a sufficient condition for a domain to have the Rellich property is narrowed from both ends. On the one hand we show that if $n \geqq 2$ then no open set whose boundary consists only of isolated points with no finite accumulation point can have the Rellich property. This settles a question raised by Clark in [3]. On the other hand we show that Condition 1 can be replaced by the following weaker Condition 2, which is still sufficient to guarantee that G has the Rellich property. In the statement $B_r(x)$ denotes the open ball of radius r about x.

CONDITION 2. There exists $R_0 \geqq 0$ such that to each $R \geqq R_0$ there correspond numbers $d(R), \delta(R) > 0$ such that

(a) $d(R) + \delta(R) \to 0$ as $R \to \infty$,

(b) $d(R)/\delta(R) < M \leqq \infty$ for all $R \geqq R_0$,

(c) for each $x \in G$ such that $|x| > R \geqq R_0$ the ball $B_{3d(R)}(x)$ is disconnected into two open components C_1 and C_2 by an $n-1$ dimensional manifold forming part of the boundary of G in such a way that each of the two open sets $C_i \cap B_{d(R)}(x)$, $i = 1, 2$, contains a ball of radius $\delta(R)$.

Roughly speaking if the $n-1$ dimensional manifolds in the boundary of G are reasonably smooth and unbroken, and bound a quasibounded domain (containing G) then G will satisfy Condition 2. Clark's "spiny urchin" is an example of such a domain. If $n = 1$ any quasibounded domain satisfies Condition 2, (but not necessarily Condition 1) and so in this case quasiboundedness is necessary and sufficient for the Rellich property.

Our principal results are as follows

THEOREM 1. *If G is open in E_n, $n \geqq 2$, and the boundary of G consists only of isolated points with no finite accumulation point, then the imbedding $H_0^1(G) \to L^2(G)$ is not compact. Thus quasibounded-*

ness is not sufficient to guarantee the Rellich property.

THEOREM 2. *If G is open in E_n and satisfies Condition 2 then it has the Rellich property.*

For the proof of Theorem 1 we require the following

LEMMA 1. *Given $\rho, \delta > 0$, $x_0 \in E_n$ $(n \geqq 2)$, there exists a function $u \in C^\infty(E_n)$ with the following properties*

(1) $u(x) = 0$ *in a neighbourhood of* x_0

(2) $0 \leqq u(x) \leqq 1$ *for all* x

(3) $u(x) = 1$ *outside the ball* $B_\rho(x_0)$

(4) $\int_{En} |\nabla u(x)|^2 \, dx^2 \leqq \delta^2$.

Proof. Let $f \in C^\infty(R)$ satisfy $0 \leqq f(t) \leqq 1$, $f(t) = 1$ for $t \geqq 1$ and $f(t) = 0$ in a neighbourhood of $t = 0$. Let m be a positive integer, put $r = |x - x_0|$ and define

$$u(x) = v(r) = f([r/\rho]^{1/m}) \, .$$

Clearly $u \in C^\infty(E_n)$ and satisfies (1), (2) and (3). Also

$$|\nabla u(x)|^2 = \sum_{i=1}^{n} |D_i u(x)|^2 = |v'(r)|^2 \, .$$

Denoting by ω_n the surface area of the unit sphere in E_n and making the change of variables $t = (r/\rho)^{1/m}$ we obtain

$$\begin{aligned}\int_{En} |\nabla u(x)|^2 \, dx &= \omega_n \int_0^\rho \left| \frac{d}{dr} f\left(\left[\frac{r}{\rho}\right]^{1/m}\right)\right|^2 r^{n-1} dr \\ &= \omega_n \rho^{n-2} m^{-1} \int_0^1 \left| \frac{d}{dt} f(t) \right|^2 t^{1+m(n-2)} dt \\ &\leqq \omega_n \rho^{n-2} m^{-1} [2 + m(n-2)]^{-1} \sup_{0 \leqq t \leqq 1} |f'(t)|^2\end{aligned}$$

which, for $n \geqq 2$, can be made less than δ^2 for a suitably large choice of m.

REMARK. If $\varphi \in C_0^\infty(E_n)$ and u is constructed as above, then $\varphi \cdot u \in C_0^\infty(E_n - \{x_0\}) \subset H_0^1(E_n - \{x_0\})$.

Proof of Theorem 1. Let Q be a fixed open ball in E_n. Let $\varphi \in C_0^\infty(Q)$ be extended to all of E_n so that $\varphi(x) = 0$ in $E_n - Q$. Suppose $\varphi(x) \geqq 0$ for all x and

$$\|\varphi\|_{0,E_n} = C > 0 \, , \qquad \|\varphi\|_{1,E_n} = K > 0 \, .$$

There exists $M > 0$ such that for all x in E_n

$$|\varphi(x)| \leqq M , \qquad |D_j\varphi(x)| \leqq M , \qquad j = 1, \cdots, n .$$

If Q contains no boundary points of G put $\psi = \varphi$. Otherwise Q contains only a finite number of boundary points of G, say $x_1, \cdots, x_k$. For $i = 1, \cdots, k$ let $B_i = B_{\rho_i}(x_i)$ where ρ_i is small enough that vol. $B_i \leqq (C/2kM)^2$. Let $\delta = K/Mk$ and let u_i be the function constructed as in Lemma 1 corresponding to the point x_i and the constants ρ_i and δ. Put $\psi = \varphi \cdot u_1 \cdots u_k$. Clearly $\psi \in H_0^1(Q - \{x_i, \cdots, x_k\}) \subset H_0^1(G)$. We have

$$\begin{aligned} \|\psi\|_{0,G} &\geqq \|\varphi\|_{0,E_n} - \sum_{i=1}^{k} \|\varphi\|_{0,B_i} \\ &\geqq C - \sum_{i=1}^{k} M(\text{vol. } B_i)^{1/2} \geqq \frac{1}{2}C . \end{aligned}$$

Also

$$\begin{aligned} \|D_j\psi\|_{0,G} &\leqq \|D_j\varphi\|_{0,E_n} + \sum_{i=1}^{k} \|\varphi u_1 \cdots D_j u_i \cdots u_k\|_{0,B_i} \\ &\leqq K + kM\delta = 2K . \end{aligned}$$

Since $\|\psi\|_{0,G} \leqq \|\varphi\|_{0,G} = C$ we have

$$\|\psi\|_{1,G} \leqq (C^2 + 4nK^2)^{1/2} = C_1 .$$

Now let $\{Q_i\}_{i=1}^{\infty}$ be a family of mutually disjoint open balls in E^n all congruent to Q. Let φ_i be a translate of φ with support in Q_i and let $\psi_i \in H_0^1(G)$ be constructed from φ_i as above, so that

$$\|\psi_i\|_{0,G} \geqq \frac{C}{2} , \qquad \|\psi_i\|_{1,G} \leqq C_1 .$$

Then the sequence $\{\psi_i\}_{i=1}^{\infty}$ is bounded in $H_0^1(G)$ but contains no subsequence convergent in $L^2(G)$ since for $i \neq j$ $\|\psi_i - \psi_j\|_{0,G} \geqq C/\sqrt{2}$. Thus the imbedding $H_0^1(G) \to L^2(G)$ is not compact.

The proof of Theorem 2 is based on the following generalization of Poincaré's inequality which is a variant on those forms appearing in Agmon [1] and Clark [2].

LEMMA 2. *Let G be open in E_n and satisfy Condition 2. Let G_R denote $G \cap \{x: |x| > R\}$. Then there exists a constant c depending only on n and M (the constant of Condition 2 (b)) such that for all $R \geqq R_0$ and every $u \in H_0^1(G)$*

$$\int_{G_R} |u(x)|^2\, dx \leqq c(d(R))^2 \int_G |\nabla u(x)|^2\, dx .$$

Proof. Fix $R \geqq R_0$ and let $d = d(R)$, $\delta = \delta(R)$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of integers let $Q_\alpha = \{x \in E_n : \alpha_k n^{-1/2} d \leqq x_k \leqq (\alpha_k + 1) n^{-1/2} d\}$. Then $E_n = \bigcup_\alpha Q_\alpha$. Let $\varphi \in C_0^\infty(G)$. Fix $x \in G_R$. Then $x \in Q_\alpha$ for some α. Let $B_d = B_d(x)$, $B_{3d} = B_{3d}(x)$. There exists an $n-1$ dimensional manifold forming part of ∂G which disconnects B_{3d} into open components C_1 and C_2 and there exist points $y_i \in C_i$ $(i = 1, 2)$ such that $B_\delta(y_i) \subset C_i$. Thus φ can be written as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_i \in C_0^\infty(G)$ and $\varphi_1 \equiv 0$ in C_2 while $\varphi_2 \equiv 0$ in C_1. Since $Q_\alpha \subset B_d$ we have

$$\int_{Q_\alpha \cap G_R} |\varphi(y)|^2 dy \leqq \int_{C_1 \cap B_d} |\varphi_1(y)|^2 dy + \int_{C_2 \cap B_d} |\varphi_2(y)|^2 dy .$$

If (r, σ) and S denote respectively spherical coordinates in E_n centered at y_2 and the surface of the unit sphere about y_2 we have

$$\begin{aligned} \int_{C_1 \cap B_d} |\varphi_1(y)|^2 dy &\leqq \int_S d\sigma \int_\delta^{2d} |\varphi_1(r, \sigma)|^2 r^{n-1} dr \\ &\leqq 2d \int_S |\varphi_1(t, \sigma)|^2 t^{n-1} d\sigma \end{aligned}$$

where $t = t(\sigma)$ satisfies $\delta \leqq t \leqq 2d$. Since $\varphi_1(\delta, \sigma) = 0$ it follows that

$$\begin{aligned} |\varphi_1(t, \sigma)^2 t^{n-1}| &= \left| \int_\delta^t \frac{d}{dr} \varphi_1(r, \sigma) dr \right|^2 t^{n-1} \\ &\leqq (2d)^n \int_\delta^{2d} \left| \frac{d}{dr} \varphi_1(r, \sigma) \right|^2 dr \\ &\leqq (2d)^n \delta^{1-n} \int_\delta^{2d} \left| \frac{d}{dr} \varphi_1(r, \sigma) \right|^2 r^{n-1} dr . \end{aligned}$$

Thus, since $d/\delta < M$,

$$\begin{aligned} \int_{C_1 \cap B_d} |\varphi_1(y)|^2 dy &\leqq (2d)^{n+1} \delta^{1-n} \int_S d\sigma \int_\delta^{2d} \left| \frac{d}{dr} \varphi_1(r, \sigma) \right|^2 r^{n-1} dr \\ &\leqq 2^{n+1} M^{n-1} d^2 \int_{\delta \leqq |y - y_2| \leqq 2d} |\nabla \varphi_1(y)|^2 dy \\ &\leqq 2^{n+1} M^{n-1} d^2 \int_{B_{3d}} |\nabla \varphi_1(y)|^2 dy . \end{aligned}$$

Combining this with a similar expression for φ_2 we obtain

$$\begin{aligned} \int_{Q_\alpha \cap G_R} |\varphi(y)|^2 dy &\leqq 2^{n+1} M^{n-1} d^2 \int_{B_{3d}} |\nabla \varphi(y)|^2 dy \\ &\leqq 2^{n+1} M^{n-1} d^2 \int_{Q'_\alpha} |\nabla \varphi(y)|^2 dy \end{aligned}$$

where Q'_α is the union of all the sets Q_α which intersect B_{3d}. There is a number N depending only on n such that any $N+1$ of the sets Q'_α have null intersection. Summing the above inequality over all α for which Q_α intersects G_R we obtain

$$\int_{G_R} |\varphi(y)|^2\, dy \leqq 2^{n+1} N M^{n-1} (d(R))^2 \int_G |\nabla\varphi(y)|^2\, dy \,.$$

This inequality extends by completion to $H_0^1(G)$.

The remaining part of the proof of Theorem 2 is similar to Clark's proof [2, Th. 3] and is included here for completeness. First, however, let $H^m(G, R)$ be the completion in the norm $\|\cdot\|_{m, G\cap K_R}$ of the space $C_0^\infty(G, R)$ of all C^∞ functions whose support is a compact subset of $G \cap K_R$ where $K_R = \overline{B_R(0)}$. Since the imbedding $H_0^{m+1}(K_R) \rightarrow H_0^m(K_R)$ is known to be compact [4, Chapter XIV] and since an element of $H^m(G, R)$ can be extended to be zero outside its support so as to belong to $H_0^m(K_R)$ it follows that the imbeddings $H^{m+1}(G, R) \rightarrow H^m(G, R)$, $m = 0, 1, 2, \cdots$ are compact.

Proof of Theorem 2. It suffices, by an inductive argument, to prove only that the imbedding $H_0^1(G) \rightarrow L^2(G)$ is compact. We make use of the following well known compactness criterion for sets in $L^2(G)$: if $G \subset E_n$ and the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(G)$ then it is compact in $L^2(G)$ provided

(a) for every bounded $G' \subset G$ the sequence $\{u_k \mid G'\}$ is compact in $L^2(G')$, and

(b) for each $\varepsilon > 0$ there exists $R > 0$ such that for all k

$$\int_{G_R} |u_k(x)|^2\, dx < \varepsilon \,.$$

Now let $\{u_k\}$ be a sequence bounded in $H_0^1(G)$, say $\|u_k\|_{1,G} \leqq K$. By Lemma 2, for $R \geqq R_0$ we have $\|u_k\|_{0,G_R} \leqq C(d(R))^2 K \rightarrow 0$ as $R \rightarrow \infty$ so condition (b) of the criterion is satisfied. To establish (a) let G' be a bounded subset of G, so that $G' \subset K_R$ for some R. Since $\{u_k \mid K_R\}$ is bounded in $H^1(G, R)$ it is compact in $H^0(G, R) = L^2(K_R \cap G)$ and so $\{u_k \mid G'\}$ is compact in $L^2(G')$. Thus $\{u_k\}$ is compact in $L^2(G)$, whence the theorem.

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Received January 8, 1968. Research partially supported by National Research Council of Canada Operating Grant A-3973.

THE UNIVERSITY OF BRITISH COLUMBIA

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

GROUPS WITH MAXIMUM CONDITIONS

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It still seems to be unknown whether there exist Noetherian groups (= groups with maximum condition on subgroups) that are not almost polycyclic, i.e., possess a soluble normal subgroup of finite index. However, the existence of even finitely generated infinite simple groups shows that in general a group whose subnormal subgroups satisfy the maximum condition need not be almost polycyclic. The following theorem gives a number of criteria for a group satisfying a weak form of the maximum condition to be almost polycyclic.

THEOREM. *The following conditions of the group G are equivalent*:

(I) *G is almost polycyclic.*

(II)
(a) *If C is a characteristic subgroup of G, then C is finitely generated.*
(b) *Every infinite epimorphic image H of G possesses a locally almost soluble characteristic subgroup $N \neq 1$.*

(III)
(a) *If C is a characteristic subgroup of G, then C is finitely generated.*
(b) *Every infinite epimorphic image H of G possesses a locally almost polycyclic accessible subgroup $E \neq 1$.*

(IV)
(a) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*
(b) *Every infinite epimorphic image H of G possesses an almost radical accessible subgroup $E \neq 1$.*

(V)
(a) *If the normal subgroup N of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of N.*
(b) *Every infinite epimorphic H of G possesses a normal subgroup $N \neq 1$ with $\mathfrak{c}_H N \neq 1$.*

(VI)
(a) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*
(b) *Every infinite epimorphic image H of G possesses a characteristic subgroup $N \neq 1$ with $\mathfrak{c}_H N \neq 1$.*

(a1) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*

(VII) (a2) *The maximum condition is satisfied by the normal subgroups of G.*
(b) *Every infinite epimorphic image H of G possesses a normal subgroup* $N \neq 1$ *with* $\mathfrak{c}_H N \neq 1$.

(VIII) (a1) *If G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of G.*
(a2) *Abelian normal subgroups of epimorphic images of G are finitely generated.*
(b) *Every infinite epimorphic image H of G possesses a normal subgroup* $N \neq 1$ with $\mathfrak{c}_H N \neq 1$.

REMARKS. G. Higman [9] has constructed an infinite finitely generated simple group. This group satisfies part (a) of every condition (II) to (VIII) of the theorem without being almost polycyclic. Hence part (b) of the conditions (II) to (VIII) is indispensable. Every group C_{p^∞} of Prüfer's type satisfies part (b) of every condition (II) to (VIII) of the theorem without being almost polycyclic. Hence part (a) of the conditions (II) to (VIII) is likewise indispensable. It is well known that a group G generated by two elements a and b with the relation $b^{-1}ab = a^2$ is metabelian and satisfies the maximum condition for normal subgroups without being almost polycyclic. This group satisfies conditions (VII. a2) and (VII. b) as well as (VIII. a1) and (VIII. b) so that (VII. a1) and (VIII. a2) are indispensable. The existence of infinite locally finite simple groups shows that conditions (II. a) and (III. a) cannot be replaced by (IV. a) or (V. a). We have been unable to decide whether or not conditions (VII. a2) and (VIII. a1) are indispensable. From the proof of the equivalence of (I) and (II) it may easily be seen that one gets a similar criteria if the word 'characteristic subgroup' in (II) is replaced by the word 'normal subgroup'.

NOTATIONS.

$\{\cdots\}$ = subgroup generated by the elements enclosed in braces.
$\mathfrak{z}G$ = center of the group G.
$\mathfrak{c}_G X$ = centralizer of the subset X of G in G.
$G^{(0)} = G$.
$G^{(i+1)} = G^{(i)}$ = commutator subgroup of $G^{(i)}$.
Factor = epimorphic image of a subgroup.

A subgroup U of the group G is Γ-admissible for the automorphism group Γ of G if every element in Γ maps U onto U.

Two subgroups A and B of G are automorphic if there exists an automorphism of G mapping A onto B.

A normal series is a well ordered set of subgroups X_ν of the group G with $0 \leqq \nu \leqq \tau$ such that X_ν is a normal subgroup of $X_{\nu+1}$ for $\nu < \tau$ and $X_\lambda = \bigcup_{\nu<\lambda} X_\nu$ for limit ordinals $\lambda \leqq \tau$; $X_{\nu+1}/X_\nu$ is a factor of the series

A subgroup U is accessible if there exists a normal series from U to G.

Soluble group = group with $G^{(i)} = 1$ for almost all i.

Noetherian group = group with maximum condition on subgroups.

Polycyclic group = Noetherian and soluble group.

Nilpotent group = group G with a finite central series from 1 to G.

Let $\mathfrak{e}$ be any group theoretical property.

A group is an $\mathfrak{e}$-group if it has the property $\mathfrak{e}$.

A group G is almost-$\mathfrak{e}$ if there exists a normal $\mathfrak{e}$-subgroup N of G with finite G/N.

A group G is locally-$\mathfrak{e}$ if every finitely generated subgroup of G is an $\mathfrak{e}$-group.

A group G is radical if every epimorphic image $H \neq 1$ of G possesses a locally nilpotent accessible subgroup $E \neq 1$.

In the proof of the theorem we need several lemmas most of which slightly extend known results. We recall that a group G is finitely presented if there exists a free group F of finite rank and a normal subgroup R of F generated by finitely many classes of elements conjugate in F such that $F/R \cong G$. Every almost polycyclic group is finitely presented; see R. Baer [3], p. 276, Folgerung 3.

LEMMA 1. *If $\mathfrak{e}$ is any class of finitely presented groups, if the finitely generated group G is not an $\mathfrak{e}$-group and if Γ is a group of automorphisms of G, then there exists a Γ-admissible normal subgroup N of G such that G/N is not an $\mathfrak{e}$-group, but G/M is an $\mathfrak{e}$-group, for every Γ-admissible normal subgroup M of G containing N properly.*

Proof. The set $\mathfrak{M}$ of all Γ-admissible normal subgroups X of G such that G/X is not an $\mathfrak{e}$-group is not empty, since it contains 1. Let $\mathfrak{X}$ be any nonempty subset of $\mathfrak{M}$ such that $X \subseteqq Y$ or $Y \subset X$ for every pair X, Y of Γ-admissible normal subgroups in $\mathfrak{X}$. If V denotes the union of the elements in $\mathfrak{X}$, then V is likewise a Γ-admissible normal subgroup of G. If V is not contained in $\mathfrak{M}$, then G/V is an $\mathfrak{e}$-group and hence finitely presented. Since G is finitely generated, there exists a finite subset T of V such that $V = \{T^G\}$; see R. Baer [3], p. 270, Satz 1. Since V is the union of the elements of $\mathfrak{X}$, for every x in T there exists a Γ-admissible normal subgroup X^* in $\mathfrak{X}$ containing x. Since the subgroups in $\mathfrak{X}$ are comparable and since T is finite, there exists a Γ-admissible normal subgroup Y in $\mathfrak{X}$ such that $X^* \subseteqq Y$ for every x in T. Thus T is a subset of the normal subgroup Y of G such that $V = \{T^G\} \subseteqq Y \subseteqq V$ so that $V = Y$ belongs to $\mathfrak{M}$. This contradiction shows that V is an element of $\mathfrak{M}$. We have shown that the maximum principle of set theory is applicable and that there exists therefore a maximal Γ-admissible normal subgroup N in $\mathfrak{M}$. Since N is contained in $\mathfrak{M}$, the epimorphic image $H = G/N$ of G is not an $\mathfrak{e}$-group. However, if M is a Γ-admissible normal subgroup of G with $N \subset M$, then the maximality of N implies that G/M is an $\mathfrak{e}$-group.

COROLLARY 2. *If the finitely generated group G is not almost polycyclic, then there exists an epimorphic image H of G which is not almost polycyclic, but every proper epimorphic image of H is almost polycyclic. Furthermore, there exists a characteristic subgroup C of G such that G/C is not almost polycyclic, but G/D is almost polycyclic for every characteristic subgroup D of G containing C properly.*

Proof. The class $\mathfrak{e}$ of almost polycyclic groups is finitely presented. Therefore the two statements follow immediately from Lemma 1 if Γ is the group of all inner automorphisms of the group G or the group of all automorphism of G respectively.

A set $\mathfrak{M}$ of normal subgroups of the group G is *independent*, if their product is direct.

LEMMA 3. *If 1 is the only finite characteristic subgroup of the group G, if 1 is the only finite Abelian accessible subgroup of G, and if independent sets of finite simple isomorphic normal subgroups of characteristic subgroups of G are finite, then 1 is the only finite accessible subgroup of G.*

Proof. If this statement is false, then there exists a finite accessible subgroup $M \neq 1$ of G, and we can assume that M is minimal. Our hypotheses imply that M is non-Abelian. If β is an automorphism of G, then the image M^β of M is automorphic to M and is likewise a finite simple non-Abelian accessible subgroup of G. Thus the subgroup M^* of G generated by all the subgroups of G which are automorphic to M is a characteristic subgroup of G which possesses a normal series with finite factors leading from 1 to G. It follows that M^* is locally finite; see for instance R. Baer [5], p. 53, bottom. *If* A and B are two different subgroups of G which are automorphic to M, then $V = \{A, B\}$ is finite, since it is a finitely generated subgroup of M^*. A and B are also accessible subgroups of the finite group V, so that A and B are subnormal subgroups of V. Application of H. Wielandt, [12], p. 463 (1. a), shows that A and B normalize each other. Thus A, B and $A \cap B$ are normal subgroups of V. Since $A \neq B$ and A and B are simple, we have $A \cap B = 1$. It follows that A and B centralize each other. Since all subgroups of G automorphic to M are finite and centralize each other pairwise, M^* is a direct product of finite simple groups automorphic to M. The hypotheses of our lemma now imply that M^* is a finite characteristic subgroup of G, which is impossible. Thus the lemma is proved.

COROLLARY 4. *If 1 is the only Abelian accesible subgroup of the group G and if independent sets of finite normal subgroups of characteristic subgroups are finite, then the product P of all finite normal subgroups of G is finite and 1 is the only almost Abelian accessible subgroup of $H = G/P$.*

Proof. Clearly the product P of all finite normal subgroups of G is a characteristic subgroup of G, so that independent sets of finite normal subgroups of P are finite. Application of R. Baer [7], p. 26, Lemma 5.1, now yields that P is finite.

If E is a finite normal subgroup of $H = G/P$, then there exists a normal subgroup X of G with $P \subseteqq X$ and $E = X/P$. Since P and E are finite, X is also finite, thus, X must be contained in P. This implies $E = 1$, and we have shown:

(1) 1 is the only finite normal subgroup of H.

Now let $h \neq 1$ be an element of H which generates an accessible subgroup $\{h\}$ of H. It follows from K. Grünberg, [8], p. 158, Th. 2, or R. Baer [5], p. 57, Satz 3.3, that the set T of all elements of H which generate accessible subgroups of H is a locally nilpotent characteristic subgroup of H. Since $h \neq 1$, we have $T \neq 1$, so that T is infinite by (1).

Let Q be the uniquely defined characteristic subgroup of G such that $P \subset Q$ and $T = Q/P$. Since 1 is the only Abelian accessible subgroup of G, we have $1 = \mathfrak{z}P = \mathfrak{c}_G P \cap P$. The finiteness of P implies the finiteness of $G/\mathfrak{c}_G P$. If $Q \cap \mathfrak{c}_G P = 1$, then

$$Q = Q/(Q \cap \mathfrak{c}_G P) \cong Q\mathfrak{c}_G P/\mathfrak{c}_G P \subseteqq G/\mathfrak{c}_G P$$

is also finite. But $T = Q/P$ infinite implies that Q is infinite. Hence $Q \cap \mathfrak{c}_G P \neq 1$. If $a \neq 1$ is an element in $Q \cap \mathfrak{c}_G P$, then Pa is an element in T and therefore $\{Pa\}$ is an accessible subgroup of T and H; see R. Baer [5], p. 59, Zusatz 3.6. It follows that $\{P, a\}/P$ is an accessible subgroup of the locally nilpotent group T; see K. Grünberg, [8], p. 158, Lemma 7, or R. Baer [5], p. 48, Lemma 1.4. Hence $\{P, a\}$ is an accessible subgroup of Q and G, and this implies that $\{P, a\} \cap \mathfrak{c}_G P$ is an accessible subgroup of $\mathfrak{c}_G P$ and G. Since $\{a\} \subseteqq \mathfrak{c}_G P$ the application of Dedekind's Modular Law yields

$$\{P, a\} \cap \mathfrak{c}_G P = P\{a\} \cap \mathfrak{c}_G P = \{a\}(P \cap \mathfrak{c}_G P) = \{a\} \neq 1 \, .$$

Thus there exists a cyclic accessible subgroup of G, which contradicts our hypotheses, and we have shown:

(2) 1 is the only Abelian accessible subgroup of H.

If U is any almost Abelian accessible subgroup of H, then (2) implies that U is finite. The statements (1) and (2) show that the hypotheses of Lemma 3 are satisfied by H. Thus $U = 1$, and our assertion is proved.

PROPOSITION 5. *Let $N \neq 1$ be a normal subgroup of the group G such that $G/\mathfrak{c}_G N$ is almost polycyclic. Then there exists an almost Abelian normal subgroup $A \neq 1$ of G. If N is a characteristic subgroup of G, then A is a characteristic subgroup of G.*

Proof. If $N \cap \mathfrak{c}_H N \neq 1$, then $\mathfrak{z}N$ is an Abelian normal subgroup of G, and clearly $\mathfrak{z}N$ is even characteristic in G whenever N is characteristic in G. If $N \cap \mathfrak{c}_H N \neq 1$, then

$$N = N/(N \cap \mathfrak{c}_G N) \cong N\mathfrak{c}_G N/\mathfrak{c}_G N \subseteqq G/\mathfrak{c}_G N\ ,$$

so that N is isomorphic to a subgroup of the almost polycyclic group $G/\mathfrak{c}_G N$. It follows that N is likewise almost polycyclic, and there exists a soluble characteristic subgroup S of N with finite N/S; see R. Baer [3], p. 276, Satz 3. If $S = 1$, then N is a nontrivial finite normal subgroup of G. If $S \neq 1$, there exists an Abelian characteristic subgroup $A \neq 1$ of N, which is a nontrivial Abelian normal subgroup of G. Clearly, A is also characteristic in S, N and G whenever N is a characteristic subgroup of G.

REMARK. The above proposition may be generalized easily.

LEMMA 6. *If 1 is the only almost Abelian normal subgroup of the group G, and if every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$ such that $\mathfrak{c}_H N \neq 1$, then every nontrivial normal subgroup of G possesses an infinite independent set of normal subgroups of G.*

Proof. If $X \neq 1$ is a normal subgroup of G, then our hypotheses imply that X is infinite. Since 1 is the only Abelian normal subgroup of G, we have $X \cap \mathfrak{c}_G X = \mathfrak{z}X = 1$. This implies that $X\mathfrak{c}_G X/\mathfrak{c}_G X \cong X/(X \cap \mathfrak{c}_G X) = X$ and therefore $G/\mathfrak{c}_G X$ are infinite. As in the proof of R. Baer [6], p. 177, Folgerung 5.2, one shows by using Lemma 5.1 of this paper that every nontrivial normal subgroup of G possesses an infinite set of independent normal subgroups of G.

COROLLARY 7. *If every independent set of infinite normal subgroups of any epimorphic image of the group G is finite, then the following two properties of G are equivalent*:

(I) *Every infinite epimorphic image H of G possesses an almost Abelian normal subgroup $N \neq 1$.*

(II) *Every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$ such that $\mathfrak{c}_H N \neq 1$.*

Proof. Let G be a group satisfying (I) and let H be an infinite epimorphic image of G. Then there exists an almost Abelian normal subgroup $N \neq 1$ of H. If N is finite, then $H/\mathfrak{c}_H N$ is finite, so that $\mathfrak{c}_H N$ is infinite. If N is infinite, then there exists an Abelian characteristic subgroup A of N with finite N/A; see R. Baer [2], p. 152, Lemma 2. Clearly A is an infinite normal subgroup of H with $\mathfrak{c}_H A \neq 1$. Thus (I) implies (II).

Conversely, let condition (II) be satisfied by G, and let H be an infinite epimorphic image of G. Then every independent set of infinite normal subgroups of H is finite, and Lemma 6 shows the existence of an almost Abelian normal subgroup $N \neq 1$ of H. Thus (II) implies (I), and our assertion is proved.

LEMMA 8. *Let G be a group satisfying the following condition*:

($\mathfrak{M}$) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*

Then the following conditions hold:

(a) *If A and B are characteristic subgroups of G with $A \subseteq B$, then B/A likewise satisfies* ($\mathfrak{M}$).

(b) *Products of independent finite normal subgroups of characteristic subgroups of G are finite.*

(c) *The product $\mathfrak{R}G$ of all almost polycyclic characteristic subgroups of G is an almost polycyclic characteristic subgroup of G.*

(d) *1 is the only almost radical accessible subgroup of $G/\mathfrak{R}G$.*

(e) *$\mathfrak{R}G$ contains every almost radical accessible subgroup of G.*

(f) *If G is not almost polycyclic, then there exists an epimorphic image H of G such that H/C is almost polycyclic for every characteristic subgroup $C \neq 1$ and 1 is the only almost radical accessible subgroup of H; H satisfies* ($\mathfrak{M}$).

Proof. It is easy to see that every characteristic subgroup and every factor group modulo a characteristic subgroup of a group with property ($\mathfrak{M}$) likewise satisfies ($\mathfrak{M}$). This implies (a).

Let C be a characteristic subgroup of G and let $\mathfrak{S}$ be an independent set of nontrivial finite normal subgroups of C. Then the product P of all finite normal subgroups of C is a locally finite characteristic subgroup of C different from 1. Thus P is finite, if it is finitely generated. If P is not finitely generated, then by $(\mathfrak{M})$ the normal subgroups of P satisfy the maximal condition. Then P is the product of finitely many finite groups and hence finite. The finiteness of P implies the finiteness of $\mathfrak{S}$, since every element of $\mathfrak{S}$ is contained in P. This proves (*b*).

Clearly the product $\mathfrak{R}G$ of all almost polycyclic characteristic subgroups of G is a characteristic subgroup of G which satisfies $(\mathfrak{M})$. Assume $\mathfrak{R}G$ is not almost polycyclic. If $\mathfrak{R}G$ is not finitely generated, then by $(\mathfrak{M})$ the normal subgroups of $\mathfrak{R}G$ satisfy the maximum condition. It follows that $\mathfrak{R}G$ is the product of finitely many almost polycyclic characteristic subgroups of G. This implies that $\mathfrak{R}G$ is likewise almost polycyclic, since every extension of an almost polycyclic group by an almost polycyclic group is almost polycyclic; see for instance W.R. Scott, [11], p. 150, 7. 1. 2. Hence $\mathfrak{R}G$ is finitely generated. Since $\mathfrak{R}G$ is not almost polycyclic, Corollary 2 shows the existence of an epimorphic image K of $\mathfrak{R}G$ with the following properties:

(1) K is not almost polycyclic, but every proper epimorphic image of K is almost polycyclic.

Clearly K is infinite. Since $\mathfrak{R}G$ is the product of almost polycyclic normal subgroups, K is likewise the product of almost polycyclic normal subgroups. Hence there exists an almost polycyclic normal subgroup $N \neq 1$ of K. By (1) K/N is almost polycyclic, and this implies that K is almost polycyclic, since every extension of an almost polycyclic group by an almost polycyclic group is almost polycyclic. Since this contradicts (1), we have proved (c).

If $C \neq 1$ is an almost polycyclic characteristic subgroup of $G/\mathfrak{R}G$, then there exists a characteristic subgroup D of G such that $\mathfrak{R}G \subset D$ and $C = D/\mathfrak{R}G$ is almost polycyclic. Since $\mathfrak{R}G$ and D are almost polycyclic, D is an almost polycyclic characteristic subgroup of G and thus contained in $\mathfrak{R}G$. This contradiction shows:

(2) 1 is the only almost polycyclic characteristic subgroup of $G/\mathfrak{R}G$.

Assume there exists a nontrivial radical accessible subgroup of $G/\mathfrak{R}G$. Then there exists also a nontrivial locally nilpotent accessible subgroup of $G/\mathfrak{R}G$, and the subgroup S generated by all locally nilpotent accessible subgroups of $G/\mathfrak{R}G$ is a nontrivial locally nilpotent characteristic subgroup of $G/\mathfrak{R}G$; see R. Baer [5], p. 57, Lemma 3. If S is finitely generated, then S is a finitely generated nilpotent group and therefore Noetherian and polycyclic; see R. Baer [1], p. 299, Satz B. This contradicts (2) so that S is not finitely generated. Since

$\mathfrak{R}G$ is a characteristic subgroup of G, by (a) $G/\mathfrak{R}G$ satisfies condition $(\mathfrak{M})$, and the normal subgroups of S fulfill the maximum condition. This implies that S is Noetherian and polycyclic, since S is locally nilpotent; see D.H. McLain, [10], Theorem 3.2, p. 10. This contradicts (1), and we have shown:

(3) 1 is the only radical accessible subgroup of $G/\mathfrak{R}G$.

By (b) independent sets of finite normal subgroups of characteristic subgroups of $G/\mathfrak{R}G$ are finite. Application of Corollary 4 yields that the product P of all finite normal subgroups of $G/\mathfrak{R}G$ is finite and that 1 is the only almost Abelian accessible subgroup of $(G/\mathfrak{R}G)/P$. It is a consequence of (2) that $P = 1$. This together with (3) implies that 1 is the only almost radical accessible subgroup of $G/\mathfrak{R}G$. We have proved (d).

If the almost radical accessible subgroup E of G is not contained in $\mathfrak{R}G$, then $E\mathfrak{R}G/\mathfrak{R}G \cong E/(E \cap \mathfrak{R}G)$ is a nontrivial almost radical accessible subgroup of $G/\mathfrak{R}G$. This contradicts (d), and thus (e) is proved.

Let G be not almost polycyclic. By condition $(\mathfrak{M})G$ is finitely generated or the normal subgroups of G satisfy the maximum condition. By Corollary 2 there exists a characteristic subgroup C of G such that G/C is not almost polycyclic, but G/D is almost polycyclic for every characteristic subgroup D of G containing C properly. By (a) $H = G/C$ satisfies $(\mathfrak{M})$. By (c) the product $\mathfrak{R}H$ of all almost polycyclic characteristic subgroups of H is an almost polycyclic characteristic subgroup of H. If $\mathfrak{R}H \neq 1$ then $H/\mathfrak{R}H$ is almost polycyclic, and this implies that H is almost polycyclic. Thus $\mathfrak{R}H = 1$, and by (d) 1 is the only almost radical accessible subgroup of H.

Proof of the theorem. If G is almost polycyclic, then G is especially Noetherian and every infinite epimorphic image of G possesses a finitely generated Abelian normal subgroup, not 1. These properties imply that the conditions (II) to (VIII) are consequences of (I).

Assume now that the group G is not almost polycyclic, but that at least one of the conditions (II) to (VIII) is satisfied. Then especially G is finitely generated or the maximum condition is satisfied by the normal subgroups of G. By Corollary 2 this implies the existence of a characteristic subgroup C of G with the following properties:

(1) $H = G/C$ is not almost polycyclic, but H/D is almost polycyclic for every characteristic subgroup $D \neq 1$ of H.

If (II) is satisfied, then H possesses a locally almost soluble characteristic subgoup $N \neq 1$ of H. Clearly H likewise satisfies condition (II. a), so that N is finitely generated. Since N is a finitely generated almost soluble group, there exists a soluble characteristic subgroup S of N and H; see for instance W.R. Scott, [11], p. 152, 7.7. If $S \neq 1$, then there exists an Abelian characteristic subgroup $A \neq 1$ of S, N and H. As a characteristic subgroup of H the group A is finitely generated and therefore Noetherian. This implies that there exists an almost polycyclic characteristic subgroup $D \neq 1$ of H. By (1) H/D is almost polycyclic, so that H is almost polycyclic. This contradicts (1), and G does not satisfy condition (II).

If (III) is satisfied, then H possesses a locally almost polycyclic accessible subgroup $E \neq 1$. Hence the subgroup R generated by all locally almost polycyclic accessible subgroup of H is a locally almost polycyclic characteristic subgroup, not 1, of H, since the product of two normal almost polycyclic subgroups is almost polycyclic; see R. Baer [4], p. 360, Folgerung 1. Since R is a characteristic subgroup of H, it is finitely generated by (III. a). Thus H is an extension of the almost polycyclic group R by H/R which is almost polycyclic by (1). But then H must be almost polycyclic, which contradicts (1). Hence G does not satisfy (III).

If one of the conditions (IV) to (VII) is satisfied, then by Lemma 8 (f) we may assume that the epimorphic image H of G satisfies, in addition to (1), the following condition:

(2) 1 is the only almost radical accessible subgroup of H.

Clearly (2) implies that G does not satify (IV).

If (V) is satisfied, (V. b) and (2) imply the existence of an infinite independent set $\mathfrak{S}$ of normal subgroups of H; see Lemma 6. Then the product P of all normal subgroups in $\mathfrak{S}$ is a normal subgroup of H, and (V. a) implies that P is finitely generated or the maximum condition is satisfied by the normal subgroups of P. In both cases $\mathfrak{S}$ must be a finite set. This contradiction shows that G does not satisfy (V).

If (VI) is satisfied, then there exists a characteristic subgroup $N \neq 1$ of H such that $\mathfrak{c}_H N \neq 1$. Since $\mathfrak{c}_H N$ is likewise a characteristic subgroup of H, $H/\mathfrak{c}_H N$ is almost polycyclic by (1). Now Proposition 5 yields the existence of an almost Abelian characteristic subgroup $A \neq 1$ of H. This contradicts (2), and G does not satisfy (VI).

If (VII) is satisfied, (VII. b) and (2) imply the existence of an infinite independent set $\mathfrak{S}$ of normal subgroups of H; see Lemma 6. But by (VII. a2) the normal subgroups of H satisfy the maximum condition. Hence $\mathfrak{S}$ must be finite, and G does not fulfill (VII).

Thus (VIII) must be satisfied. By (VIII. a1) and Lemma 2 there exists an epimorphic image H of G with the following properties:

(3) H is not almost polycyclic, but every proper epimorphic image of H is almost polycyclic.

By (VIII. b) there exists a normal subgroup $N \neq 1$ of H such that $\mathfrak{c}_H N \neq 1$. Condition (3) yields that $H/\mathfrak{c}_H N$ is almost polycyclic. Application of Proposition 5 shows the existence of an almost Abelian normal subgroup $B \neq 1$ of H. If B is infinite, then there exists an Abelian characteristic subgroup $C \neq 1$ of B which is an Abelian normal subgroup of H; see R. Baer [2], p. 152, Lemma 2. By (VIII. a2) C is finitely generated and therefore Noetherian. Thus there exists a Noetherian almost Abelian normal subgroup $A \neq 1$ of H. Since H/A is almost polycyclic by (3), H must be almost polycyclic also. This contradiction finally proves our theorem.

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Received May 15, 1969.

THE UNIVERSITY OF TEXAS

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

MÖBIUS FUNCTIONS OF ORDER k

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Let k denote a fixed positive integer. We define an arithmetical function μ_k, the Möbius function of order k, as follows:

$$\mu_k(1) = 1 ,$$
$$\mu_k(n) = 0 \text{ if } p^{k+1} \mid n \text{ for some prime } p ,$$
$$\mu_k(n) = (-1)^r \text{ if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{a_i} , \qquad 0 \leqq a_i < k ,$$
$$\mu_k(n) = 1 \text{ otherwise} .$$

In other words, $\mu_k(n)$ vanishes if n is divisible by the $(k+1)$st power of some prime; otherwise, $\mu_k(n)$ is 1 unless the prime factorization of n contains the kth powers of exactly r distinct primes, in which case $\mu_k(n) = (-1)^r$. When $k = 1$, $\mu_k(n)$ is the usual Möbius function, $\mu_1(n) = \mu(n)$.

This paper discusses some of the relations that hold among the functions μ_k for various values of k. We use these to derive an asymptotic formula for the summatory function

$$M_k(x) = \sum_{n \leqq x} \mu_k(n)$$

for each $k \geqq 2$. Unfortunately, the analysis sheds no light on the behavior of the function $M_1(x) = \sum_{n \leqq x} \mu(n)$.

It is clear that $|\mu_k|$ is the characteristic function of the set Q_{k+1} of $(k+1)$-free integers (positive integers whose prime factors are all of multiplicity less than $k+1$). Further relations with Q_{k+1} are given in §'s 4 and 5.

The asymptotic formula for $M_k(x)$ is given in the following theorem.

THEOREM 1. *If $k \geqq 2$ we have*

$$\sum_{n \leqq x} \mu_k(n) = A_k x + O(x^{1/k} \log x) , \tag{1}$$

where

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \prod_{p \mid n} \frac{1 - p^{-1}}{1 - p^{-k}} . \tag{2}$$

Note. In (2), $\zeta(k)$ is the Riemann zeta function. The formula for A_k can also be expressed in the form

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(n)}{nJ_k(n)} \tag{3}$$

where $\varphi(n)$ and $J_k(n)$ are the totient functions of Euler and Jordan, given by

$$\varphi(n) = n \prod_{p \mid n} (1 - p^{-1}),\ J_k(n) = n^k \prod_{p \mid n} (1 - p^{-k}) .$$

We also have the Euler product representation

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right) . \tag{4}$$

2. **Lemmas.** The proof of Theorem 1 is based on a number of lemmas.

LEMMA 1. *If $k \geqq 1$ we have $\mu_k(n^k) = \mu(n)$.*

LEMMA 2. *Each function μ_k is multiplicative. That is,*

$$\mu_k(mn) = \mu_k(m)\mu_k(n) \quad \textit{whenever} \quad (m, n) = 1 .$$

LEMMA 3. *Let f and g be multiplicative arithmetical functions and let a and b be positive integers, with $a \geqq b$. Then the function h defined by the equation*

$$h(n) = \sum_{d^a \mid n} f\left(\frac{n}{d^a}\right) g\left(\frac{n}{d^b}\right)$$

is also multiplicative. (The sum is extended over those divisors d of n for which d^a divides n.)

The first two lemmas follow easily from the definition of the function μ_k. The proof of Lemma 3 is a straightforward exercise.
The next lemma relates μ_k to μ_{k-1}.

LEMMA 4. *If $k \geqq 2$ we have*

$$\mu_k(n) = \sum_{d^k \mid n} \mu_{k-1}\left(\frac{n}{d^k}\right) \mu_{k-1}\left(\frac{n}{d}\right) .$$

Proof. By Lemmas 2 and 3, the sum on the right is a multiplicative function of n. To complete the proof we simply verify that the sum agrees with $\mu_k(n)$ when n is a prime power.

LEMMA 5. *If $k \geqq 1$ we have*

$$| \mu_k(n) | = \sum_{d^{k+1} \mid n} \mu(d) .$$

Proof. Again we note that both members are multiplicative functions of n which agree when n is a prime power.

LEMMA 6. *If $k \geqq 2$ and $r \geqq 1$, let*

$$F_r(x) = \sum_{n \leqq x} \mu_{k-1}(n)\mu_{k-1}(r^{k-1}n) .$$

Then we have the asymptotic formula

$$F_r(x) = \frac{x}{\zeta(k)} \frac{\mu(r)\varphi(r)r^{k-1}}{J_k(r)} + O(x^{1/k}\sigma_{-s}(r)) ,$$

where $\sigma_\alpha(r)$ is the sum of the αth powers of the divisors of r, and s is any number satisfying $0 < s < 1/k$. (The constant implied by the O-symbol is independent of r.)

Proof. In the sum defining $F_r(x)$ the factor $\mu_{k-1}(r^{k-1}n) = 0$ if r and n have a prime factor in common. Therefore we need consider only those n relatively prime to r. But if $(r, n) = 1$ the multiplicative property of μ_{k-1} gives us

$$\mu_{k-1}(n)\mu_{k-1}(r^{k-1}n) = \mu_{k-1}(n)^2\mu_{k-1}(r^{k-1}) = | \mu_{k-1}(n) | \mu(r) ,$$

where in the last step we used Lemma 1. Therefore we have

$$F_r(x) = \mu(r) \sum_{\substack{n \leqq x \\ (n,r)=1}} | \mu_{k-1}(n) | .$$

Using Lemma 5 we rewrite this in the form

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{\substack{n \leqq x \\ (n,r)=1}} \sum_{d^k \mid n} \mu(d) = \mu(r) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d) \sum_{\substack{q \leqq x/d^k \\ (q,r)=1}} 1 \\ &= \mu(r) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d) \sum_{t \mid r} \mu(t)\left[\frac{x}{td^k}\right] \\ &= \mu(r) \sum_{t \mid r} \mu(t) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d)\left[\frac{x}{td^k}\right] . \end{aligned}$$

At this point we use the relation $[x] = x + O(x^s)$, valid for any fixed s satisfying $0 \leqq s < 1$, to obtain

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{t \mid r} \mu(t) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d)\left\{\frac{x}{td^k} + O\left(\frac{x^s}{t^s d^{ks}}\right)\right\} \\ &= x\mu(r) \sum_{t \mid r} \frac{\mu(t)}{t} \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \frac{\mu(d)}{d^k} + O\left(x^s \sum_{t \mid r} \frac{1}{t^s} \sum_{d \leqq x^{1/k}} \frac{1}{d^{ks}}\right) . \end{aligned}$$

If we choose s so that $0 < ks < 1$ we have

$$\sum_{d\leqq x^{1/k}} \frac{1}{d^{ks}} = O\left(\int_1^{x^{1/k}} \frac{dt}{t^{ks}}\right) = O(x^{-s+1/k}) ,$$

and the O-term in the last formula for $F_r(x)$ is $O(x^{1/k}\sigma_{-s}(r))$. To complete the proof of Lemma 6 we use the relations

$$\sum_{t\mid r} \frac{\mu(t)}{t} = \frac{\varphi(r)}{r}$$

and

$$\begin{aligned}\sum_{\substack{d^k\leqq x\\(d,r)=1}} \frac{\mu(d)}{d^k} &= \sum_{\substack{d=1\\(d,r)=1}}^{\infty} \frac{\mu(d)}{d^k} + O\Big(\sum_{d>x^{1/k}} d^{-k}\Big)\\ &= \frac{1}{\zeta(k)}\prod_{p\mid r}\frac{1}{1-p^{-k}} + O(x^{(1-k)/k})\\ &= \frac{1}{\zeta(k)}\frac{r^k}{J_k(r)} + O(x^{(1-k)/k}) .\end{aligned}$$

3. Proof of Theorem 1. In the sum defining $M_k(x)$ we use Lemma 4 to write

$$\begin{aligned}M_k(x) &= \sum_{n\leqq x}\mu_k(n) = \sum_{n\leqq x}\sum_{d^k\mid n}\mu_{k-1}\Big(\frac{n}{d^k}\Big)\mu_{k-1}\Big(\frac{n}{d}\Big)\\ &= \sum_{d^k\leqq x}\sum_{m\leqq x/d^k}\mu_{k-1}(m)\mu_{k-1}(d^{k-1}m)\\ &= \sum_{d^k\leqq x} F_d(x/d^k) = \sum_{r\leqq x^{1/k}} F_r(x/r^k) .\end{aligned}$$

Using Lemma 6 we obtain

$$M_k(x) = \frac{x}{\zeta(k)}\sum_{r\leqq x^{1/k}}\frac{\mu(r)\varphi(r)}{rJ_k(r)} + O\Big(x^{1/k}\sum_{r\leqq x^{1/k}}\frac{\sigma_{-s}(r)}{r}\Big) . \tag{5}$$

The sum in the first term is equal to

$$\begin{aligned}\sum_{r\leqq x^{1/k}}\frac{\mu(r)}{r^k}\prod_{p\mid r}\frac{1-p^{-1}}{1-p^{-k}} &= \sum_{r=1}^{\infty}\frac{\mu(r)}{r^k}\prod_{p\mid r}\frac{1-p^{-1}}{1-p^{-k}} + O\Big(\sum_{r>x^{1/k}}\frac{1}{r^k}\Big)\\ &= \sum_{r=1}^{\infty}\frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{(1-k)/k}) .\end{aligned}$$

The sum in the O-term in (5) is equal to

$$\begin{aligned}\sum_{r\leqq x^{1/k}}\frac{\sigma_{-s}(r)}{r} &= \sum_{r\leqq x^{1/k}} r^{-1}\sum_{d\delta=r} d^{-s} = \sum_{\delta\leqq x^{1/k}}\delta^{-1}\sum_{d\leqq x^{1/k}/\delta} d^{-1-s}\\ &= O\Big(\sum_{\delta\leqq x^{1/k}}\delta^{-1}\Big) = O(\log x) .\end{aligned}$$

Therefore (5) becomes

$$M_k(x) = \frac{x}{\zeta(k)} \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{1/k}\log x) ,$$

which completes the proof of Theorem 1.

To deduce (4) from (2) we note that (2) has the form

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} f(n)$$

where $f(n)$ is multiplicative and $f(p^a) = 0$ for $a \geqq 2$. Hence we have the Euler product decomposition: [see 3, Th. 286]

$$A_k = \frac{1}{\zeta(k)} \prod_p \{1 + f(p)\} = \prod_p (1 - p^{-k}) \prod_p \left\{1 - \frac{1}{p^k}\frac{1 - p^{-1}}{1 - p^{-k}}\right\}$$
$$= \prod_p \left\{1 - p^{-k} - \frac{1 - p^{-1}}{p^k}\right\} = \prod_p \left\{1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right\} .$$

4. **Relations to k-free integers.** Let Q_k denote the set of k-free integers (positive integers whose prime factors are all of multiplicity less than k), and let q_k denote the characteristic function of Q_k:

$$q_k(n) = \begin{cases} 1 & \text{if} \quad n \in Q_k, \\ 0 & \text{otherwise.} \end{cases}$$

Gegenbauer [2, p. 47] has proved that the number of k-free integers $\leqq x$ is given by

$$(6) \qquad \sum_{n \leqq x} q_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k}) , \qquad (k \geqq 2) .$$

From the definition of μ_k it follows that $q_{k+1}(n) = |\mu_k(n)|$, so Gegenbauer's theorem implies the asymptotic formula

$$(7) \qquad \sum_{n \leqq x} |\mu_k(n)| = \frac{x}{\zeta(k+1)} + O(x^{1/(k+1)}) , \qquad (k \geqq 1) .$$

From our Theorem 1 we have

$$(8) \qquad \sum_{n \leqq x} \mu_k(n) = A_k x + O(x^{1/k}\log x) \qquad (k > 1) .$$

The two formulas (7) and (8) show that among the $(k + 1)$-free integers, $k > 1$, those for which $\mu_k(n) = 1$ occur asymptotically more frequently than those for which $\mu_k(n) = -1$; in particular, these two sets of integers have, respectively, the densities

$$\frac{1}{2}\left(\frac{1}{\zeta(k+1)} + A_k\right) \quad \text{and} \quad \frac{1}{2}\left(\frac{1}{\zeta(k+1)} - A_k\right) .$$

This is in contrast to the case $k = 1$ for which it is known that

$$\sum_{n\leqq x} |\mu(n)| = \frac{x}{\zeta(2)} + O(x^{1/2}), \quad \text{but} \quad \sum_{n\leqq x} \mu(n) = o(x) ,$$

so the square-free integers with $\mu(n) = 1$ occur with the same asymptotic frequency as those with $\mu(n) = -1$ [see 3, p. 270].

Our Theorem 1 can also be derived very simply from an asymptotic formula of Cohen [1, Th. 4.2]. Following the notation of Cohen, let Q_k^* denote the set of positive integers n with the property that the multiplicity of each prime divisor of n is not a multiple of k. Let q_k^* denote the characteristic function of Q_k^*. Then $q_k^*(1) = 1$, and for $n > 1$ we have

$$q_k^*(n) = \begin{cases} 1 & \text{if} \quad n = \prod_{i=1}^{r} p_i^{a_i}, \quad \text{with each} \quad a_i \not\equiv 0 \pmod{k} , \\ 0 & \text{otherwise.} \end{cases}$$

The functions q_k^* and μ_k are related by the following identity:

$$q_k^*(n) = \sum_{d^k \mid n} \mu_k\left(\frac{n}{d^k}\right) . \tag{9}$$

This is easily verified by noting that both members are multiplicative functions of n that agree when n is a prime power, or by equating coefficients in the Dirichlet series identity (14) given below in § 5. Inversion of (9) gives us

$$\mu_k(n) = \sum_{d^k \mid n} \mu(d) q_k^*\left(\frac{n}{d^k}\right) . \tag{10}$$

Cohen's asymptotic formula states that for $k \geqq 2$ we have

$$\sum_{n\leqq x} q_k^*(n) = A_k \zeta(k) x + O(x^{1/k}) , \tag{11}$$

where A_k is the same constant that appears in our Theorem 1. To deduce Theorem 1 from (11) we use (10) to obtain

$$\begin{aligned} \sum_{n\leqq x} \mu_k(n) &= \sum_{n\leqq x} \sum_{d^k \mid n} \mu(d)\, q_k^*\left(\frac{n}{d^k}\right) = \sum_{d^k\leqq x} \mu(d) \sum_{m\leqq x/d^k} q_k^*(m) \\ &= \sum_{d^k\leqq x} \mu(d)\left\{A_k \zeta(k) \frac{x}{d^k} + O\left(\frac{x^{1/k}}{d}\right)\right\} \\ &= A_k \zeta(k) x \sum_{d\leqq x^{1/k}} \frac{\mu(d)}{d^k} + O\left(x^{1/k} \sum_{d^k\leqq x} \frac{1}{d}\right) \\ &= A_k \zeta(k) x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} + O\left(\sum_{d > x^{1/k}} d^{-k}\right) + O(x^{1/k} \log x) \\ &= A_k x + O(x^{1/k} \log x) . \end{aligned}$$

Conversely, if we start with equation (9) and use Theorem 1 we can deduce Cohen's asymptotic formula (11) but with an error term $O(x^{1/k}\log x)$ in place of $O(x^{1/k})$.

5. Generating functions. The generating function for the k-free integers is known to be given by the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{q_k(n)}{n^s}=\frac{\zeta(s)}{\zeta(ks)}\qquad(s>1) \tag{12}$$

[see 3, Th. 303, p. 255]. It is not difficult to determine the generating functions for the functions μ_k and q_k^* as well. Straightforward calculations with Euler products show that we have

$$\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\zeta(s)\prod_p\left\{1-\frac{2}{p^{ks}}+\frac{1}{p^{(k+1)s}}\right\} \tag{13}$$

and

$$\sum_{n=1}^{\infty}\frac{q_k^*(n)}{n^s}=\zeta(ks)\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s} \tag{14}$$

for $s>1$. Equation (14) is also equivalent to equations (9) and (10). From (12) and (14) we obtain the following identity relating μ_k, q_k, and q_k^*:

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\left(\sum_{n=1}^{\infty}\frac{q_k(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{q_k^*(n)}{n^s}\right).$$

This shows [see 3, §17.1] that the numerical integral of μ_k is the Dirichlet convolution of q_k and q_k^*:

$$\sum_{d\mid n}\mu_k(d)=\sum_{d\mid n}q_k(d)q_k^*\left(\frac{n}{d}\right).$$

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Received April 11, 1969.

CALIFORNIA INSTITUTE OF TECHNOLOGY

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

ON AN INITIAL VALUE PROBLEM IN THE THEORY OF TWO-DIMENSIONAL TRANSONIC FLOW PATTERNS

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In the case of the differential equation

$$\mathbf{L}(\psi) = \frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\theta^2} + N\frac{\partial\psi}{\partial\lambda} = 0, N \equiv N(\lambda, \theta) ,$$

where N is an analytic function, the integral operator of the first kind

$$\mathbf{P}(f) \overset{\text{def}}{=} \int_{t=-1}^{1} E(\lambda, \theta, t)f(\zeta(1-t^2)/2)dt/\sqrt{1-t^2}$$

transforms analytic functions of a complex variable $\zeta = \lambda + i\theta$ into solutions of $\mathbf{L}(\psi) = 0$. Here E is a fixed function which depends only on L, while $f(\zeta)$ is an arbitrary analytic function of the complex variable ζ; f is assumed to be regular at $\zeta = 0$. Using this operator, one shows that many theorems valid for analytic functions of the complex variable can be generalized for the solutions ψ of $\mathbf{L}(\psi) = 0$. Continuing $\psi(\lambda, \theta)$ to complex values $U = \lambda + i\Lambda$ and setting $\lambda = 0$, one shows that many theorems in the theorems in the theory of functions of a real variable can be generalized to the case of solutions of

$$\mathbf{H}(\psi) \equiv -\frac{\partial^2\psi}{\partial\Lambda^2} + \frac{\partial^2\psi}{\partial\theta^2} - iN\frac{\partial\psi}{\partial\Lambda} = 0 .$$

By change of the variables,

$$\mathbf{M}(\psi) \equiv \frac{\partial^2\psi}{\partial x^2} + l(x)\frac{\partial^2\psi}{\partial y^2} = 0 ,$$

$l(x) > 0$ for $x < 0$, $l(x) < 0$ for $x > 0$, $l(0) = 0$, when considered for $x < 0$ can be reduced to the equation $\mathbf{L}(\psi) = 0$. The variables can be chosen so that $U = 0$ corresponds to $x = 0$. However, in this case the function $N(\lambda)$ becomes singular at $\lambda = 0$. Nevertheless, one can apply the theory of the so-called integral operators of the second kind. If $\psi(0, \theta) = \chi_1(\theta)$ and

$$\lim_{M\to 1^-} \psi_M(M, \theta) = \chi_2(\theta)$$

are given, one can determine the function f. Here M is the Mach number. In this way one can determine from χ_1 and χ_2 the location and character of singularities of ψ in the subsonic region. When considering ψ in the supersonic region, one can show that some theorems on functions of one real variable can be generalized to the case of certain sets of particular solutions $\psi_\nu(\Lambda, \theta)$, $\nu = 1, 2, \cdots$, of $\mathbf{H}(\psi) = 0$.

Suppose the streamfunction ψ of a transonic two-dimensional compressible fluid flow is given by the values of ψ and of ψ_M on a segment of the sonic line. Here ψ_M is the derivative with respect to the Mach number M.

One of the problems which arises is to determine the regularity domain, say $\mathscr{R}$, and the location and properties of the singularities of ψ in the subsonic region. Finally, it is of interest to determine ψ in a given domain $\mathscr{D}$, $\mathscr{D} \subset \mathscr{R}$. This problem complex will be called *the initial value problem in the large.*

ψ, when considered in the physical plane is a solution of a nonlinear partial differential equation. However, by introducing conveniently chosen new variables (instead of the coordinates x, y of the physical plane), we obtain for ψ a linear partial differential equation (see Chaplygin [12] and Molenbroek [23]).

The linear equation which we obtain in this way, see (1.4), is of mixed type. However, it is possible to use the theory of integral operators in the study of the behavior of ψ in the *subsonic* region.

The theory of integral operators investigates the solutions of linear partial differentiation equations of the form

$$\Delta\psi + \sum_{\nu=1}^{n} a_\nu \frac{\partial\psi}{\partial x_\nu} + a_{n+1}\psi = 0 . \tag{1.1}$$

$\Delta = \sum_{\nu=1}^{n} \partial^2/\partial x_\nu^2$ is the Laplace differential operator and a_ν are analytic functions of $x_1, \cdots, x_n$ regular in a sufficiently large domain.[1] Suppose the solution $\psi(x_1, \cdots, x_n)$ is given in the small, say in the neighborhood of the origin in the form of a series development

$$\psi(x_1, \cdots, x_n) = \sum_{\nu_1,\nu_2,\cdots,\nu_n=0}^{\infty} a_{\nu_1\cdots\nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n} . \tag{1.2}$$

Then this approach reduces the study whether ψ is regular in a domain $\mathscr{D}$ to the investigation whether or not an analytic function $f(Z_1, \cdots, Z_m)$ of m complex variables $Z_k = x_k + iy_k$, $k = 1, 2, \cdots, m$, given by its power series development

$$f(Z_1, \cdots, Z_m) = \sum_{\nu_1,\nu_2,\cdots,\nu_m=0}^{\infty} A_{\nu_1\cdots\nu_m} Z_1^{\nu_1} \cdots Z_m^{\nu_m} \tag{1.3}$$

is regular in a domain $\mathscr{D}$. (See [1], [2], [9], [13], [20], [15], [16], [17].)

In the case of one variable, i.e., if $f(Z) = \sum A_\nu Z^\nu$ is given, two methods can be used to determine the regularity domain and the

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[1] In the case of differential equations of the form (1.1) and for $n = 2$ one uses integral operators of the *first* kind, see [**9**], pp. 9-27. If the coefficient $4F$ of the equation (see (1.9)) has the singularity indicated in (1.9a), we use the integral operator of the *second* kind (see [**5**], p. 869, and [**6**], p. 452 ff.).

location and character of the singularities of f from given A_ν, $\nu = 1, 2, \cdots$. (I) the Hadamard-Polya-Mandelbrojt approach, (II) the theory of Hilbert spaces possessing a kernel function. Two possibilities should be mentioned proceeding along the lines of (II): (a) the use of functions which are simultaneously orthogonal in two domains $\mathscr{B}$ and $\mathscr{D}$, $\bar{\mathscr{B}} \subset \mathscr{D}$, see [8], (b) some results by Schiffer, Siciak and the author which give conditions for the coefficients A_ν in order that an analytic function given by its series development $\sum_{\nu=1}^{\infty} A_\nu Z^\nu$ is regular and square integrable in a given domain $\mathscr{D}$ (possessing a kernel function), see [11], [30].

The streamfunction ψ of a two-dimensional compressible fluid flow satisfies an equation of mixed type, namely

$$\mathbf{M}(\psi) \equiv \frac{\partial^2 \psi}{\partial H^2} + l(H)\frac{\partial^2 \psi}{\partial \theta^2} = 0 , \qquad l(0) = 0 , \tag{1.4}$$

where $l(H)$ is an analytic function of H, which is real for real H and such that

$$l(H) > 0 \qquad \text{for } H < 0 , \tag{1.5a}$$

$$l(H) < 0 \qquad \text{for } H > 0 . \tag{1.5b}$$

$l(H)$ is supposed to be regular in a sufficiently large domain including $H = 0$. Further we assume that, if we reduce (1.4) to the normal form (1.7), $l(H)$ is chosen in such a way that N considered as a function of λ, see (1.6), has a development of the type indicated in (1.7a). The study of (1.4) can be reduced to the study of the equation (1.1) with *singular* coefficient a_{n+1}. By the transformation

$$-\lambda = \int_{\tau=0}^{-H} [l(-\tau)]^{1/2} d\tau , \tag{1.6}$$

the equation (1.4) in the region $H < 0$ is transformed into

$$\mathbf{L}(\psi) \equiv \psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N\psi_\lambda = 0 , \tag{1.7}$$

$$N = -\frac{1}{8} l^{-3/2} l_H = \frac{1}{12\lambda}[1 + \beta_1(-\lambda)^{2/3} + \cdots], \beta_1 > 0, \lambda < 0 , \tag{1.7a}$$

(See (2.4), p. 860 of [5].)

Introducing

$$\psi^* = \frac{\psi}{H^*} , \tag{1.8}$$

$$\begin{aligned} H^* &= \exp\left[-\int_{-\infty}^{2\lambda} 2N(\tau)d\tau\right] \\ &= S_0(-2\lambda)^{-1/6}[1 + S_1(-2\lambda)^{2/3} + S_2(-2\lambda)^{4/3} + \cdots] , \end{aligned} \tag{1.8a}$$

(1.7) becomes

$$\mathbf{L}^*(\psi^*) \equiv \psi^*_{\lambda\lambda} + \psi^*_{\theta\theta} + 4F\psi^* = 0 , \tag{1.9}$$

$$F = \frac{5}{144}(-\lambda)^{-2} + \tilde{A}_{-2}(-\lambda)^{-2/3} + \tilde{A}_0 + \tilde{A}_2(-\lambda)^{2/3} + \cdots , \tag{1.9a}$$

see [4], [5], [6], [9, p. 106 ff.]

In the next section we shall discuss an integral representation for the solution ψ of (1.7) in terms of a function of one variable.

In the previous papers [5], [6] the conditions for the associate $f(Z) = Z^{1/6} \sum c_\nu Z^\nu$, $Z = \lambda + i\theta$, in order that ψ satisfies the relations (4.2), (4.3) on the segment of the sonic line, have been determined. However, the proof which shows that the relation (4.1′) is a sufficient condition that ψ satisfies (4.2) and (4.3) can be simplified. (see §§ 3 and 4 of this paper.)

REMARK. Formula (7.15) of [6] has been obtained in replacing c_ν by (4.1a) of the present paper and applying some further transformations. It should be noted that in formula (7.15) of [6] (as well as in (4) of [9], p. 121) $J_\nu^{(\kappa)}$ should be replaced by

$$J_\nu^{(\kappa)} \exp\left[-\frac{\pi(6-2\kappa)i}{3}\right], \kappa = 1, 2 .$$

A representation of ψ in the supersonic region is derived in § 7 by the use of integral operators.

2. An integral representation for the analytic solution of (1.4) in terms of functions of several complex variables. In this section we shall derive an integral representation for the solution ψ of equation (1.4). This representation is valid in a subdomain of the subsonic region.

DEFINITION. $\mathscr{E}(\mathscr{B}) = \bigcup_{Z \in \mathscr{B}} \mathscr{K}(Z)$ where $\mathscr{K}(Z) = \{\zeta \mid |\zeta - Z| \leq \left|\frac{Z}{2}\right|\}$.

In the following we assume that $\mathscr{B}$ is a stardomain with respect to the origin.

THEOREM 2.1. *Let $E(Z, Z^*, t)$ be a function of three complex variables Z, Z^*, t, $Z = \lambda + i\theta$, $Z^* = \lambda - i\theta$, which is defined for $t \in N(\mathscr{C})$ and $(Z, Z^*) \in \mathscr{G}$. Here $N(\mathscr{C})$ is a domain which includes the rectifiable (oriented) curve $\mathscr{C}$, with initial point $t = 1$ and end point $t = -1$, and $\mathscr{G}$ denotes a sufficiently small neighborhood of the origin $O = [Z = Z^* = 0]$. $\mathscr{C}$ is a curve of the complex t-plane, namely*

$$\mathscr{C} = \{|t| = 1\} . \tag{2.1}$$

We assume that E satisfies the following conditions:

(1) *E possesses continuous partial derivatives with respect to all three of its arguments up the second order for $(Z, Z^*, t) \in \mathscr{G} \times N(\mathscr{C})$.*

(2) *E satisfies the partial differential equation*

$$(2.2) \qquad (1 - t^2)(E_{Z^*t} + NE_t) - t^{-1}E_{Z^*} + 2tZ\mathbf{L}(E) = 0 ,$$

concerning $\mathbf{L}$ *see* (1.7).

If $f(\zeta) = (\zeta)^{1/6}p(\zeta/2)$, where $p(\zeta/2)$ is an analytic function of ζ which is defined in a simply connected domain $\mathscr{P}$, $\mathscr{P} \supset \mathscr{E}(\mathscr{B})$, then

$$(2.3) \qquad \psi(\lambda, \theta) = \mathbf{P}_2(f) \equiv \mathrm{Im} \int_{\mathscr{C}} E(Z, Z^*, t) f\left(\frac{1}{2}Z(1 - t^2)\right)\frac{dt}{\sqrt{1 - t^2}}$$

$$\mathrm{Im} = \textit{imaginary part} , \qquad Z = \lambda + i\theta, Z^* = \lambda - i\theta ,$$

is a solution of $\mathbf{L}(\psi) = 0$.

The function ψ is defined in $\mathscr{W} \cap \mathscr{B}$,

$$(2.4) \qquad \mathscr{W} = \{(\lambda, \theta) \mid 3\lambda^2 < \theta^2, \theta > 0, -s_0^{3/2} < \lambda < 0\} .$$

The proof of the above theorem is given in [5, p. 878 ff.] and [6] see also [1] and [9], Chapters I and V].

DEFINITION. $E(Z, Z^*, t)$ and f are denoted as *the generating and associate functions*, respectively, of the integral operator $\mathbf{P}_2$.

After certain auxiliary lemmas are obtained in § 3, we shall prove Theorem 4.1. The latter theorem will enable us to solve the problem mentioned in the introduction.

3. Auxiliary lemma. In this section we shall at first evaluate certain integrals which we shall need in § 4.

LEMMA 3.1.[2]

$$(3.1) \qquad \begin{aligned} I_\nu^{(1)} &\equiv \int_{\mathscr{C}} t^{-1/3}(1 - t^2)^{\nu+1/6}\frac{dt}{\sqrt{1 - t^2}} \\ &= -\frac{1}{2}(1 - e^{2\pi i/3})\frac{\Gamma(1/3)\Gamma(\nu + 2/3)}{\Gamma(\nu + 1)} , \end{aligned}$$

$$(3.2) \qquad \begin{aligned} I_\nu^{(2)} &\equiv \int_{\mathscr{C}} t^{-5/3}(1 - t^2)^{\nu+5/6}\frac{dt}{\sqrt{1 - t^2}} \\ &= \frac{1}{2}(1 - e^{-2\pi i/3})\frac{\Gamma(-1/3)\Gamma(\nu + 4/3)}{\Gamma(\nu + 1)} . \end{aligned}$$

[2] The counterclockwise orientation of $\mathscr{C}$ yields the negative sign in (3.1).

In accordance with our assumptions, $\mathscr{C}$ is a rectifiable (oriented) curve connecting the points 1 and -1 and lying in $[|t| \geqq 1]$.

Proof. Applying Cauchy's theorem to the integral of (3.1), we can reduce the curve $\mathscr{C}$ to the segment $(1, -1)$ of the real t-axis.[3] Thus

$$I_\nu^{(1)} = I_\nu^{(11)} + I_\nu^{(12)} ,$$
$$(3.3)\qquad I_\nu^{(11)} = \int_1^0 t^{-1/3}(1 - t^2)^{\nu+1/6}\frac{dt}{\sqrt{1-t^2}} ,$$
$$I_\nu^{(12)} = \int_0^{-1} t^{-1/3}(1 - t^2)^{\nu+1/6}\frac{dt}{\sqrt{1-t^2}} .$$

Introducing $\tau = t^2$, we obtain

$$(3.4)\qquad I_\nu^{(11)} = -\frac{1}{2}\int_0^1 \tau^{-2/3}(1-\tau)^{\nu-1/3}d\tau = -\frac{1}{2}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\nu + \frac{2}{3}\right)}{\Gamma(\nu+1)} .$$

When considering $I_\nu^{(12)}$, we note that for $-1 < t < 0$, $t = re^{i\pi}$, $r > 0$, and therefore

$$(3.5)\qquad I_\nu^{(12)} = e^{-4\pi i/3}\int_0^1 r^{-1/3}(1-r^2)^{\nu-1/3}dr = \frac{1}{2}e^{-4\pi i/3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\nu + \frac{2}{3}\right)}{\Gamma(\nu+1)} .$$

Thus (3.1) is obtained. (Γ = Gamma function.)

When evaluating (3.2), we assume at first that $t = 0$ does not belong to the integration curve denoted by $\mathscr{C}$. Integrating by parts yields

$$(3.6)\qquad \begin{aligned} I_\nu^{(2)} &= -\frac{3}{2}\int_{\mathscr{C}} (1 - t^2)^{\nu+1/3}d(t^{-2/3}) \\ &= -\left(\frac{3}{2}\right)[t^{-2/3}(1-t^2)^{\nu+1/3}]_{t=1}^{t=-1} - 3\left(\nu + \frac{1}{3}\right)\int_{\mathscr{C}} t^{1/3}(1-t^2)^{\nu-2/3}dt . \end{aligned}$$

The first term on the right-hand side of (3.6) vanishes. In the second term we replace $\mathscr{C}$ by the segment $(1, -1)$ of the real t-axis. Introducing $\tau = t^2$, we obtain

$$(3.7)\qquad \begin{aligned} -3\left(\nu+\frac{1}{3}\right)\int_1^0 t^{-1/3}(1-t^2)^{\nu+1/3}dt &= \frac{3}{2}\left(\nu + \frac{1}{3}\right)\int_0^1 \tau^{-2/3}(1-\tau)^{\nu+1/3}d\tau \\ &= \frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\nu+\frac{4}{3}\right)}{\Gamma(\nu+1)} . \end{aligned}$$

[3] We replace $\mathscr{C}$ at first by the sum of segments $[1 > \operatorname{Re} t > \varepsilon]$, $[t = \varepsilon e^{i\theta}, 0 < \theta < \pi]$, $[-\varepsilon > \operatorname{Re} t > -1]$, $\varepsilon > 0$. Then we consider the limit of the integrals for $\varepsilon \to 0$.

When integrating from 0 to -1, we introduce $t = re^{i\pi}$ and thus obtain

$$I_\nu^{(2)} = \frac{1}{2}(1 - e^{-2\pi i/3})\frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\nu + \frac{4}{3}\right)}{\Gamma(\nu + 1)} . \tag{3.8}$$

The generating function E yielding the representation (2.3) has been determined in [5], [6], [7], [10]. In particular, it has been shown that two functions

$$E^{(k)} = H^*E^{*(k)},\ E^{*(k)} = \sum_{n=0}^{\infty} \frac{q^{(n,k)}(\lambda)}{(-t^2Z)^{n-(1/2)+(2/3)k}},\ k = 1, 2\ , \tag{3.9}$$

$$q^{(n,k)}(\lambda) = \sum_{\nu=0}^{\infty} C_\nu^{(n,k)}(-\lambda)^{n-(1/2)+(2/3)(k+\nu)},\ C_\nu^{(n,k)} = \text{const.}, \quad C_0^{(01)} = 2^{1/6},\ C_0^{(02)} = 2^{5/6}, \tag{3.10}$$

(see (1.8a) and [6], p. 453) are solutions of (2.2) for $(\lambda, \theta) \in \mathscr{W}$ (see (2.4)). Let ψ be equal to the right-hand side of (2.3) where

$$E = \tilde{E}(\lambda, \theta, t) \overset{\text{def}}{\equiv} A_1E^{(1)} + A_2\left[Z\frac{(1 - t^2)}{2}\right]^{2/3} A_2E^{(2)}\ , \tag{3.11}$$

then obviously $\mathbf{L}(\psi) = 0$. Here A_1 and A_2 are two complex numbers such that

$$\operatorname{Im}(\bar{A}_1A_2) \neq 0\ . \tag{3.12}$$

In the following considerations we need Lemma 3.2 yielding the limit relations for the generating function $\tilde{E}$ introduced in (3.11).

LEMMA 3.2.

$$\lim_{\lambda\to 0^-} \tilde{E}(\lambda, \theta, t) = -d_2t^{-1/3}\theta^{-1/6}\ , \tag{3.13}$$

$$\lim_{\lambda\to 0^-} (-\lambda)^{1/3}\tilde{E}_\lambda(\lambda, \theta, t) = [d_1t^{-1/3} + d_0t^{-5/3}(1 - t^2)^{2/3}]\theta^{-1/6} \tag{3.14}$$

where

$$d_0 = -\frac{2}{3}i^{3/2}S_0A_2,\ d_1 = -\frac{2^{5/3}}{3}i^{1/6}S_0S_1A_1,\ d_2 = -i^{1/6}S_0A_1\ , \tag{3.15}$$

S_0, S_1 *positive.*

Proof. By (1.8), (3.9),

$$\begin{aligned} A_1E^{(1)} &= A_1[S_0(-2\lambda)^{-1/6} + S_0S_1(-2\lambda)^{1/2} + \cdots]\left[\frac{q^{(0,1)}(\lambda)}{(-t^2Z)^{1/6}} + \cdots\right] \\ &= \frac{A_1S_0C_0^{(01)}2^{-1/6} + A_1S_0S_1C_0^{(01)}2^{1/2}(-\lambda)^{2/3} + \cdots}{(-t^2(\lambda + i\theta))^{1/6}}\ , \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
A_2E^{(2)} &= A_2(S_0(-2\lambda)^{-1/6} + S_0S_1(-2\lambda)^{1/2} + \cdots]\left[\frac{q^{(0,2)}(\lambda)}{(-t^2Z)^{5/6}} + \cdots\right] \\
&= \frac{A_2C_0^{(02)}S_0 2^{-1/6}(-\lambda)^{2/3} + A_2S_0S_1C_0^{(02)}2^{1/2}(-\lambda)^{8/6} + \cdots}{[-t^2(\lambda + i\theta)]^{5/6}} + \cdots ,
\end{aligned}
\tag{3.17}
$$

$$
\left[\frac{1}{2}Z(1 - t^2)\right]^{2/3} A_2E^{(2)} = \frac{A_2C_0^{(02)}S_0 2^{-5/6}(-\lambda)^{2/3}(1-t^2)^{2/3}(\lambda+i\theta)^{2/3} + \cdots}{[-t^2(\lambda + i\theta)]^{5/6}} . \tag{3.18}
$$

Thus

$$
\begin{aligned}
\tilde{E} &= \frac{A_1S_0C_0^{(01)}2^{-1/6} + A_1S_0S_1C_0^{(01)}2^{1/2}(-\lambda)^{2/3} + \cdots}{[-t^2(\lambda + i\theta)]^{1/6}} \\
&\quad + \frac{\left[\frac{1}{2}Z(1 - t^2)\right]^{2/3} A_2C_0^{(02)}S_0 2^{-1/6}(-\lambda)^{2/3} + \cdots}{[-t^2(\lambda + i\theta)]^{5/6}} + \cdots ,
\end{aligned}
\tag{3.19}
$$

$C_\nu^{(0,k)}$ have been introduced in (3.10). (see also [6] p. 453.) From (3.19) the limit relations (3.13) and (3.14) follow. The justifications of the above operations follow from considerations in [5], p. 882, and [10], p. 336. To derive (3.14), we note that after differentiation of (3.18) with respect to λ, the only terms of $\tilde{E}_\lambda$ which contribute to $\lim_{\lambda\to 0^-} (-\lambda)^{1/3}\tilde{E}_\lambda$ are the coefficients of $(-\lambda)^{-1/3}$. Hence

$$
\begin{aligned}
\lim_{\lambda\to 0^-} (-\lambda)^{1/3}\tilde{E}_\lambda &= -\frac{2}{3} i^{3/2} S_0A_2 t^{-5/3}(1 - t^2)^{2/3}\theta^{-1/6} \\
&\quad - \frac{2^{5/3}}{3} i^{1/6} S_0S_1A_1 t^{-1/3}\theta^{-1/6}
\end{aligned}
\tag{3.20}
$$

which implies (3.14).

4. The determination of the associate f in (2.3) from given values of ψ and ψ_M on the sonic line.

THEOREM 4.1. *Let* $\chi_k(\theta) = \sum_{\nu=0}^{\infty} a_\nu^{(k)}\theta^\nu$, $k = 1, 2$, ($a_\nu^{(k)}$ real) *be two power series which converge uniformly for* $0 \leqq \theta \leqq \theta_1$, $\theta_1 > 0$. *Suppose further that* $\tilde{E}(Z, Z^*, t)$ *is given by* (3.11), *and let*

$$
f(Z) = Z^{1/6} \sum_{\nu=1}^{\infty} c_\nu Z^\nu , \tag{4.1}
$$

$$
c_\nu = (-2i)^{\nu+1/6} \frac{(\bar{d}_0\bar{I}_\nu^{(2)} + \bar{d}_1\bar{I}_\nu^{(1)})a_\nu^{(1)} + \bar{d}_2\bar{I}_\nu^{(1)}a_\nu^{(2)}}{\operatorname{Im}[d_0\bar{d}_2\bar{I}_\nu^{(1)}I_\nu^{(2)}]} , \tag{4.1'}
$$

where $f(Z)$ *is the associate function of the integral operator* $\mathbf{P}_2(f)$ *in* (2.3). *Then* ψ *given by* (2.3) *is a solution of* (1.7) *satisfying the conditions*

$$
\lim_{\lambda\to 0^-} \psi(\lambda, \theta) = \sum_{\nu=0}^{\infty} a_\nu^{(1)}\theta^\nu , \tag{4.2}
$$

$$\lim_{\lambda\to 0^-}(-\lambda)^{1/3}\psi_\lambda(\lambda,\theta)=\sum_{\nu=0}^{\infty}a_\nu^{(2)}\theta^\nu\ . \tag{4.3}$$

Proof. In order to prove the theorem, it is sufficient to show that

$$\psi_\nu(\lambda,\theta)=\operatorname{Im}\int_{\mathscr{C}}E(Z,Z^*,t)\Big(\frac{1}{2}Z(1-t^2)\Big)^{\nu+1/6}c_\nu\frac{dt}{\sqrt{1-t^2}} \tag{4.4}$$

satisfies the relations (4.2), (4.3) with the right-hand side replaced by $a_\nu^{(k)}\theta^\nu$, $k=1,2$, respectively. Using Lemma 3.2, we obtain the general term

$$\frac{-\{\bar d_0 d_2 I_\nu^{(1)}\bar I_\nu^{(2)}+\bar d_1 d_2 I_\nu^{(1)}\bar I_\nu^{(1)}\}a_\nu^{(1)}-d_2\bar d_2 I_\nu^{(1)}\bar I_\nu^{(1)}a_\nu^{(2)}}{\operatorname{Im}[d_0\bar d_2\bar I_\nu^{(1)}I_\nu^{(2)}]}\theta^\nu\ . \tag{4.5}$$

Since $\bar d_1 d_2$ is real, from (4.4), (3.12) and (4.5) we infer (4.2). It is easy to see that $\operatorname{Im}[d_0\bar d_1\bar I_\nu^{(1)}I_\nu^{(2)}]\neq 0$ in the case under consideration.

We now consider the second condition. We note that

$$\lim_{\lambda\to 0^-}(-\lambda)^{1/3}\frac{\partial\psi}{\partial\lambda}=\operatorname{Im}\int_{\mathscr{C}}\lim_{\lambda\to 0^-}(-\lambda)^{1/3}\tilde E_\lambda f\frac{dt}{\sqrt{1-t^2}} \tag{4.6}$$

since

$$\lim_{\lambda\to 0^-}(-\lambda)^{1/3}\tilde E f_\lambda=0\ . \tag{4.7}$$

Concerning the interchange of $\lim_{\lambda\to 0^-}$ and $\int_{\mathscr{C}}$ compare [10, pp. 336, 339], and [5, (5.28), p. 882 ff]. Using Lemmas 3.1 and 3.2, we obtain the general term

$$\begin{aligned}&\frac{[\bar d_0 d_1 I_\nu^{(1)}\bar I_\nu^{(2)}+|d_0^2|\,|I_\nu^{(2)}|^2+|d_1|^2\,|I_\nu^{(1)}|^2+d_0\bar d_1\bar I_\nu^{(1)}I_\nu^{(2)}]a_\nu^{(1)}}{\operatorname{Im}[d_0\bar d_2\bar I_\nu^{(1)}I_\nu^{(2)}]}\theta^\nu\\ &\qquad+\frac{[d_1\bar d_2\,|I_\nu^{(1)}|^2+d_0\bar d_2\bar I_\nu^{(1)}I_\nu^{(2)}]a_\nu^{(2)}}{\operatorname{Im}[d_0\bar d_2\bar I_\nu^{(1)}I_\nu^{(2)}]}\theta^\nu\ .\end{aligned} \tag{4.8}$$

Noting (4.6), we infer (4.3) from (4.8). This completes the proof of Theorem 4.1.

5. The conditions imposed on the coefficients $a_\nu^{(k)}$ in order that ψ has singularities of specific types. In § 4 we expresed the associate function $f(Z)$ in terms of values $a_\nu^{(k)}$, $k=1,2$, $\nu=0,1,2,\cdots$ (see (4.2) and (4.3)), which appear in the initial value problem considered here. Suppose $f(Z)$ is regular, say in some simply connected domain $\mathscr{D}$, $\mathscr{D}\subset\mathscr{W}$, the stream function ψ will be regular there. As mentioned in the introduction, there exist various procedures for the determination of the location of singularities of a function $Z^{1/6}f(Z)$ given by its series development $Z^{1/6}\sum c_\nu Z^\nu$ at the origin.

In the following we shall discuss a procedure using the approach indicated by (I), see [18], [21].

THEOREM 5.1. *Suppose that the solution* $\psi(\lambda, \theta)$ *is defined in a sufficiently small neighborhood of the origin and on a segment* $(\lambda = 0, -\theta_1 \leqq \theta \leqq \theta_1)$, $\theta_1 > 0$, *and satisfies the conditions* (4.2) and (4.3).

Here $\sum_{\nu=0}^{\infty} a_\nu^{(k)} \theta^\nu$ *are power series converging absolutely and uniformly for* $|\theta| \leqq \theta_1$. *Let*

$$(5.1) \qquad \frac{1}{\rho} = 2 \overline{\lim_{\nu\to\infty}} \left[\left| \frac{a_\nu^{(1)} \sum_{k=1}^{2} \bar{d}_{k-1} \bar{I}_\nu^{(3-k)} + a_\nu^{(2)} \bar{d}_2 I_\nu^{(1)}}{\operatorname{Im}[d_0 \bar{d}_2 \bar{I}_\nu^{(1)} I_\nu^{(2)}]} \right| \right]^{1/\nu} < \infty ,$$

where d_k, $I_\nu^{(k)}$, $k = 1, 2$, *have been introduced in* (3.15), (3.1), (3.2), *and let*

$$(5.2) \qquad \cos \varphi = \lim_{h\to 0^+} \frac{\sigma(h) - 1}{h}, \ \sigma(h) = \overline{\lim_{n\to\infty}} [|\boldsymbol{d}_n(h)|]^{1/n} ,$$

$$(5.3) \qquad \boldsymbol{d}_n(h) = \sum_{N=0}^{n} \frac{2C_n^N h^{n-N} (-2i)^{N+1/6} \left[a_N^{(1)} \sum_{k=1}^{2} \bar{I}_N^{(3-k)} \bar{d}_{k-1} + a_N^{(2)} I_N^{(2)} \right]}{\operatorname{Im}[d_0 \bar{d}_2 \bar{I}_N^{(1)} I_N^{(2)}]} \frac{1}{\rho} .$$

(C_n^N are binomial coefficients.) *Suppose*

$$(5.4) \qquad -\frac{1}{2} < \lim_{h\to 0^+} \frac{\sigma(h) - 1}{h} < 0 .$$

Let us denote by $\mathscr{D}^2$ *the domain*

$$(5.5) \qquad \{(\lambda, \theta) \mid 3^{1/2} |\lambda| < \theta, -s_0^{3/2} < \lambda \leqq 0, \lambda^2 + \theta^2 < \rho^2\} .$$

and let

$$(5.6) \qquad s^1 = \left\{ (\lambda, \theta) \mid \lambda = \rho \cos \varphi, \theta = \rho \sin \varphi, \right. \\ \left. \lim_{h\to 0^+} \frac{\sigma(h) - 1}{h} < \cos \varphi < 0, \lambda^2 + \theta^2 = \rho^2 \right\} .$$

Then $\psi(\lambda, \theta)$ *is regular in* $\mathscr{D}^2 \cup s^1$. *Concerning* s_0 *see* [5], p. 878.

Proof. Since $E(Z, Z^*, t)$ is regular in $\mathscr{W}$, see (2.4), the solution ψ is regular in every subdomain of $\mathscr{W}$ which does not include $\theta = 0$ in which $f(Z)/Z^{1/6}$ is regular. Here $f(Z) = Z^{1/6} \sum_{\nu=0}^{\infty} c_\nu Z^\nu$. By the theorems of Hadamard and Mandelbrojt, the function $g(Z) = \sum_{\nu=0}^{\infty} c_\nu Z^\nu$ is regular in the circle $|Z| < \rho, \rho = 1/\overline{\lim}_{\nu\to\infty} |c_\nu|^{1/\nu} > 0$ and on the arc $[(0, \varphi), |Z| = \rho$ where

$$(5.7) \qquad \cos \varphi = \lim_{\sigma\to 0^+} \left\{ \frac{\sigma(h) - 1}{h} \right\}, \ \sigma(h) = \overline{\lim_{n\to\infty}} [|d_n(\lambda)|]^{1/n} .$$

6. The representation of ψ in a simply connected domain $\mathscr{D}$, $\mathscr{D} \subset \mathscr{W}$. As indicated in [2], [9], the integral operators enable us to translate many theorems in the theory of analytic functions of complex variables into theorems on functions ψ satisfying a linear partial differential equation of elliptic type. As an example of an application of this method, we shall determine for the domain $\mathscr{D}$ systems $\{\psi_\nu(\lambda, \theta)\}$ of solutions of (1.7) such that every solution ψ regular in $\overline{\mathscr{D}}_1$, $\overline{\mathscr{D}}_1 = \mathscr{E}(\overline{\mathscr{D}})$, can be represented in $\mathscr{D}$ in the form

$$\psi(\lambda, \theta) = \sum_{\nu=0}^{\infty} A_\nu \psi_\nu(\lambda, \theta) . \tag{6.1}$$

Given a simply connected domain $\mathscr{D}_1$, there exist various systems $\{\varphi_\nu(Z)\}$ of analytic functions of one complex variable such that a function $g(Z)$ regular in $\overline{\mathscr{D}}_1$ can be developed in $\mathscr{D}_1$ in the form

$$g(Z) = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu(Z) . \tag{6.2}$$

For instance, one can choose for $\{\varphi_\nu(Z)\}$, $\nu = 1, 2, \cdots$, the system of functions which are orthogonal in $\mathscr{D}_1$, or functions $\{[\tilde{g}(Z)]^\nu\}$, $\nu = 0, 1, 2, 3, \cdots$, $[(\tilde{g}(Z))^0 \overset{\text{def}}{=} \text{const}]$, where $\tilde{g}(Z)$ maps $\mathscr{D}_1$ onto the unit circle. Suppose now that $\mathscr{D}$ is a star domain with respect to $Z = O$, $\overline{\mathscr{D}} \subset \mathscr{W}$ and $g(Z/2)$ is regular in $\overline{\mathscr{D}}_1 = \mathscr{E}(\overline{\mathscr{D}})$, every solution ψ regular for $\lambda, \theta \in \overline{\mathscr{D}}_1$ can be represented in $\mathscr{D}$ in the form (6.1), where

$$\psi_{2\nu}(\lambda, \theta) = \operatorname{Im} \int_{\mathscr{C}} E(Z, Z^*, t) Z^{1/6} \varphi_\nu\Big(\frac{1}{2} Z(1 - t^2)\Big) \frac{dt}{(1 - t^2)^{1/3}} , \tag{6.3}$$

and $\psi_{2\nu-1}(\lambda, \theta)$ are the real parts of the above integral.

Proof. Since we assumed that $Z^{-1/6} f(Z/2)$ is regular in $\overline{\mathscr{D}}_1$, the representation

$$Z^{-1/6} f\Big(\frac{Z}{2}\Big) = \sum a_\nu \varphi_\nu\Big(\frac{Z}{2}\Big) \tag{6.4}$$

converges for $(Z/2) \in \mathscr{D}_1$ uniformly and absolutely.

In the development

$$\begin{aligned}\psi(\lambda, \theta) = \operatorname{Im} \int_{\mathscr{C}} & E(Z, Z^*, t)(Z(1 - t^2))^{1/6} \times \\ & \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu\Big(\frac{Z}{2}(1 - t^2)\Big) \frac{dt}{\sqrt{1 - t^2}}\end{aligned} \tag{6.5}$$

we can interchange the order of summation and integration, and we obtain the development (6.1), where $\psi_\nu(\lambda, \theta)$ are given by the real and imaginary parts of $\int_{\mathscr{C}} \cdots$ in (6.3).

7. A representation of a stream function ψ in the supersonic region. As indicated in [10], the approach of the present paper can be generalized to the case where λ is replaced by the complex variable

$$U = \lambda + i\Lambda \,, \tag{7.1}$$

and under some assumptions about

$$\chi_1(\theta) = \lim_{U\to 0} \psi(U, \theta) \quad \text{and} \quad \chi_2(\theta) = \lim_{U\to 0} U^{1/3}\frac{\partial\psi(U, \theta)}{\partial U}$$

one can determine the associate f in terms of χ_1 and χ_2. Consequently, the method of integral operators can also be used to consider the initial value problem in the supersonic region. Replacing λ by U, see (7.1), and setting $\lambda = 0$, we obtain

$$\begin{gathered} \Lambda = h^{-1} \arctan\,[h(M^2 - 1)^{1/2}] - \arctan\,[(M^2 - 1)^{1/2}] \,, \\ h = \left(\frac{k-1}{k+1}\right)^{1/2}, k > 1 \,. \end{gathered} \tag{7.2}$$

Here M is the Mach number and $p = c\rho^k$ is the pressure density relation, k, c are constants. (See [22] and [5], p. 861). Equation (1.7) assumes the form

$$\begin{gathered} \mathbf{H}(\psi) \equiv \psi_{\Lambda\Lambda} - \psi_{\theta\theta} + 4N_1\psi_\Lambda = 0 \,, \\ N_1 = \frac{k+1}{8}\frac{M^4}{(M^2-1)^{3/2}}, \ M > 1 \,. \end{gathered} \tag{7.3}$$

In the supersonic case it is convenient to introduce the variable

$$\tilde{\tilde{\tilde{T}}}{}^2 = M^2 - 1 \,. \tag{7.3a}$$

If we write in analogy to (1.8)

$$\psi(\Lambda, \theta) = \tilde{\tilde{\tilde{H}}}_1(\Lambda)\tilde{\psi}^*(\Lambda, \theta) \,, \qquad \tilde{\psi}^*(\Lambda, \theta) \equiv \psi^*(i\Lambda, \theta) \,, \tag{7.3b}$$

where

$$\tilde{\tilde{\tilde{H}}}_1(\Lambda) = \exp\left[-2\int_{\Lambda_0}^{\Lambda} N_1(\tau)d\tau\right] \tag{7.3c}$$

and

$$\Lambda = \Lambda(M) \,, \qquad \Lambda_0 = \Lambda(2^{1/2}) \,,$$

then ψ^* satisfies

$$\mathbf{H}^*(\psi^*) \equiv \psi^*_{\Lambda\Lambda} - \psi^*_{\theta\theta} - 4F_1\psi^* = 0 \,, \tag{7.4}$$

$$F_1 = \frac{(k+1)M^4}{64}\left[\frac{-(3k-1)M^4 - 4(3-2k)M^2 + 16}{(1-M^2)^3}\right] \tag{7.4$'$}$$

(see [5], p. 861). A formal computation yields that

$$(7.4'')\qquad \tilde{\tilde{H}}(\tilde{\tilde{T}}) = \frac{1}{\tilde{\tilde{T}}^{1/2}}\left[\frac{2k}{(k+1)+(k-1)\tilde{\tilde{T}}^2}\right]^{1/2(k-1)} = \tilde{\tilde{H}}_1(\Lambda(M))\ .$$

In general, we use in the following the same notations as in [5] and [6]. As a rule we write $\tilde{p}(\Lambda) \equiv p(i\Lambda)$, e.g., $\tilde{q}^{(nk)}(\Lambda) \equiv q^{(n,\kappa)}(i\Lambda)$. Further, instead of $H(T)$ (see [5] (4.3), p. 870) we introduced here $\tilde{\tilde{H}}(\tilde{\tilde{T}})$ (see (7.4'') and (7.3a)). Consequently the generating function $\tilde{\tilde{E}}$ *differs* from $E(i\Lambda, i\Lambda, \theta)$.

In defining the operation[4]

$$(7.5)\qquad \begin{gathered} \mathbf{P}(f) \overset{\text{def}}{=} \operatorname{Im}\left[\tilde{\tilde{H}}_1(\Lambda)\int_{\mathscr{C}} \tilde{E}^*_{22}(\Lambda, \theta, t)\tilde{f}\Big(\frac{1}{2}(\Lambda+\theta)(1-t^2)\Big)\frac{dt}{\sqrt{1-t^2}}\right], \\ \tilde{\tilde{H}}_1(\Lambda) \equiv \tilde{\tilde{H}}(\Lambda(T)),\ \Lambda > 0,\ 2\Lambda < |\Lambda+\theta|\ , \\ \tilde{E}^*_{22}(\lambda + i\Lambda, \theta, t) = E^*_{22}(Z, Z^*, t)\ , \end{gathered}$$

it was assumed that $\mathscr{C}$ is a curve connecting $t=1$ with $t=-1$ and lying in $|t| \geqq 1$ (see [5], p. 872). As in the subsonic case (see [6] (7.12) p. 468), we set

$$(7.6)\qquad \begin{aligned} \tilde{E}^*_{22}(\Lambda, \theta, t) &= A_1\tilde{E}^{*(1)}(i\Lambda, \theta, t) \\ &\quad + [(i\Lambda + i\theta)/(1-t^2)/2]^{2/3}A_2\tilde{E}^{*(2)}(i\Lambda, \theta, t)\ , \\ &\qquad \operatorname{Im}(A_2\bar{A}_1) \neq 0\ . \end{aligned}$$

$$(7.6')\qquad \tilde{E}_{22}(\Lambda, \theta, t) = \tilde{\tilde{H}}_1(\Lambda)\tilde{E}^*_{22}(\Lambda, \theta, t)\ .$$

We shall show that by imposing some additional restrictions on the domain of definition of $\tilde{E}(\Lambda, \theta, t)$ we can use for $\mathscr{C}$ the curve

$$(7.7)\qquad \mathscr{C}^* = \mathscr{C}_{21} \cup \mathscr{C}_{22} \cup \mathscr{C}_{23}\ ,$$

$$(7.7')\qquad \begin{aligned} \mathscr{C}_{21} &= (-1 \leqq t \leqq -t_0)\ , \\ \mathscr{C}_{22} &= (t = t_0e^{i\varphi},\ t_0 > 0,\ -\pi \leqq \varphi \leqq 0)\ , \\ \mathscr{C}_{23} &= (t_0 \leqq t \leqq 1)\ . \end{aligned}$$

Analogously to [5, (5.5), p. 878], or [6, (4.7), p. 453],

$$(7.8)\qquad \begin{aligned} \tilde{\tilde{E}}^{(\kappa)}(U, \theta, t) &= \tilde{\tilde{H}}_1(U)\tilde{E}^{*(\kappa)}(U, \theta, t)\ , \\ \tilde{E}^{*(\kappa)}(U, \theta, t) &= \sum_{n=0}^{\infty}\frac{q^{(n,\kappa)}(U)}{(-t^2Z)^{n-1/2+2\kappa/3}}\ , \end{aligned}$$

where $q^{(n,\kappa)}(U)$, $\kappa = 1, 2$, $n = 0, 1, 2, \cdots$, are solutions of the equations

[4] Both the real and imaginary parts of $\tilde{\tilde{H}}_1(\Lambda)\int_{\mathscr{C}}\cdots$ are solutions of (7.3). In accordance with the previous definition we choose here "Im". In view of definition (2.4) it is sufficient to assume that $|t| \geqq 1$. If $0 < t_0 \leqq |t|$, where $t_0 < 1$, then $\mathscr{W}$ has to be replaced by $\{2|\lambda| < |Z|t_0^2\}$.

$$q_{UU}^{(0,\kappa)} + 4F(U)q^{(0,\kappa)} = 0 , \tag{7.8'}$$

$$2\left[n + \left(\frac{2\kappa}{3}\right)\right]q_U^{(n,\kappa)} + q_{UU}^{(n+1,\kappa)} + 4F(U)q^{(n+1,\kappa)} = 0 \tag{7.8''}$$

(see [6, p. 453, (4.4)])[5]. These solutions can be written in the form

$$q^{(n,\kappa)} = \sum_{\nu=0}^{\infty} C_\nu^{(n,\kappa)}(-U)^{n-1/2+2/3(\kappa+\nu)} \tag{7.9}$$

where

$$C_0^{(01)} = 2^{1/6},\ C_0^{(n1)}\frac{\left(\frac{1}{6}\right)_n\left(\frac{2}{3}\right)_n 2^{n+1/6}}{n!\left(\frac{1}{3}\right)_n},\ n \geqq 1,\ C_1^{(n1)} = 0,\ n \geqq 0 , \tag{7.10}$$

$$C_0^{(0,2)} = 2^{5/6},\ C_0^{(n,2)} = \frac{\left(\frac{5}{6}\right)_n\left(\frac{4}{3}\right)_n 2^{n+5/6}}{n!\left(\frac{5}{3}\right)_n},\ n \geqq 1 , \tag{7.11}$$

$$(a)_n \equiv a(a+1)\cdots(a+n-1)$$

(see [6, (4.4), (4.5), (4.6a), (4.6b), (4.6c), (3.11), (3.12)] or [9, p. 113, (1a), (1b), (2), (3a), (3b), 3c)].)

REMARK. *One* initial value condition determines uniquely $q^{(n,2)}(U)$. Indeed, the general solution of (7.8′) and (7.8″) can be written in the form

$$C_{-1}^{(n,2)}(-U)^{n+1/6} + \sum_{\nu=0}^{\infty} C_\nu^{(n,2)}(-U)^{n-1/2+2/3(n+1)} . \tag{7.12}$$

From (7.9) follows that the second initial value condition used for (7.12) is $C_{-1}^{(n,2)} = 0$.

In [6, p. 459, (4.43)], it has been shown that

$$|\widetilde{E}_{2\kappa}^*(U, \theta, t)| \leqq \sum_{n=0}^{\infty}\left|\frac{w^{(\kappa)}(U)u^{(n\kappa)}(U)}{(-t^2Z)^{n-1/2+2\kappa/3}}\right| . \tag{7.13}$$

and that

$$|u^{(n\kappa)}(U)| \leqq C\frac{2^n\Gamma\left(n+\frac{2\kappa}{3}\right)}{\Gamma\left(\frac{2\kappa}{3}\right)\Gamma(n+1)}(1+\varepsilon)^n|U|^n , \qquad \varepsilon > 0 . \tag{7.14}$$

LEMMA 7.1. *Let* $U = i\Lambda$, $0 < \Lambda < t_1^2\theta/(1-t_1^2)$, *where* $(2+\varepsilon)t_1^2 < t_0^2$,

[5] In the first equation of [6, (4.4)], q is missing after $4F(\lambda)$.

$0 < t_0 < 1, \varepsilon > 0$ $[0 < t_0 \leqq |t|, t \in \mathscr{C}^*]$, *see* (7.6). *Then the series on the right-hand side of* (7.13) *converges absolutely and uniformly for* $t \in \mathscr{C}^*$ *and* $2|U| < t_0^2 |Z|$.

Proof. Since $\{\Gamma(n + 4/3)/\Gamma(n + 1)\} \leqq n + 1$, it is sufficient to show that $2\Lambda/|t^2(\Lambda + \theta)| < 1$. From our assumptions follows that $\theta/\Lambda > (1 - t_1^2)/t_1^2$, therefore, for $t \in \mathscr{C}^*$ and $\Lambda > 0$,

$$\begin{aligned} \frac{\Lambda}{|t^2(\Lambda + \theta)|} &= \frac{1}{\left|t^2\left(1 + \dfrac{\theta}{\Lambda}\right)\right|} \leqq \frac{1}{|t|^2 \left|1 + \dfrac{1 - t_1^2}{t_1^2}\right|} \\ &= \frac{|t_1|^2}{|t|^2} \leqq \frac{t_0^2}{(2 + \varepsilon)|t|^2} \leqq \frac{1}{2 + \varepsilon} . \end{aligned} \tag{7.15}$$

Analogously to Theorem 4.1 we can express the associate $\tilde{f}(\Lambda + \theta)$ in terms of

$$\lim_{\Lambda \to 0^+} \tilde{\psi}(\Lambda, \theta) \quad \text{and} \quad \lim_{\Lambda \to 0^+} [\Lambda^{1/3} \tilde{\psi}_\Lambda(\Lambda, \theta)] , \tag{7.16}$$

see also [10], [28] and [29]. In formula (7.5) with $U = i\Lambda$ the associate $f \equiv f((1/2)i(\Lambda + \theta)(1 - t^2)) \equiv \tilde{f}((1/2)(\Lambda + \theta)(1 - t^2))$ is a function of a *real* variable $(\Lambda + \theta)$. In this case we obtain some modifications of our results.

Operating with functions of one *real* variable, it is convenient for many purposes to represent them in form of trigonometric series. Analogously, in the case of solutions of (7.3) we introduce a set of solutions

$$\begin{aligned} C_n(\Lambda, \theta) = \mathrm{Re} \int_{\mathscr{C}^*} &\tilde{E}_{22}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\ &\times \cos [n(\Lambda + \theta)(1 - t^2)/2] \frac{dt}{(1 - t^2)^{1/3}} , \qquad n = 0, 1, 2, \cdots , \end{aligned} \tag{7.17}$$

$$\begin{aligned} S_n(\Lambda, \theta) = \mathrm{Re} \int_{\mathscr{C}^*} &\tilde{E}_{22}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\ &\times \sin [n(\Lambda + \theta)(1 - t^2)/2] \frac{dt}{(1 - t^2)^{1/3}} , \end{aligned} \tag{7.18}$$

of (7.3)

REMARK. We note that when introducing a set of particular solutions of the equation of elliptic type, we used $\{U^n\}$, $n = 0, 1, 2, \cdots$, as associates (see, e.g., [9], p. 22]); here we use

$$\frac{1}{2}(e^{in(\Lambda+\theta)} + e^{-in(\Lambda+\theta)}) \quad \text{and} \quad \frac{1}{2i}(e^{in(\Lambda+\theta)} - e^{-in(\Lambda+\theta)}) , \qquad n = 0, 1, 2, \cdots . \tag{7.19}$$

Further, instead of using $i \cos [n(\Lambda+\theta)(1-t^2/2]$ and $i \sin [\Lambda+\theta)(1-t^2)/2]$ as associates, we take the *real* part of the integral in (7.17) (7.18). The integral operator (7.5) assumes the form

$$\begin{aligned}\psi(\Lambda, \theta) = \operatorname{Re} \int_{\mathscr{C}^*} & \tilde{E}_{22}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\ & \times f\Big[\frac{1}{2}(\Lambda + \theta)(1 - t^2)\Big]\frac{dt}{(1 - t^2)^{1/3}} .\end{aligned} \tag{7.20}$$

When considering the integration along $\mathscr{C}_2^*$, it is useful to introduce the auxiliary variable $\tau = \tilde{t}^{-1}(t)$ given by

$$\begin{aligned} \tau &= \operatorname{Re} t + 1 && \text{for } t \in \mathscr{C}_{21} = [-1 \leqq t \leqq t_0] \\ \tau &= \frac{1}{i}\Big(\log \frac{t}{t_0}\Big) && \text{for } t \in \mathscr{C}_{22} = [t = t_0 e^{i\varphi}, 0 \leqq \varphi \leqq \pi] \\ \tau &= \operatorname{Re} t - t_0 + \pi t_0 && \text{for } t \in \mathscr{C}_{23} = [-t_0 \leqq t \leqq 1] . \end{aligned} \tag{7.21}$$

LEMMA 7.2. *$d\tau/dt = 1$ for $t \in (\mathscr{C}_{21} - P_1) \cup (\mathscr{C}_{23} - P_2)$* and *$dt/d\tau = ie^{i\tau} t_0$ for $t \in \mathscr{C}_{22} - P_1 - P_2$, $P_1 = \mathscr{C}_{21} \cap \mathscr{C}_{22}$, $P_2 = \mathscr{C}_{22} \cap \mathscr{C}_{23}$.*

THEOREM 7.1. *Suppose that the coefficients $\{a_n\}$, $\{b_n\}$, $n = 0, 1, 2, \cdots$, of the series*

$$\sum_{n=9}^{\infty} (a_n \cos nx + b_n \sin nx) \sim g(x) . \tag{7.22}$$

are chosen in such a way that

(7.23)

$$\int_{\mathscr{C}^*} \sum_{n=0}^{\infty} \left[\frac{\left|a_n \cos\left(\frac{n}{2}(\Lambda + \theta)(1 - \tilde{t}^2(\tau))\right)\right| + \left|b_n \sin\left(\frac{n}{2}(\Lambda + \theta)(1 - \tilde{t}^2(\tau))\right)\right|}{|\sqrt{1 - \tilde{t}^2(\tau)}|} \right] \times d\tau < \infty .$$

Then if

$$\begin{aligned}\psi(\Lambda, \theta) = \operatorname{Re} \int_{\mathscr{C}^*} & \tilde{E}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\ & \times g\Big(\frac{1}{2}(\Lambda + \theta)(1 - t^2)\Big)\frac{dt}{(1 - t^2)^{1/3}} ,\end{aligned} \tag{7.24}$$

it holds

$$\psi(\Lambda, \theta) = \sum_{n=0}^{\infty} (a_n C_n(\Lambda, \theta) + b_n S_n(\Lambda, \theta)) . \tag{7.25}$$

Proof. Since for $0 < 2\Lambda < |\Lambda + \theta|\, t_0^2$, $t \in \mathscr{C}^*$, $E(Z, Z^*, t) \equiv \tilde{E}(\Lambda, \theta, t)$

and $dt/d\tau$ are uniformly bounded on $\mathscr{C}^* - P_1 - P_2$ and $|d\tau/dt| \geqq A \geqq 0$ by the Lebesgue theorem (see, e.g., [31, p. 347]), it follows that we can interchange the order of summation and integration in (7.24). Using (7.17) and (7.18), we obtain (7.25).

The above investigations suggest that we write the solution[6] ψ as a sum of two operators, namely,

$$\psi(\Lambda, \theta) = \psi_\tau(\Lambda, \theta) + \Psi_\tau(\Lambda, \theta) , \tag{7.26}$$

$$\psi_\tau(\Lambda, \theta) = \int_{\mathscr{C}_{21} \cup \mathscr{C}_{23}} \tilde{E}(\Lambda, \theta, t)\zeta^{1/6} f(\zeta) \frac{dt}{(1-t^2)^{1/2}}, \; \zeta \equiv \frac{1}{2}(\Lambda + \theta)(1 - t^2) \tag{7.26a}$$

$$\Psi_\tau(\Lambda, \theta) = \int_{\mathscr{C}_{22}} \tilde{E}(\Lambda, \theta, t)\zeta^{1/6} f(\zeta) \frac{dt}{(1-t^2)^{1/2}} , \tag{7.26b}$$

$$\mathscr{C}_{21} \cup \mathscr{C}_{23} = [-1 \leqq t < -\tau] \cup [\tau \leqq t \leqq 1] ,$$
$$\mathscr{C}_{22} = [t = \tau e^{i\varphi}, \; -\pi \leqq \varphi \leqq 0] ,$$
$$0 < \tau < 1 .$$

When considering $\{\psi_\tau(\Lambda, \theta)\}$, one can apply various results in the theory of trigonometrical series, while considering $\{\Psi_\tau(\Lambda, \theta)\}$, we apply theorems on analytic functions of one complex variable.

The author wishes to thank Paul Rosenthal for his assistance in preparing this paper.

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Received February 10, 1969. This investigation was supported in part by contract AEC, AT 04-3-326 PA -22.

STANFORD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

CONCERNING SEMI-STRATIFIABLE SPACES

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In this paper, a class of spaces, called semi-stratifiable spaces is introduced. This class of spaces lies between the class of semi-metric spaces and the class of spaces in which closed sets are G_δ. This class of spaces is invariant with respect to taking countable products, closed maps, and closed unions. In a semi-stratifiable space, bicompactness and countable compactness are equivalent properties. A semi-stratifiable space is F_σ-screenable.

A T_1-space is semi-metric if and only if it is semi-stratifiable and first countable. A completely regular space is a Moore space if and only if it is a semi-stratifiable p-space.

The concept of semi-stratifiable spaces as a generalization of semi-metric spaces (see Corollary 1.4) is due to E. A. Michael. It appears that all properties of semi-metric spaces which do not depend on first countability also hold in semi-stratifiable spaces. The class of semi-stratifiable spaces contains all stratifiable spaces [3], all cosmic spaces [13], and all spaces with a σ-locally finite [15] or σ-discrete [2] network.

Some of the results of this paper were announced in [5].

Most terms which are not defined in this paper are used as in Kelley [10].

1. Preliminaries.

DEFINITION 1.1. A topological space X is a *semi-stratifiable* space if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

(a) $\bigcup_{n=1}^{\infty} U_n = U$,

(b) $U_n \subset V_n$ whenever $U \subset V$, where $\{V_n\}_{n=1}^{\infty}$ is the sequence assigned to V.

A correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *semi-stratification* for the space X whenever it satisfies conditions (a) and (b) of Definition 1.1.

By comparing the above definition with Definition 1.1 of [3], one can see that, if the correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a stratification for X, then $U \rightarrow \{U'_n\}_{n=1}^{\infty}$, where $U'_n = \mathrm{Cl}\ U_n$, is a semi-stratification for X. In [8], Heath gives an example of a (paracompact) semi-stratifiable space which is not stratifiable.

THEOREM 1.2. *A necessary and sufficient condition for a topological space X to be semi-stratifiable is that there be a sequence*

$\{g_i\}_{i=1}^{\infty}$ *of functions from* X *into the collection of open sets of* X *such that* (i) $\bigcap_{i=1}^{\infty} g_i(x) = Cl\{x\}$ *for each* x, *and* (ii) *if* y *is a point of* X *and* $\{x_i\}_{i=1}^{\infty}$ *is a sequence of points in* X, *with* $y \in g_i(x_i)$ *for all* i, *then* $\{x_i\}_{i=1}^{\infty}$ *converges to* y.

Proof. Let $U \rightarrow \{U_n\}_{n=1}^{\infty}$ be a semi-stratification for X. For each i, define the function g_i by $g_i(x) = X - (X - \mathrm{Cl}\,\{x\})_i$. The sequence $\{g_i\}_{i=1}^{\infty}$ satisfies conditions (i) and (ii) of the theorem.

Conversely, let $\{g_i\}_{i=1}^{\infty}$ satisfy conditions (i) and (ii) of the theorem. For each n and each open set U, let $U_n = X - \bigcup\{g_n(x) \colon x \in X - U\}$. Then correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a semi-stratification for X.

DEFINITION 1.3. A topological space X is *semi-metric* if there is a distance function d defined on X such that

(1) $d(x, y) = d(y, x) \geqq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) x is a limit point of a set M if and only if $\inf\{d(x, y) \colon y \in M\} = 0$.

See [7, 11].

With the aid of Theorem 3.2 of [7] we have the following relationship between semi-stratifiable spaces and semi-metric spaces.

COROLLARY 1.4. *A* T_1*-space is a semi-metric space if and only if it is a first countable semi-stratifiable space.*

2. Properties of semi-stratifiable spaces.

THEOREM 2.1. *The countable product of semi-stratifiable spaces is semi-stratifiable.*

Proof. For each i, let X_i be a semi-stratifiable space and $\{g_{ij}\}_{j=1}^{\infty}$ be a sequence of functions on X_i satisfying the conditions of Theorem 1.2. Let $X = \prod_{i=1}^{\infty} X_i$ and let π_i be the projection of X onto X_i. For each i, j and each x in X, let $h_{ij}(x) = g_{ij}(\pi_i(x))$ if $j \leqq i$ and $h_{ij}(x) = X_i$ if $j > i$. Now let $g_j(x) = \prod_{i=1}^{\infty} h_{ij}(x)$ for each j and x. The sequence $\{g_i\}_{i=1}^{\infty}$ satisfies the conditions of Theorem 1.2 and, hence, X is semi-stratifiable.

THEOREM 2.2. *A semi-stratifiable space is hereditarily semi-stratifiable.*

Theorem 2.2 can be proved by taking the natural restriction to the subspace of a semi-stratification of the larger space. In the case of closed subspaces, all semi-stratifications on the subspace can be constructed in this manner.

THEOREM 2.3. *If Y is a closed subspace of a semi-stratifiable space X and $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a semi-stratification for Y, then there is a semi-stratification $V \rightarrow \{V_n\}_{n=1}^{\infty}$ for X such that $(V \cap Y)_n = (V_n \cap Y)$.*

Proof. If $W \rightarrow \{W_n\}_{n=1}^{\infty}$ is any semi-stratification for X, then let $V_n = (V \cap Y)_n \cup (V - Y)_n$. The correspondence $V \rightarrow \{V_n\}_{n=1}^{\infty}$ is a semi-stratification for X satisfying $(V \cap Y)_n = V_n \cap Y$.

By applying Theorem 2.3 with respect to the common subspace, we obtain the following theorem:

THEOREM 2.4. *The union of two closed (in the union) semi-stratifiable spaces is semi-stratifiable.*

DEFINITION 2.5. A topological space is *F_σ-screenable* if every open cover has a σ-discrete closed refinement which covers the space.

Theorem 2.6 generalizes McAulely's Lemma 1 of [12].

Theorem 2.6. *A semi-stratifiable space is F_σ-screenable.*

Proof. Let X be a semi-stratifiable space with a semi-stratification $U \rightarrow \{U_n\}_{n=1}^{\infty}$. Let $\{O_\alpha : \alpha \in I\}$ be an open cover of X and let I be well-ordered. For each natural number n, define: $H_{1n} = (O_1)_n$ and, for each $\alpha > 1$, $H_{\alpha n} = (O_\alpha)_n - \cup \{O_\beta : \beta \in I, \beta < \alpha\}$. For each natural number n, let $\mathscr{H}_n = \{H_{\alpha n} : \alpha \in I\}$. Then $\mathscr{H}_n$ is a discrete collection of closed sets. By the well-ordering on I, $\mathscr{H} = \cup_{n=1}^{\infty} \mathscr{H}_n$ covers X.

DEFINITION 2.7. A topological space is *$\aleph_1$-compact* if every uncountable subset has a limit point.

THEOREM 2.8. *In a semi-stratifiable T_1-space X, the following are equivalent* (1) *X is Lindelöf,* (2) *X is hereditarily separable, and* (3) *X is $\aleph_1$-compact.*

Proof. (1) $\Rightarrow$ (2) Let X be a Lindelöf semi-stratifiable space. Since a Lindelöf space in which open sets are F_σ is hereditarily Lindelöf, it is sufficient to show that X is separable. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence of functions satisfying the conditions of Theorem 1.2. For each i, $\{g_i(x) : x \in X\}$ is an open cover of X and, since X is Lindelöf, there is a countable subset D_i of X such that $\{g_i(x) : x \in D_i\}$ is an open cover of X. The set $D = \cup_{i=1}^{\infty} D_i$ is a countable dense subset of X.

(2) $\Rightarrow$ (3) The proof of this part is well-known.

(3) $\Rightarrow$ (1) Let X be an $\aleph_1$-compact semi-stratifiable T_1-space. Let $\mathscr{G}$ be an open cover of X and suppose that $\mathscr{G}$ has no countable subcover. By Theorem 2.6, $\mathscr{G}$ has a closed refinement $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ where each $\mathscr{H}_n$ is discrete. Since $\mathscr{G}$ has no countable subcover, there is an n such that $\mathscr{H}_n$ is uncountable. Let X' be a subset of X consisting of exactly one point of each nonempty element of $\mathscr{H}_n$. The set X' is uncountable and has no limit point.

Theorem 2.8 cannot be strengthened by replacing hereditarily separable by separable. The example of a Moore space which is not metrizable due to R. L. Moore (see [9]) is an example of a separable semi-stratifiable space which is not Lindelöf.

Since, in a Lindelöf space, bicompact is equivalent to countably compact, Theorem 2.8 has the following corollary:

COROLLARY 2.9. *In a semi-stratifiable T_1-space, bicompact is equivalent to countably compact.*

3. Mappings. It is a well-known theorem [17] that the closed compact image of a separable metric space is a separable metric space. However, there are closed images of separable metric spaces which are not first countable. The following theorem gives a property of metric spaces which is preserved by closed maps.

THEOREM 3.1. *The closed image of a semi-stratifiable space is semi-stratifiable.*

Proof. Let f be a closed continuous function from a semi-stratifiable space X onto a topological space Y. Let $U \rightarrow \{U_n\}_{n=1}^{\infty}$ be a semi-stratification for X. For each open set V of Y and each natural number n, let $V_n = f((f^{-1}[V])_n)$. The correspondence $V \rightarrow \{V_n\}_{n=1}^{\infty}$ is a semi-stratification for Y.

Theorem 3.1 does not remain true if closed is replaced by open.

Theorem 3.1 and Corollary 1.4 imply that the closed image of a semi-metric space is semi-stratifiable. However, it can be shown that the subspace of βN (the Stone-Čech compactification of the natural numbers) consisting of N together with one point of $\beta N - N$ is a semi-stratifiable space which cannot be the closed image of a semi-metric space. It is an open question whether the spaces which are closed images of semi-metric spaces are precisely the semi-stratifiable Fréchet-Urysohn spaces [2].

4. Moore spaces. In this section, we wish to give necessary and sufficient conditions for a semi-stratifiable space to be a Moore space.

DEFINITION 4.1. A sequence $\{\mathscr{G}_n\}_{n=1}^{\infty}$ of open covers of a topological space X is a *development* for X if (1) $\mathscr{G}_{i+1}$ is a refinement for $\mathscr{G}_i$ and (2), if x is a point of X and U is an open set in X containing x, then there is a natural number k such that $\mathrm{St}\,(x, \mathscr{G}_k) \subset U$. A *Moore space* is a regular T_1-space which has a development. See [7, 14].

A Moore space is semi-metric and, hence, is semi-stratifiable. The following example, due to McAuley [11], shows that this implication cannot be reversed.

EXAMPLE 4.2. Let X be the x-axis of the Cartesian plane E^2. Let d denote the usual distance function in E^2 and, if $p \neq q$, let $\alpha(p, q)$ denote the nonobtuse angle (in radians) formed by X and the line through p and q. Define a distance function D on E^2 as follows: $D(p, p) = 0$ and, if $p \neq q$, $D(p, q) = d(p, q) + \alpha(p, q)$. A basis for the topology on E^2 is $\{U_\varepsilon(p): p \in E^2, \varepsilon > 0\}$ where $U_\varepsilon(p) = \{q: D(p, q) < \varepsilon\}$. Let S denote E^2 with this topology. If S were a Moore space, it would be second countable, since it is Lindelöf. But S is not second countable since any basis contains uncountably many elements.

DEFINITION 4.3. A T_1-space X is said to be *quasi-complete* provided that there is a sequence $\{\mathscr{B}_n\}_{n=1}^{\infty}$ of open covers of X with the following property: if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed subsets of X and if there exists an element $x_0 \in X$ such that, for each n, there is a $\mathscr{B}_n \in \beta_n$ with $A_n \cup \{x_0\} \subset B_n$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

DEFINITION 4.4 (Borges [4]). A T_1-space X is a *wΔ-space* if there exists a sequence $\{\mathscr{B}_n\}_{n=1}^{\infty}$ of open covers of X such that, if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed subsets of X and there exists $x_0 \in X$ for which $A_n \subset \mathrm{St}\,(x_0, \mathscr{B}_n)$ for all n, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Definition 4.3 is at least formally weaker then Definition 4.4. It is an open question whether all quasi-complete spaces are $w\Delta$-spaces.

Theorem 4.5, due to Heath [7], gives a sufficient condition for a space to be a Moore space.

THEOREM 4.5. *A regular T_1-space X is a Moore space provided that there is a sequence $\{g_i\}_{i=1}^{\infty}$ of functions from X into the topology on X with the following properties*: (A) *For each x in X, $\{g_i(x)\}_{i=1}^{\infty}$ is a decreasing local base at x.* (B) *If y is a point of X and $\{x_i\}_{i=1}^{\infty}$ is a sequence in X with $y \in g_i(x_i)$ for each i, then $\{x_i\}_{i=1}^{\infty}$ converges to y.* (C) *If y is a point of X, U is an open subset of X containing y, and $\{x_i\}_{i=1}^{\infty}$ is a sequence in X such that, for each n, $y \in g_n(x_n)$ and there* *is a natural number k with* $\mathrm{Cl}\,[g_{n+k}(x_{n+k})] \subset g_n(x_n)$, *then there is a natural number m with* $g_m(x_m) \subset U$.

THEOREM 4.6. *A regular T_1-space is a Moore space if it is a quasi-complete semi-stratifiable space.*

Proof. Let X be a regular quasi-complete semi-stratifiable T_1-space. Let $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ be a sequence satisfying the conditions of Definition 4.3 and let $\{h_n\}_{n=1}^{\infty}$ be a sequence satisfying the conditions of Theorem 1.2. For each x in X, let $B_n(x)$ be a member of $\mathfrak{B}_n$ containing x. For each x, let $g_1(x)$ be an open subset of X containing x such that $\mathrm{Cl}\, g_1(x) \subset B_1(x) \cap h_1(x)$ and let $g_{n+1}(x)$ be an open subset of X containing x such that $\mathrm{Cl}\, g_{n+1}(x) \subset B_{n+1}(x) \cap h_{n+1}(x) \cap g_n(x)$. The sequence $\{g_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 4.5 and, hence, X is a Moore space.

By Proposition 2.8 of [4], we have the following corollary:

COROLLARY 4.7. *If X is a regular T_1-space, then the following are equivalent*:

(1) *X is a Moore space.*

(2) *X is a semi-stratifiable $w\Delta$-space.*

(3) *X is a semi-stratifiable quasi-complete space.*

If X is a completely regular T_1-space, let βX denote its Stone-Čech compactfication. The following definition is due to Arhangel'skii [1, 2].

DEFINITION 4.8. A completely regular T_1-space X is a *p-space* provided that there is a sequence $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ of collections of open subsets of βX such that each $\mathfrak{B}_n$ covers X and $\bigcap_{n=1}^{\infty} \mathrm{St}\,(x, \mathfrak{B}_n) \subset X$ for each point x in X.

LEMMA 4.9. *A p-space is quasi-complete.*

Proof. Let X be a p-space and let $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ satisfy the conditions of Definition 4.8. For each n, let $\mathfrak{B}'_n$ be an open cover of X such that, if $B \in \mathfrak{B}'_n$, then $\mathrm{Cl}_{\beta X}\, B$ is contained in some member of $\mathfrak{B}_n$. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed subsets of X and x be a point of X such that there is a $B_n \in \mathfrak{B}'_n$ with $A_n \cup \{x\} \subset B_n$ for each n. Since $\mathrm{Cl}_{\beta X}\, A_n$ is compact, $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta X}\, A_n \neq \varnothing$. But $\mathrm{Cl}_{\beta X}\, A_n \subset \mathrm{St}\,(x, \mathfrak{B}_n)$ and $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta X}\, A_n \subset X$. Thus, $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta X}\, A_n$. Hence, X is quasi-complete.

It can be seen that, in completely regular spaces, the concepts of $w\Delta$-spaces, p-spaces, and quasi-complete spaces are related. The exact relationship between these three concepts is an open problem.

THEOREM 4.10. *A completely regular T_1-space is a Moore space if and only if it is a semi-stratifiable p-space.*

Proof. Lemma 4.9 and Theorem 4.6 show that a semi-stratifiable p-space is a Moore space.

Conversely, let X be a completely regular Moore space and let $\{\mathscr{G}_n\}_{n=1}^\infty$ be a development for X. By the remark following Definition 4.1, X is semi-stratifiable. For each n, let

$$\mathscr{B}_n = \{\beta X - \mathrm{Cl}_{\beta X}(X - G): G \in \mathscr{G}_n\} .$$

The sequence $\{\mathscr{B}_n\}_{n=1}^\infty$ satisfies the conditions of Definition 4.8. Since $G \subset \beta X - \mathrm{Cl}_{\beta X}(X - G)$, $\mathscr{B}_n$ covers X. If $x \in X$ and $y \in \beta X - X$, let U and V be disjoint open subsets of βX containing x and y, respectively. There is a k where $\mathrm{St}(x, \mathscr{G}_k) \subset U \cap X$. Then $y \notin \mathrm{St}(x, \mathscr{B}_k)$. Hence, X is a p-space.

Since a locally compact Hausdorff space is a p-space, we have the following corollary:

COROLLARY 4.11. *A locally compact semi-stratifiable Hausdorff space is a Moore space.*

In Theorem 4.10, the condition of complete regularity can be replaced with regularity by using the Wallman compactification [6, 16] instead of the Stone-Čech compactification. Appropriate changes will also have to be made in Definition 4.8.

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Received March 17, 1969. This paper is a part of the author's doctoral dissertation written under the direction of R. W. Heath. The research was supported by a NASA Traineeship and a NSF Fellowship.

ARIZONA STATE UNIVERSITY AND
LOUISIANA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

MATRIC POLYNOMIALS WHICH ARE HIGHER COMMUTATORS

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Let A be an $n \times n$ matrix defined over a field F of characteristic greater than n. For each $n \times n$ matrix X we define

$$X_1 = [A, X]_0 = X \tag{1}$$
$$X_{h+1} = [A, X]_h = [A, X_h] = AX_h - X_hA$$

for each positive integer h. Then X is defined to be k-commutative with A if and only if

$$[A, X]_k = 0 , \qquad [A, X]_{k-1} \neq 0 . \tag{2}$$

Let $P(x)$ be a polynomial such that $P(A) \neq 0$. Specifically, assume that

$$P(A) = \sum_{i=p}^{n-1} \lambda_i A^i \neq 0 \tag{3}$$

where p is a positive integer, each λ_i is a scalar from F, and $\lambda_p \neq 0$. In this paper we study, for each positive integer k, the matrices X such that

$$[A, X]_k = P(A) . \tag{4}$$

We specify a polynomial $P(A)$ in the form (3) and show how the maximal value of k for which (4) has a solution depends on the polynomial $P(A)$. In Theorem 3 it is assumed that A is nonderogatory. Since the only matrices which commute with A in this case are polynomials in A, we are, in effect, establishing a more precise bound for k in (2) by predetermining X_k.

In the derogatory case, a matrix which is not a polynomial in A may commute with A. However, Theorem 4 shows that if we choose a polynomial $P(A)$ as X_k, then the maximal value of k depends on the polynomial P.

The problem of determining the maximal value of k for which (2) has a solution has been studied by Roth [8] and others. Roth's results are stated in terms of the maximal degrees of the elementary divisors of the matrix A. In particular, he showed that there exists a matrix X satisfying (2) for some A if $k \leqq 2n - 1$.

Nilpotent case. Throughout the paper we assume that A is in Jordan canonical form, since $[a, X]_k = P(A)$ if and only if

$$[BAB^{-1}, BXB^{-1}]_k = BP(A)B^{-1} .$$

The following notation introduced by W. V. Parker is used to simplify the proofs of the theorems.

DEFINITION. Let M_s for any integer s such that $-n+1 \leqq s \leqq m-1$ be the set of all $n \times m$ matrices in which all elements are zero except those for which $j-i=s$ (i denotes the row and j denotes the column in which the element appears). If $s > m-1$, M_s is defined to be the set consisting of only the zero matrix. A particular member of M_s will be denoted by D_s and will be called an s-stripe matrix. Note that if X is any $n \times m$ matrix then X can be written uniquely as $X = \sum_{s=-n+1}^{m-1} D_s$ where D_s is an element of M_s.

If A_1 and A_2 are $n \times n$ and $m \times m$ nilpotent nonderogatory matrices in Jordan canonical form and if $D_s = (d_{ij})$ is an $n \times m$ element of M_s where s is any integer such that $-n+1 \leqq s \leqq m-1$, let $f(D_s) = A_1D_s - D_sA_2$ and $f^k(D_s) = A_1f^{k-1}(D_s) - f^{k-1}(D_s)A_2$. It is easily seen that $f^k(D_s)$ is an element of M_{s+k}. Notice that the element in the ij position of $f(D_s)$, where $j-i=s+1$, is $d_{i+1,j} - d_{i,j-1}$ for $i \neq 1$. The element in the nj position is $-d_{n,j-1}$ if $j \neq 1$; the element in the $i1$ position is $d_{i+1,1}$ if $i \neq n$; and the element in the $n1$ position is zero.

LEMMA 1. *If A is an $n \times n$ nilpotent nonderogatory matrix in Jordan canonical form, if X is an $n \times n$ matrix, and if*

$$M = [A, X] = AX - XA ,$$

then the trace of M is zero and the trace of every subdiagonal stripe of M is zero.

Proof. Any $n \times n$ matrix X may be written as $\sum_{s=-n+1}^{n-1} D_s$ where D_s is an element of M_s. Thus

$$[A, X] = \left[A, \sum_{s=-n+1}^{n-1} D_s\right] = \sum_{s=-n+1}^{n-1} [A, D_s] .$$

If $s < 0$, then $[A, D_s]$ is a matrix such that the sum of the nonzero elements is zero. The matrix $[A, D_s]$ forms the $(s+1)$-stripe of M. This completes the proof of the lemma.

If A is an $n \times n$ nilpotent nonderogatory matrix in Jordan canonical form then for any positive integer $s < n$, $(A^T)^sA^s$ plays the part of a "lower identity" which we denote by L_s. That is,

$$(5) \qquad (A^T)^sA^s = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-s} \end{pmatrix} = L_s .$$

Similarly,

$$A^s(A^T)^s = \begin{pmatrix} I_{n-s} & 0 \\ 0 & 0 \end{pmatrix} = U_s \tag{6}$$

which we call an "upper identity".

Using the above, we prove the following lemma.

LEMMA 2. *Let A be an $n \times n$ nilpotent nonderogatory matrix in Jordan canonical form. Let L_s and U_s be as defined above. Then*

$$L_s(I - A)L_{s+k} = (I - A)L_{s+k} \tag{7}$$

and

$$U_{s+k}(I - A)U_s = U_{s+k}(I - A) , \tag{8}$$

where k is any positive integer less than $n - s$.

Proof. If we partition $I - A$ as follows:

$$(I - A) = \begin{pmatrix} M & 0 \\ * & N \end{pmatrix}$$

where M is $s \times (s + k)$, then

$$L_s(I - A)L_{s+k} = \begin{pmatrix} 0 & 0 \\ * & N \end{pmatrix} L_{s+k} = \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix} = (I - A)L_{s+k} .$$

The proof of (8) is similar.

Let $V = (1, 1, \cdots, 1)$, a $1 \times n$ vector, and let $V_s = VD_s$. That is, V_s is the vector in which each element represents a column sum in D_s, and since the columns in D_s have at most one nonzero element, V_s simply displays these elements in the form of a row vector. To simplify the notation we will let $V_{s+k} = VD_{s+k}$ where $D_{s+k} = [A, D_s]_k$ for some matrix D_s. In other words, the added subscript, k, implies that V_{s+k} is the result of k commutations. From now on, s will denote a nonnegative integer, $0 \leqq s \leqq n - 1$, and subdiagonal stripes of X will be denoted by D_{-s}. Also, the nontrivial subvector in V_s will be denoted by w_{n-s}, and the nontrivial subvector in V_s will be denoted by $\hat{w}_{n-s}$. Thus

$$V_s = (0, 0, \cdots, 0, d_{1,s+1}, d_{2,s+2}, \cdots, d_{n-s,n}) = (0_s, w_{n-s}) . \tag{9}$$

Similarly,

$$V_{-s} = (d_{s+1,1}, d_{s+2,2}, \cdots, d_{n,n-s}, 0, \cdots, 0) = (\hat{w}_{n-s}, 0_s) . \tag{10}$$

The following lemma is a vital part of the proof of Theorem 1.

Lemma 3. *If k is a positive integer and if V_s, A, U_s, and L_s are as defined above, then*

(i) $V_{s+k} = V_s(I - A)^k L_k$,

(ii) $V_{-s+k} = V_{-s}U_s(I - A)^k$ *if* $k \leqq s$,

(iii) $V_{-s+k} = V_{-s}U_s(I - A)^k L_{k-s}$ *if* $k > s$.

Proof. Case (i). If $k = 1$, from (7) and (9)

$$V_s(I - A)L_{s+1} = (0_s, w_{n-s})\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}.$$

In this case N has dimensions $(n - s) \times (n - s - 1)$, so N has (-1)'s on the diagonal and 1's on the first subdiagonal. But

$$(0_s, w_{n-s})\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix} = (0_s, w_{n-s})N = (0_{s+1}, w_{n-s-1})$$

where w_{n-s-1} has only $n-s-1$ elements of the form $(d_{i+1,s+i+1} - d_{i,s+i})$, and this is V_{s+1}. Therefore

$$V_{s+1} = V_s(I - A)L_{s+1} .$$

Similarly,

$$V_{s+2} = V_{s+1}(I - A)L_{s+2} = V_s(I - A)L_{s+1}(I - A)L_{s+2} .$$

But by Lemma 2,

$$L_{s+1}(I - A)L_{s+2} = (I - A)L_{s+2} .$$

Thus $V_{s+2} = V_s(I - A)^2 L_{s+2}$, and by induction it follows that

$$V_{s+k} = V_s(I - A)^k L_{s+k} . \tag{11}$$

In particular,

$$V_{0+k} = V_0(I - A)^k L_k . \tag{12}$$

Case (ii). From (10),

$$V_{-s}U_s(I - A) = V_{-s}\begin{pmatrix} I_{n-s} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} M & 0 \\ * & N \end{pmatrix} = (\hat{w}_{n-s}, 0_s)\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$$

where M has dimensions $(n - s) \times (n - s + 1)$ and so has 1's on the diagonal and (-1)'s on the first superdiagonal. But

$$(\hat{w}_{n-s+1}, 0_s)\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} = (\hat{w}_{n-s+1}, 0_{s-1})$$

where $\hat{w}_{n-s+1}$ has $n - s + 1$ elements

$$d_{s+i+1,i+1} - d_{s+i,i}, \ (i = 0, 1, \cdots, n - s + 1) ,$$

and $d_{s,0} = d_{n+1,n-s+1} = 0$. This is $V[A, D_{-s}] = V_{-s+1}$. Similarly,

$$V_{-s+2} = V_{-s+1}U_{s-1}(I - A) = V_{-s}U_s(I - A)U_{s-1}(I - A) .$$

But by Lemma 2, $U_s(I - A)U_{s-1} = U_s(I - A)$. Thus

$$V_{-s+2} = V_{-s}U_s(I - A)^2 ,$$

and by induction it follows that if $k \leqq s$,

$$V_{-s+k} = V_{-s}U_s(I - A)^k . \tag{13}$$

In particular,

$$V_{-s+s} = V_{-s}U_s(I - A)^s . \tag{14}$$

Case (iii). When $k > s$, we divide the problem into two parts. Using case (i) we have

$$V_{-s+k} = V_{-s+s}(I - A)^{k-s}L_{k-s} . \tag{15}$$

But by case (ii), $V_{-s+s} = V_{-s}U_s(I - A)^s$. Thus

$$\begin{aligned} V_{-s+k} &= V_{-s}U_s(I - A)^s(I - A)^{k-s}L_{k-s} \\ &= V_{-s}U_s(I - A)^k L_{k-s} . \end{aligned}$$

This completes the proof of the lemma.

Using the above lemmas we prove Theorem 1, which establishes a precise upper bound for k in the case where A is nilpotent and $[A, X]_k = P(A) \neq 0$.

THEOREM 1. *Let A be an $n \times n$ nilpotent nonderogatory matrix. Let p be a positive integer such that $p < n$. Let*

$$\lambda_i (i = p, p + 1, \cdots, n - 1)$$

be scalars from F such that $\lambda_p \neq 0$. Then there exists a matrix X such that

$$[A, X]_k = \sum_{i=p}^{n-1} \lambda_i A^i \neq 0 \tag{16}$$

if and only if $k \leqq 2p$.

Proof. We first prove the case where $\lambda_i = 0$ for all $i > p$. We may assume without loss of generality that $\lambda_p = 1$ since $[A, X]_k = A^p$ if and only if $[A, \lambda_p X]_k = \lambda_p A^p$.

If there exists a matrix X satisfying (16) where A is nilpotent, then $[A, X]_k = [A, \sum_{s=-n+1}^{n-1} D_s]_k = A^p$. Thus we must have

$$[A, D_{s-k}]_k = \begin{cases} 0 & \text{if } s \neq p \\ A^p & \text{if } s = p \end{cases} . \tag{17}$$

Therefore, for $s = p$,

$$\begin{aligned} V[A, D_{p-k}]_k &= V_{(p-k)+k} = VD_p = VA^p \\ &= (0, 0, \cdots, 0, 1, 1, \cdots, 1) , \end{aligned}$$

which we will call $(0_p, E_{n-p})$. If $k \leqq p$, from (11),

$$V_{(p-k)+k} = V_{p-k}(I - A)^k L_p .$$

Using an argument similar to that used in proving lemma 2, we find that $(I - A)^k L_p$ can be written as $\begin{pmatrix} 0 & 0 \\ 0 & N_k \end{pmatrix}$ where N_k has dimensions $(n - p + k) \times (n - p)$. Since this matrix has a square submatrix of order $n - p$ with 1's on the diagonal, zeros below, it has rank $n - p$.

Now rewriting (12) as

$$(0_p, E_{n-p}) = (0_{p-k}, w_{n-p+k}) \begin{pmatrix} 0 & 0 \\ 0 & N_k \end{pmatrix}$$

we see that solving this equation is equivalent to solving $E_{n-p} = (w_{n-p+k})N_k$. The augmented matrix for this equation is $\begin{pmatrix} N_k \\ E_{n-p} \end{pmatrix}$, and since N_k has rank $n - p$, the augmented matrix also has rank $n - p$. Thus the system has a solution with $(n - p + k) - (n - p) = k$ parameters.

Now if $k > p$ we refer to equation (15) and set

$$V_{(p-k)+k} = V_{p-k} U_{k-p} (I - A)^k L_p . \tag{18}$$

But the product on the right may be written as $\begin{pmatrix} 0 & H_k \\ 0 & 0 \end{pmatrix}$.

If $k = 2p$ then H_k is square of order $n - p$. Since it has minus signs in a checkerboard pattern, we may transform it into a matrix with nonnegative elements or nonpositive elements (depending on whether p is even or odd) by multiplying on the left and right by the matrix $D = \text{diag.}\ (-1, 1, -1, \cdots, (-1)^{n-p})$. Thus the determinant of H_k will be unchanged and the resulting matrix has determinant

$$(-1)^p \prod_{i=0}^{n-p-1} \frac{\binom{2p+i}{p}}{\binom{p+i}{p}} \neq 0$$

(see Muir, Vol. 3, p. 451). Hence H_k is nonsingular. Furthermore,

$(-1)^p H_k$ is positive definite since the principal subdeterminants are all positive by the same argument.

Thus if $k = 2p$ we may rewrite the equation (18) as

$$(0_p, E_{n-p}) = (\hat{w}_{n-p}, 0_p)\begin{pmatrix} 0 & H_k \\ 0 & 0 \end{pmatrix}.$$

But solving this system is equivalent to solving

$$E_{n-p} = \hat{w}_{n-p} H_k \,, \tag{19}$$

and since H_k is nonsingular, this system has a unique solution. A solution for $k = 2p$ implies the existence of matrices X satisfying $[A, X]_k = A^p$ for all $k < 2p$.

Next we show that there is no solution for $k = 2p + 1$, and thus for any $k > 2p$, by the following argument. Since H_k is nonsingular, equation (19) is equivalent to $E_{n-p} H_k^{-1} = \hat{w}_{n-p}$. Multiplying both sides of this equation by the $(n - p) \times 1$ column vector E_{n-p}^T gives

$$E_{n-p} H_k^{-1} E_{n-p}^T = \hat{w}_{n-p} E_{n-p}^T = \sum_{i=1}^{n-p} d_{p+i,i} \,. \tag{20}$$

This is the sum of the nonzero elements in D_{-p}. By Lemma 1, if $[A, X] = D_{-p}$, then $\sum_{i=1}^{n-p} d_{p+i,i} = 0$. But since $(-1)^p H_k$ is positive definite, $(-1)^p H_k^{-1}$ is also. Thus the product on the left in (20) is not zero and there does not exist a solution for $k > 2p$.

This completes the proof in the case where $[A, X]_k = \lambda A^p$. In the case where $[A, X]_k = \lambda_p A^p + \lambda_{p+1} A^{p+1} + \cdots + \lambda_{n-1} A^{n-1}$, we see that X may be written as $\sum_{i=p}^{n-1} X_{1i}$ where $[A, X_{1i}]_k = \lambda_i A^i$.

If A is derogatory then the Jordan canonical form for A is diag. $(A_1, A_2, \cdots, A_s)$ where $s > 1$. Theorem 1 can also be extended to the derogatory case. The method of proof is similar to that used in Theorem 1.

THEOREM 2. *Let A be an $n \times n$ nilpotent matrix. Let p be a positive integer such that $p < n_i$ where n_i is the dimension of the largest block in the Jordan canonical form for A. Let λ_i $(i = p, p + 1, \cdots, n - 1)$ be scalars from F such that $\lambda_p \neq 0$. Then there exists a matrix X such that*

$$[A, X]_k = \sum_{i=p}^{n_i - 1} \lambda_i A^i \neq 0 \tag{23}$$

if and only if $k \leqq 2p$.

Some remarks about the integer p are in order here. If the Jordan canonical form for A is diag. $(A_1, A_2, \cdots, A_s)$ we may assume without loss of generality that the dimension n_i of A_i is greater than or equal to the dimension n_{i+1} of A_{i+1} for $i = 1, 2, \cdots, s-1$. Since $A^p = \text{diag.}\ (A_1^p, A_2^p, \cdots, A_s^p)$, p must be less than n_1 if A^p is to be different from zero. However, A_i^p may be zero for some $i > 1$.

Notice that since the Jordan canonical form for a nilpotent matrix is the same as the rational canonical form for that matrix, the constructions for the matrices X in Theorems 1 and 2 may be done with rational operations.

The general case. Here it is not assumed that A is nilpotent. We assume that A is in Jordan canonical form. Again we choose a polynomial $P(A)$ which we desire to write as a higher commutator of A. Theorems 3 and 4 establish the maximal value for k in equation (4).

THEOREM 3. *Let A be an $n \times n$ nonderogatory matrix in Jordan canonial form $\alpha I + N$ where N is the nilpotent matrix with 1's on the first superdiagonal and zeros elsewhere. Let $P(A)$ be a polynomial in A such that $P(A) \neq 0$. Let t be the multiplicity of α as a root of $P(x)$. Then there exists an $n \times n$ matrix X such that*

$$[A, X]_k = P(A) \tag{24}$$

if and only if $k \leqq 2t$.

Proof. If $A = (\alpha I + N)$ then

$$[A, X]_k = [(\alpha I + N), X]_k = [\alpha I, X]_k + [N, X]_k = [N, X]_k\ .$$

Thus condition (24) becomes $[N, X]_k = P(\alpha I + N) = \sum_{i=1}^{n-1} \lambda_i N^i$ where $\lambda_i = p^{(i)}(\alpha)/i!$. Now by Theorem 1, (24) has a solution if and only if $k \leqq 2t$.

THEOREM 4. *Let $A = \text{diag.}\ (A_1, A_2, \cdots, A_s)$ where $A_i = (\alpha_i I + N_i)$ $(i = 1, 2, \cdots, s)$ where each N_i is as in Theorem 3. Let P be a polynomial such that $P(A) \neq 0$. Let $A_{i_1}, A_{i_2}, \cdots, A_{i_t}$ be the blocks of A such that $P(A_{i_j}) \neq 0$. Let m_{i_j} be the multiplicity of $(x - \alpha_{i_j})$ in $P(x)$. Let $m = \text{min.}\ \{m_{i_j}\}$. Then there exists an $n \times n$ matrix X such that*

$$[A, X]_k = P(A) \tag{25}$$

if and only if $k \leqq 2m$.

Proof. If $A = \text{diag.}\ (A_1, A_2, \cdots, A_s)$ then

$$P(A) = \text{diag.}\ (P(A_1), P(A_2), \cdots, P(A_s))\ .$$

If $P(A_t) = 0$ for some A_t, then there exists a matrix $X_t \neq 0$ such that

$[A_t, X_t]_k = P(A_t) = 0$ for any positive integer k. Thus we need only consider those A_i for which $P(A_i) \neq 0$. Assume that $P(A_i) \neq 0$ for all $i = 1, 2, \cdots, s$. Then if we let

$$X = \text{diag.}\ (X_1, X_2, \cdots, X_s)$$

where $[A_i, X_i]_k = P(A_i)$, the matrix X will satisfy (25). Assume without loss of generality that the degree of $(x - \alpha_1)$ in $P(x)$ is $m = \min. \{m_i\}$. Then $[A_1, X_1] = P(A_1)$ if and only if $k \leqq 2m$. Thus $[A, X]_k = P(A)$ if and only if $k \leqq 2m$.

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Received December 6, 1968.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN DENSITY II

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Let S denote the set of all infinite increasing sequences of positive integers. For all $A \cong \{a_n\}$ and $B = \{b_n\}$ in S define the metric $\rho(A, B) = 0$ if $A = B$; i.e., if $a_n = b_n$ for all n and $\rho(A, B) = 1/k$ otherwise, where k is the smallest value of n for which $a_n \neq b_n$. The main object of this note is to show that the set of points of continuity of the Schnirelmann density $d(A)$ is a residual set and that this is the best possible result of this type.

The space S and some of the properties of densities defined on it have been discussed previously [2, 3, 4]. In particular, it has been shown that the set of points of continuity of $d(A)$ is the set of all points having density zero. Let $L_a = \{A \in S \mid d(A) = a\}(0 \leqq a \leqq 1)$ denote the level sets of $d(A)$ and define $M_a = \{A \in S \mid d(A) \geqq a\}$. Then $\bar{L}_a = M_a$ so that M_a is closed and L_a is dense in M_a [4]. These results are required in the sequel. A brief and lucid account of all other necessary topological results is given in [1].

THEOREM 1. *The family of all sets of the form $S(m, n) = \{A \in S \mid a_n = m\}$ is a sub-basis for the topology of S.*

Proof. If $A \in S(m, n)$ and $B \notin S(m, n)$, then $\rho(A, B) \geqq 1/n$. Hence $S - S(m, n)$ is closed and $S(m, n)$ is open. Also, the spheres $S_\varepsilon(A) = \{B \in S \mid \rho(A, B) < \varepsilon\}$, $0 < \varepsilon \leqq 1$, constitute a basis for S and the desired result follows since

$$S_\varepsilon(A) = \bigcap_{n=1}^{[1/\varepsilon]} S(a_n, n) .$$

COROLLARY. *S has a countable basis.*

COROLLARY. *S is separable.*

It is also clear that S is a subspace of $\times_{n=1}^{\infty} P_n$, where P_n is the set of all positive integers with the discrete topology for each n.

THEOREM 2. *S is complete.*

Proof. Let $A_n = \{a_{n,\nu}\}_{\nu=1}^{\infty}$ and suppose that $\{A_n\}$ is a Cauchy sequence in S. Also, let n_k be the smallest positive integer such that

$\rho(A_m, A_n) < 1/k$ for all $m, n \geqq n_k$ and define $A = \{a_{n_k,k}\}_{k=1}^{\infty}$. Since all of the A_n's have the same first k terms for $n \geqq n_k$, it is clear that $A \in S$ and $\rho(A_n, A) < 1/k$ for all $n \geqq n_k$. Hence $\lim_{n\to\infty} \rho(A_n, A) = 0$ and S is complete.

The following corollaries are a consequence of the Baire category theorem and the fact that M_a is a closed subset of S.

COROLLARY. *M_a is complete.*

COROLLARY. *M_a is a set of the second category in itself.*

The following result would be of no interest for those values of a for which the second of the above corollararies fails to hold.

THEOREM 3. *L_a is residual in M_a.*

Proof. $M_a - L_a = \bigcup_{k=1}^{\infty} M_{a+1/k}$. Since $\bar{L}_a = M_a$, L_a is dense in M_a and, since $M_{a+1/k} \subset M_a$, L_a is dense in $M_{a+1/k}$. Also, since $M_{a+1/k}$ is closed, $M_{a+1/k}$ is nowhere dense in M_a and $M_a - L_a$ is a set of the first category in M_a.

Since the set of points of continuity of $d(A)$ is L_0 and $M_0 = S$, the following result ensues.

COROLLARY. *The set of points of continuity of $d(A)$ is residual in S.*

The following theorem shows that the above corollary is a best possible result in the following sense. In the true statement, $S - L_0$ is a countable union of nowhere dense sets, the word countable can not be replaced by finite.

THEOREM 4. *$\overline{M_a - L_a}$ is open if and only if $a = 0$ or 1.*

Proof. $\overline{M_1 - L_1}$ is the empty set and hence open. Also, it is easily seen that $\overline{M_0 - L_0} = S(1, 1)$ in the notation of Theorem 1 and hence open.

Suppose that $\overline{M_a - L_a}$ is open for $a > 0$. Then $\overline{M_a - L_a} \subset M_a$, since M_a is closed, and it follows that $L_0 \subset S - \overline{M_a - L_a}$. Since $\bar{L}_0 = S$ and $S - \overline{M_a - L_a}$ is closed, we have $S - \overline{M_a - L_a} = S$ and $\overline{M_a - L_a}$ is the empty set. Thus $a = 1$ and the proof is complete.

The following result is included in the preceding proof.

COROLLARY. *The support of* d (A) *is the set of all sequences with first term one.*

The final result concerns the asymptotic density

$$\delta(A) = \liminf A(k)/k ,$$

where $A(k)$ denotes the number of elements of A which do not exceed k.

THEOREM 5. $\delta(A)$ *is a function of Baire class two.*

Proof. Let $\delta_n(A) = \inf_{k \geqq n} A(k)/k$. Then $\delta_n(A)$ is a function of Baire class one [4, Th. 3]. Also, $\delta(A) = \lim_{n \to \infty} \delta_n(A)$. Now $\delta(A)$ is obviously everywhere discontinuous on S. Suppose $\delta(A)$ is a function of Baire class one. Then the set of points of discontinuity of $\delta(A)$ is a set of the first category [5, Th. 36]. But S is a set of the second category and the desired result follows.

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Received February 6, 1968, and in revised form July 15, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

COEFFICIENT MULTIPLIERS OF H^p AND B^p SPACES

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This paper describes the coefficient multipliers of $H^p(0 < p < 1)$ into $\ell^q(p \leqq q \leqq \infty)$ and into $H^q(1 \leqq q \leqq \infty)$. These multipliers are found to coincide with those of the larger space B^p into $\ell^q(1 \leqq q \leqq \infty)$ and into $H^q(1 \leqq q \leqq \infty)$. The multipliers of H^p and B^p into $B^q(0 < p < 1, 0 < q < 1)$ are also characterized.

A function f analytic in the unit disk is said to be of class $H^p(0 < p < \infty)$ if

$$M_p(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}$$

remains bounded as $r \to 1$. H^∞ is the space of all bounded analytic functions. It was recently found ([2], [4]) that if $p < 1$, various properties of H^p extend to the larger space B^p consisting of all analytic functions f such that

$$\int_0^1 (1 - r)^{1/p-2} M_1(r, f) dr < \infty \ .$$

Hardy and Littlewood [8] showed that $H^p \subset B^p$.

A complex sequence $\{\lambda_n\}$ is called a *multiplier* of a sequence space A into a sequence space B if $\{\lambda_n a_n\} \in B$ whenever $\{a_n\} \in A$. A space of analytic functions can be regarded as a sequence space by identifying each function with its sequence of Taylor coefficients. In [4] we identified the multipliers of H^p and $B^p(0 < p < 1)$ into ℓ^1. We have also shown ([2], Th. 5) that the sequence $\{n^{1/q-1/p}\}$ multiplies B^p into B^q. We now extend these results by describing the multipliers of $H^p(0 < p < 1)$ into $\ell^q(p \leqq q \leqq \infty)$, of B^p into $\ell^q(1 \leqq q \leqq \infty)$, and of both H^p and B^p into $B^q(0 < q < 1)$. We also extend a theorem of Hardy and Littlewood (whose proof was never published) by characterizing the multipliers of H^p and B^p into $H^q(0 < p < 1 \leqq q \leqq \infty)$. In almost every case considered, the multipliers of B^p into a given space are the same as those of H^p.

2. Multipliers into ℓ^q.

We begin by describing the multipliers of H^p and B^p into ℓ^∞, the space of bounded complex sequences.

THEOREM 1. *For $0 < p \leqq 1$, a sequence $\{\lambda_n\}$ is a multiplier of H^p into ℓ^∞ if and only if*

$$(1) \qquad \lambda_n = O(n^{1-1/p}) \,.$$

For $p < 1$, the condition (1) *also characterizes the multipliers of B^p into ℓ^∞.*

Proof. If $f(z) = \sum a_n z^n$ is in B^p, then by Theorem 4 of [2],

$$(2) \qquad a_n = o(n^{1/p-1}) \,.$$

If $f \in H^1$, then $a_n \to 0$ by the Riemann-Lebesgue lemma. This proves the sufficiency of (1). Conversely, suppose $\{\lambda_n\}$ is a multiplier of H^p into ℓ^∞. Then the closed linear operator

$$\Lambda : f \longrightarrow \{\lambda_n a_n\}$$

maps H^p into ℓ^∞. Thus Λ is bounded, by the closed graph theorem (which applies since H^p is a complete metric space with translation invariant metric; see [1], Chapter 2). In other words,

$$(3) \qquad \sup_n |\lambda_n a_n| = ||\Lambda(f)|| \leqq K ||f|| \,.$$

Now let

$$g(z) = (1 - z)^{-1-1/p} = \sum b_n z^n \,,$$

where $b^n \sim B n^{1/p}$; and choose $f(z) = g(rz)$ for fixed $r < 1$. Then by (3)

$$|\lambda_n| \, n^{1/p} r^n \leqq C(1 - r)^{-1} \,.$$

The choice $r = 1 - 1/n$ now gives (1). Note that $\{\lambda_n\}$ multiplies H^p or B^p into ℓ^∞ if and only if it multiplies into c_0 (the sequences tending to zero).

As a corollary we may show that the estimate (2) is best possible in a rather strong sense. For functions of class H^p, this estimate is due to Hardy and Littlewood [8]. Evgrafov [6] later showed that if $\{\delta_n\}$ tends monotonically to zero, then there is an $f \in H^p$ for which $a_n \neq O(\delta_n n^{1/p-1})$. A simpler proof was given in [5]. The result may be reformulated: if $a_n = O(d_n)$ for all $f \in H^p$, then $d_n n^{1-1/p}$ cannot tend monotonically to zero. We can now sharpen this statement as follows.

COROLLARY. *If $\{d_n\}$ is any sequence of positive numbers such that $a_n = O(d_n)$ for every function $\sum a_n z^n$ in H^p, then there is an $\varepsilon > 0$ such that*

$$d_n n^{1-1/p} \geqq \varepsilon > 0 \,, \qquad n = 1, 2, \cdots \,.$$

Proof. If $a_n = O(d_n)$ for every $f \in H^p$, then $\{1/d_n\}$ multiplies H^p into ℓ^∞. Thus $1/d_n = O(n^{1-1/p})$, as claimed.

We now turn to the multipliers of H^p and B^p into $\ell^q(q < \infty)$, the space of sequences $\{c_n\}$ with $\sum |c_n|^q < \infty$. The following theorem generalizes a previously known result [4] for ℓ^1.

THEOREM 2. *Suppose* $0 < p < 1$.

(i) *A complex sequence* $\{\lambda_n\}$ *is a multiplier of* H^p *into* $\ell^q(p \leqq q < \infty)$ *if and only if*

$$\sum_{n=1}^{N} n^{q/p} |\lambda_n|^q = O(N^q) . \tag{4}$$

(ii) *If* $1 \leqq q < \infty$, $\{\lambda_n\}$ *is a multiplier of* B^p *into* ℓ^q *if and only if* (4) *holds.*

(iii) *If* $q < p$, *the condition* (4) *does not imply that* $\{\lambda_n\}$ *multiplies* H^p *into* ℓ^q; *nor does it imply that* $\{\lambda_n\}$ *multiplies* B^p *into* ℓ^q *if* $q < 1$.

Proof. (i) A summation by parts (see [4]) shows that (4) is equivalent to the condition

$$\sum_{n=N}^{\infty} |\lambda_n|^q = O(N^{q(1-1/p)}) . \tag{5}$$

Assume without loss of generality that $\lambda_n \geqq 0$ and $\sum_{n=1}^{\infty} \lambda_n^q = 1$. Let $s_1 = 0$ and

$$s_n = 1 - \left\{\sum_{k=n}^{\infty} \lambda_k^q\right\}^{1/\beta}, \qquad n = 2, 3, \cdots ,$$

where $\beta = q(1/p - 1)$. Note that s_n increases to 1 as $n \to \infty$. By a theorem of Hardy and Littlewood ([8], p. 412), $f \in H^p(0 < p < 1)$ implies

$$\int_0^1 (1 - r)^{\beta-1} M_1^q(r, f)dr < \infty , \qquad p \leqq q < \infty . \tag{6}$$

Thus if $f(z) = \sum a_n z^n$ is in H^p and $\{\lambda_n\}$ satisfies (4) with $p \leqq q < \infty$, it follows that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} \int_{s_n}^{s_{n+1}} (1 - r)^{\beta-1} M_1^q(r, f)dr \\ &\geqq \sum_{n=1}^{\infty} |a_n|^q \int_{s_n}^{s_{n+1}} (1 - r)^{\beta-1} r^{nq} dr \\ &\geqq \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \int_{s_n}^{s_{n+1}} (1 - r)^{\beta-1} dr \\ &= \frac{1}{\beta} \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \{(1 - s_n)^\beta - (1 - s_{n+1})^\beta\} \\ &= \frac{1}{\beta} \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \lambda_n^q ,\end{aligned}$$

by the definition of s_n. But by (5),

$$\left\{\sum_{k=n}^{\infty} \lambda_k^q\right\}^{1/\beta} \leqq \frac{C}{n},$$

which shows, by the definition of s_n, that

$$(s_n)^{nq} \geqq (1 - C/n)^{nq} \longrightarrow e^{-Cq} > 0 .$$

Since these factors $(s_n)^{nq}$ are eventually bounded away from zero, the preceding estimates show that $\sum |a_n|^q \lambda_n^q < \infty$. In other words, $\{\lambda_n\}$ is a multiplier of H^p into ℓ^q if it satisfies the condition (4).

(ii) The above proof shows that $\{\lambda_n\}$ multiplies B^p into ℓ^1 under the condition (4) with $q = 1$. (This was also shown in [4].) The more general statement (ii) now follows by showing that if $\{\lambda_n\}$ satisfies (4), then the sequence $\{\mu_n\}$ defined by

$$\mu_n = |\lambda_n|^q \, n^{(1/p-1)(q-1)}$$

satisfies (4) with $q = 1$. Hence $\{\mu_n\}$ is a multiplier of B^p into ℓ^1, and in view of (2), $\{\lambda_n\}$ is a multiplier of B^p into ℓ^q. Alternatively, it can be observed that $f \in B^p$ implies (6) for $1 \leqq q < \infty$, so that the foregoing proof applies directly. Indeed, if $f \in B^p$, then (as shown in [2], proof of Theorem 3)

$$M_1(r, f) = O((1 - r)^{1-1/p}) ;$$

hence, if $1 \leqq q < \infty$,

$$\int_0^1 (1 - r)^{q(1/p-1)-1} M_1^q(r, f)dr \leqq C\int_0^1 (1 - r)^{1/p-2} M_1(r, f)dr < \infty .$$

(iii) That (4) does not imply $\{\lambda_n\}$ multiplies H^p into $\ell^q (q < p)$ or B^p into $\ell^q (q < 1)$, follows from the fact [4] that the series

$$\sum_{n=1}^{\infty} n^{q(1-1/p)-1} |a_n|^q$$

may diverge if $f \in H^p$ and $q < p$, or if $f \in B^p$ and $q < 1$.

To show the necessity of (4), we again appeal to the closed graph theorem. If $\{\lambda_n\}$ multiplies H^p into $\ell^q (0 < p < \infty, 0 < q < \infty)$, then

$$\Lambda : f \longrightarrow \{\lambda_n a_n\}$$

is a bounded operator:

$$\left\{\sum_{n=0}^{\infty} |\lambda_n a_n|^q\right\}^{1/q} \leqq C\,||f|| , \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p .$$

Choosing $f(z) = g(rz)$ as in the proof of Theorem 1, we now find

$$\left\{\sum_{n=1}^{\infty} n^{q/p} |\lambda_n|^q r^{nq}\right\}^{1/q} \leqq C(1-r)^{-1} ;$$

and (4) follows after terminating this series at $n = N$ and setting $r = 1 - 1/N$. Note that the argument shows (4) is necessary even if $p \geqq 1$ or $q < p$.

COROLLARY 1. *If $\{n_k\}$ is a lacunary sequence of positive integers $(n_{k+1}/n_k \geqq Q > 1)$, and if $f(z) = \sum a_n z^n$ is in $H^p(0 < p < 1)$, then*

$$\sum_{k=1}^{\infty} n_k^{q(1-1/p)} |a_{n_k}|^q < \infty , \qquad p \leqq q < \infty .$$

COROLLARY 2. *If $f(z) = \sum a_n z^n$ is in $H^p(0 < p < 1)$, then $\sum n^{p-2} |a_n|^p < \infty$.*

The first corollary extends a theorem of Paley [13] that $f \in H^1$ implies $\{a_{n_k}\} \in \ell^2$. The second is a theorem of Hardy and Littlewood [7]. It is interesting to ask whether the converse to Corollary 1 (with $q = p$) is valid. That is, if $\{c_k\}$ is a given sequence for which

$$\sum_{k=1}^{\infty} n_k^{p-1} |c_k|^p < \infty ,$$

then is there a function $f(z) = \sum a_n z^n$ in H^p with $a_{n_k} = c_k$? We do not know the answer.

Hardy and Littlewood [9] also proved that $\{\lambda_n\}$ multiplies H^1 into H^2 (alias ℓ^2) if (and only if)

$$\sum_{n=1}^{N} n^2 |\lambda_n|^2 = O(N^2) .$$

From this it is easy to conclude that (4) characterizes the multipliers of H^1 into ℓ^q, $2 \leqq q < \infty$. Indeed, let $\{\lambda_n\}$ satisfy (4) and let $\mu_n = |\lambda_n|^{q/2}$. Then, by the Hardy-Littlewood theorem, $\{\mu_n\}$ multiplies H^1 into ℓ^2 (see [3], p. 253). Hence $\{\lambda_n\}$ multiplies H^1 into ℓ^q. (See also Hedlund [12].)

On the other hand, the condition (4) is *not* sufficient if $p = 1$ and $q < 2$. This may be seen by choosing a lacunary series

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k} , \qquad n_{k+1}/n_k \geqq Q > 1 ,$$

with $\sum |c_k|^2 < \infty$ but $\sum |c_k|^q = \infty$ for all $q < 2$. The sequence $\{\lambda_n\}$ with $\lambda_n = 1$ if $n = n_k$ and $\lambda_n = 0$ otherwise then satisfies (4) but does not multiply H^1 into ℓ^q, $q < 2$.

3. Multipliers into B^q. The following theorem may be regarded as a generalization of our previous result ([2], Th. 5) that if $f \in B^p$, then its fractional integral of order $(1/p - 1/q)$ is in B^q. (A fractional integral of negative order is understood to be a fractional derivative.)

THEOREM 3. *Suppose* $0 < p < 1$ *and* $0 < q < 1$. *Let* ν *be the positive integer such that* $(\nu + 1)^{-1} \leqq p < \nu^{-1}$. *Then* $\{\lambda_n\}$ *is a multiplier of* H^p *or* B^p *into* B^q *if and only if* $g(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ *has the property*

$$M_1(r, g^{(\nu)}) = O((1 - r)^{1/p-1/q-\nu}) . \tag{7}$$

Proof. Let $\{\lambda_n\}$ satisfy (7), let $f(z) = \sum a_n z^n$ be in B^p, and let $h(z) = \sum \lambda_n a_n z^n$. Then

$$h(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it}) g(z e^{-it}) dt , \qquad 0 < \rho < 1 .$$

Differentiation with respect to z gives

$$\rho^\nu h^{(\nu)}(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it}) g^{(\nu)}(z e^{-it}) e^{-i\nu t} dt . \tag{8}$$

Hence

$$\begin{aligned} \rho^\nu M_1(r\rho, h^{(\nu)}) &\leqq M_1(r, g^{(\nu)}) M_1(\rho, f) \\ &\leqq C(1 - r)^{1/p-1/q-\nu} M_1(\rho, f) , \end{aligned}$$

where $r = |z|$. Taking $r = \rho$, we now see that $f \in B^p$ implies $h^{(\nu)} \in B^s$, $1/s = 1/q + \nu$. Thus $h \in B^q$, by Theorem 5 of [2].

Conversely, let $\{\lambda_n\}$ multiply H^p into B^q. Then by the closed graph theorem,

$$\Lambda\colon \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n$$

is a bounded operator from H^p to B^q. If $(\nu + 1)^{-1} \leqq p < \nu^{-1}$, let

$$f(z) = \nu!\, z^\nu (1 - z)^{-\nu-1} = \sum_{n=\nu}^{\infty} a_n z^n ,$$

where $a_n = n!/(n - \nu)!$, and observe that

$$h(z) = \sum_{n=\nu}^{\infty} \lambda_n a_n z^n = z^\nu g^{(\nu)}(z) . \tag{9}$$

Let $f_r(z) = f(rz)$ and $h_r(z) = h(rz)$. Since Λ is bounded, there is a constant C independent of r such that

$$\|h_r\|_{B^q} = \|\Lambda(f_r)\| \leqq C \|f_r\|_{H^p} .$$

In other words,

$$\int_0^1 (1-t)^{1/q-2} M_1(tr, h)dt \leqq CM_p(r, f)$$
$$= O((1-r)^{1/p-\nu-1}) .$$

It follows that

$$M_1(r^2, h)\int_r^1 (1-t)^{1/q-2}dt = O((1-r)^{1/p-\nu-1}) ,$$

or

$$M_1(r^2, h) = O((1-r)^{1/p-1/q-\nu}) .$$

But in view of (9), this proves (7).

COROLLARY. *The sequence $\{\lambda_n\}$ multiplies B^p into B^p if and only if*

$$(10) \qquad M_1(r, g') = O\left(\frac{1}{1-r}\right) .$$

Proof. If $p = q$, the condition (10) is equivalent to (7). (see [8], p. 435.) This corollary is essentially the same as a result of Zygmund ([14], Th. 1), who found the multipliers of the Lipschitz space Λ_α or λ_α into itself. Because of the duality between these spaces and B^p (see [2], §§3, 4), the multipliers from Λ_α to Λ_α and from λ_α to λ_α $(0 < \alpha < 1)$ are the same as those from B^p to B^p. Similar remarks apply to the spaces Λ_* and λ_*, also considered in [14].

4. **Multipliers into H^q.** By combining Theorem 3 with the simple fact that $f' \in B^{1/2}$ implies $f \in H^1$, it is possible to obtain a sufficient condition for $\{\lambda_n\}$ to multiply H^p into H^q, $0 < p < 1 \leqq q \leqq \infty$. However, this method leads to a sharp result only in the case $q = 1$. The following theorem provides the complete answer.

THEOREM 4. *Suppose $0 < p < 1 \leqq q \leqq \infty$, and let $(\nu+1)^{-1} \leqq p < \nu^{-1}$, $\nu = 1, 2, \cdots$. Then $\{\lambda_n\}$ is a multiplier of H^p or B^p into H^q if and only if $g(z) = \sum_{n=0}^\infty \lambda_n z^n$ has the property*

$$(11) \qquad M_q(r, g^{(\nu+1)}) = O((1-r)^{1/p-\nu-2}) .$$

Hardy and Littlewood ([9], [10]) stated in different terminology that (11) implies $\{\lambda_n\}$ is a multiplier of H^p into $H^q (0 < p < 1 \leqq q < \infty)$, but they never published the proof. Our proof will make use of the following lemma.

LEMMA. *Let f be analytic in the unit disk, and suppose*

$$\int_0^1 (1-r)^{\alpha} M_q(r, f')dr < \infty ,$$

where $\alpha > 0$ *and* $1 \leqq q \leqq \infty$. *Then*

$$\int_0^1 (1-r)^{\alpha-1} M_q(r, f)dr < \infty .$$

Proof of Lemma. Without loss of generality, assume $f(0) = 0$, so that

$$f(re^{i\theta}) = \int_0^r f'(se^{i\theta})e^{i\theta}ds .$$

The continuous form of Minkowski's inequality now gives

$$M_q(r, f) \leqq \int_0^r M_q(s, f')ds . \tag{12}$$

Hence an interchange of the order of integration shows that

$$\int_0^1 (1-r)^{\alpha-1} M_q(r, f)dr \leqq \frac{1}{\alpha}\int_0^1 (1-s)^{\alpha} M_q(s, f')ds ,$$

which proves the lemma.

Proof of Theorem 4. Suppose first that $\{\lambda_n\}$ satisfies (11). Given $f(z) = \sum a_n z^n$ in B^p, we are to show that $h(z) = \sum \lambda_n a_n z^n$ belongs to H^q. By (8), with ν replaced by $(\nu + 1)$, we have

$$\rho^{\nu+1} \,|\, h^{(\nu+1)}(\rho z) \,| \leqq \frac{1}{2\pi}\int_0^{2\pi} |\, f(\rho e^{it}) \,|\,|\, g^{(\nu+1)}(ze^{-it}) \,|\, dt .$$

Since $q \geqq 1$, it follows from Jensen's inequality ([11], § 6.14) that

$$\begin{aligned} \rho^{\nu+1} M_q(r\rho, h^{(\nu+1)}) &\leqq M_1(\rho, f) M_q(r, g^{(\nu+1)}) \\ &\leqq C(1-r)^{1/p-\nu-2} M_1(\rho, f) , \end{aligned}$$

where $r = |z|$ and (11) has been used. Now set $r = \rho$ and use the hypothesis $f \in B^p$ to conclude that

$$\int_0^1 (1-r)^{\nu} M_q(r, h^{(\nu+1)})dr < \infty .$$

But by successive applications of the lemma, this implies

$$\int_0^1 M_q(r, h')dr < \infty .$$

Thus, in view of the inequality (12), it follows that $h \in H^q$, which was to be shown.

Conversely, suppose $\{\lambda_n\}$ is a multiplier of H^p into H^q for arbitrary $q (0 < q \leqq \infty)$. Then by the closed graph theorem,

$$\Lambda: \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n$$

is a bounded operator from H^p to H^q. An argument similar to that used in the proof of Theorem 3 now leads to the estimate (11).

COROLLARY. *If $0 < p < 1 \leqq q \leqq \infty$ and $f \in B^p$, then its fractional integral $f_\alpha \in H^q$, where $\alpha = 1/p - 1/q$. This is false if $q < 1$.*

This corollary can also be proved directly. Indeed, since ([2], Th. 5) the fractional integral of order $(1/p - 1/s)$ of a B^p function is in B^s $(0 < s < 1)$, and since ([8], p. 415) the fractional integral of order $(1 - 1/q)$ of an H^1 function is in $H^q (1 \leqq q \leqq \infty)$, it suffices to show that $f' \in B^{1/2}$ implies $f \in H^1$. But this is easy; it follows from (12) with $q = 1$. That the corollary is false for $q < 1$ is a consequence of the fact ([2], Th. 5) that the fractional derivative of order $(1/p - 1/q)$ of every B^q function is in B^p.

The converse is also false. That is, if $f \in H^q$, its fractional derivative of order $(1/p - 1/q)$ need not be in $B^p (0 < p < 1 \leqq q \leqq \infty)$. As before, this reduces to showing that $f \in H^1$ does not imply $f' \in B^{1/2}$. To see this, let $f(z) = \sum c_k z^{n_k}$, where $\{n_k\}$ is lacunary, $\{c_k\} \in \ell^2$, and $\{c_k\} \notin \ell^1$. Then $f \in H^2 \subset H^1$, but $f' \notin B^{1/2}$, since it was shown in [4] (Th. 3, Corollary 2) that

$$\sum_{k=1}^{\infty} n_k^{1-1/p} |a_{n_k}| < \infty$$

whenever $\sum a_n z^n \in B^p$ and $\{n_k\}$ is a lacunary sequence.

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Received December 23, 1968. Supported in part by the National Science Foundation under Contract GP-7234.

UNIVERSITY OF MICHIGAN

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

ON A CLASS OF DIFFERENTIAL EQUATIONS FOR VECTOR-VALUED DISTRIBUTIONS

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The aim of this paper is to seek necessary and sufficient conditions on the linear operator A in a linear topological space in order that the Cauchy problem for the equation $U^{(\alpha)} - AU = T$ should be well set in the sense of distributions (see definition in § 2). Here $0 < \alpha < \infty$, $U^{(\alpha)}$ is the fractional derivative of U of order α. Such conditions are obtained for α integer $\geqq 3$ and then for any $\alpha > 0$, this time with the additional assumption of (at most) exponential growth of the solutions at infinity.

Throughout this paper E will be a quasi-complete, barreled locally convex linear topological space over the field C of complex numbers ([1], Chapter II, § 4; [2], Chapter II, § 1 and § 2), A a closed linear operator with domain $D(A)$ dense in E and range in E.

The equation

$$u^{(\alpha)}(t) = Au(t) \tag{1.1}$$

was studied in [3]. If α is an integer $\geqq 3$ and the Cauchy problem for (1.1) is "well posed" (strong solutions exist for a dense set of initial data, are unique and depend continuously on them) it was shown that

(a) $D(A) = E$ and A is continuous.

(b) The series $\sum_{j=0}^{\infty} t^j A^j u/(\alpha j)!$ converges in E for all t, u.

The solutions of (1.1) are actually holomorphic and can be expressed as $u(t) = \sum_{k=0}^{\alpha-1} S_k(t)u^{(k)}(0)$, where $S_k(t) = \sum_{j=0}^{\infty} t^{\alpha j+k} A^j u/(\alpha j + k)!$, $0 \leqq k \leqq \alpha - 1$. Conversely, conditions (a) and (b) imply that the Cauchy problem for (1.1) is well posed (see [3], Th. 3.1). A necessary and sufficient condition for the solutions of (1.1) to increase at most exponentially at ∞ is the existence of $R(\lambda; A)$ for large $|\lambda|$ and its analyticity at ∞ ([3], 3.3). It has been suggested by J. L. Lions that the above results will still hold if we only assume the Cauchy problem for (1.1)-or, rather, for its inhomogeneous version—to be well set in the sense of distributions (this notion was introduced by him in [7] for the case $\alpha = 1$). Lions also raised the question of whether the results could be extended to the case of *noninteger* $\alpha > 2$. We give here some partial answers to these questions. Under a special assumption, Theorem 3.1 of [3] is extended to the distribution setting, although only for α integer; Theorem 3.3 is also extended for all values of $\alpha > 2$. (Theorems 4.1 and 5.2 respectively.) We also examine the case

$\alpha < 2$, always with the assumption of exponential increase at infinity and give conditions on $R(\lambda; A)$ that insure the Cauchy problem to be well set (Theorem 6.1). For $\alpha = 1$, we obtain the condition of Lions for generation of distribution semigroups.

2. The Cauchy problem. We denote by R the real numbers. The symbol $\mathscr{E}$ stands for a family $\{|\cdot|, \cdots\}$ of semi-norms determining the topology of E (see [1], Chapter II, § 4; for instance, $\mathscr{E}$ can be taken as the family of all continuous semi-norms in E), i.e., such that the (generalized) sequence $\{u_\gamma\}$ converges to zero if and only if $|u_\gamma| \to 0$ for all $|\cdot| \in \mathscr{E}$. We assume $D(A)$, the domain of A endowed with the topology generated by the semi-norms $u \to |u|, u \to |Au|$, $|\cdot| \in \mathscr{E}$. $D(A)$ is under this topology a quasi-complete locally convex linear topological space.

In the following remarks the spaces $F, G, \cdots$ are as E, quasi-complete, barreled locally convex linear topological spaces. The space $\mathscr{L}(F, G)$ consists of all linear continuous operators from F into G endowed with the topology of uniform convergence on bounded sets of F. $\mathscr{L}(F, G)$ is a locally convex, quasi-complete linear topological space (see [2], Chapter III, § 3, no. 7; it is not necessary for this result that G be barreled). We shall write $\mathscr{L}(F)$ instead of $\mathscr{L}(F, F)$, F^* instead of $\mathscr{L}(F, C)$, application of an element $u^* \in F^*$ to $u \in F$ being denoted by $\langle u^*, u\rangle$ or $\langle u, u^*\rangle$.

We recall that the "equi-continuity principle" ([2], Chapter III, § 3, Théorème 2) holds in $\mathscr{L}(F, G)$ (thus in particular in $\mathscr{L}(F), F^*$); if $\{B_\gamma\}$ is a family of elements of $\mathscr{L}(F, G)$ such that $\{B_\gamma u\}$ is bounded in G that is, such that $\sup_\gamma |B_\gamma u| < \infty$ for any continuous seminorm $|\cdot|$ in G for every $u \in F$, then $\{B_\gamma\}$ is an equicontinuous family. This principle will be used many times in what follows, sometimes without explicit mention.

The space $\mathscr{D}_t$ (or simply $\mathscr{D}$) consists of all complex-valued functions $t \to \varphi(t)$ defined in R, infinitely differentiable there and with compact support; the space $\mathscr{D}_{s,t}$ is similarly defined but with reference to functions $(s, t) \to \varphi(s, t)$ of two variables. Both spaces will be endowed with their L. Schwartz topologies ([10], Chapter III). $\mathscr{D}_0$ consists of all $\varphi \in \mathscr{D}$ with support in $(0, \infty)$. By definition, a generalized sequence $\{\varphi_\gamma\}$ in $\mathscr{D}_0$ converges to zero if and only if the supports of the φ_γ are contained in a fixed compact subset of $(0, \infty)$ and $\varphi_\gamma^{(m)}(t) \to 0$ uniformly in $(0, \infty)$ for all $m \geqq 0$. The space $\mathscr{D}'(F)$ (or $\mathscr{D}'_t(F)$) of F-valued distributions of one variable is $\mathscr{L}(\mathscr{D}; F)$; similarly, $\mathscr{D}'_{s,t}(F) = \mathscr{L}(\mathscr{D}_{s,t}; F)$. The space $\mathscr{D}_0'(F)$ (F-valued distributions defined in $(0, \infty)$) is $\mathscr{L}(\mathscr{D}_0, F)$. For any real a, the subspace of $\mathscr{D}'(F)$ consisting of distributions with support in $[a, \infty)$ will be denoted by $\mathscr{D}'_{[a,\infty)}(F)$; it inherits the topology of $\mathscr{D}'(F)$. Finally,

$$\overline{\mathscr{D}}'_{+}(F) = \bigcup_{n=-\infty}^{\infty} \mathscr{D}'_{[n,\infty)}(F)^{1}$$

and we assign to $\overline{\mathscr{D}}'_{+}(F)$ the inductive limit of the topologies of the $\mathscr{D}'_{[n,\infty)}(F)$ ([1], Chapter II, § 4). As customary, we write $\mathscr{D}'(C) = \mathscr{D}'$ and similarly for other distribution spaces.

Let S be a distribution in, say, $\overline{\mathscr{D}}'_{+}(\mathscr{L}(E, F))$, $T \in \overline{\mathscr{D}}'_{+}(E)$. We recall briefly the definition of the convolution $S * T$ as given in [9] (Proposition 39, p. 167). It is a consequence of Proposition 33, p. 145, that there exists a unique distribution $V \in \mathscr{D}'_{s,t}(F)$ such that

$$V(\varphi \otimes \psi) = S(\varphi)T(\psi)$$

for any $\varphi, \psi \in \mathscr{D}$ (here $\varphi \otimes \psi$ denotes the function in $\mathscr{D}_{s,t}$ defined by $(\varphi \otimes \psi)(s, t) = \varphi(s)\psi(t))$ and whose support is contained in the Cartesian product $\operatorname{supp}(S) \times \operatorname{supp}(T)$. $S*T$ is then defined by

$$(S*T)(\varphi) = V(\hat{\varphi})$$

for any $\varphi \in \mathscr{D}$, where $\hat{\varphi}(s, t) = \varphi(s + t)$ (note that, since the intersection of the supports of V and $\hat{\varphi}$ is compact the expression $V(\hat{\varphi})$ has a sense although $\hat{\varphi} \notin \mathscr{D}_{s,t}$ if $\varphi \neq 0$). For any S, T we have

$$\operatorname{supp}(S*T) \subseteqq \operatorname{supp}(S) + \operatorname{supp}(T) ;$$

the convolution, as a linear map (we assume S fixed) from $\overline{\mathscr{D}}'_{+}(\mathscr{L}(E, F)) \times \overline{\mathscr{D}}'_{+}(E)$ into $\overline{\mathscr{D}}'_{+}(E)$ is continuous (see again [9], Proposition 39 for proofs of these and other facts). The convolution $S * T$ can be defined in a similar way when $S \in \overline{\mathscr{D}}'_{+}$ and $T \in \overline{\mathscr{D}}'_{+}(F)$, when $S \in \overline{\mathscr{D}}'_{+}(\mathscr{L}(E, F))$ and $T \in \overline{\mathscr{D}}'_{+}(\mathscr{L}(G, E))$, $\cdots$ and enjoys the same properties as in the previous case.

Fractional derivatives will be defined by means of convolutions. For any complex β we write

$$Y_{\beta} = \frac{1}{\Gamma(\beta)} \operatorname{Pf.} (t^{\beta-1})_{t>0} .$$

This distribution (see [10], Chapter II, § II, p. 43) coincides with the function $(h(t)t)^{\beta-1}/\Gamma(\beta)$ for $\operatorname{Re}\beta > 0$, h the Heaviside function ($h(t) = 0$ for $t < 0$, $h(t) = 1$ for $t > 1$). The function of β that results applying it to any $\varphi \in \mathscr{D}$ admits of an analytic extension to the entire

[1] Let $\mathscr{D}_{-}$ be the space of all infinitely differentiable functions in R with support bounded above endowed with its usual Schwartz topology ([**10**], Chapter VI, § 5, p. 172). Then $\overline{\mathscr{D}}'_{+}(F) \subseteqq \mathscr{D}'_{+}(F) = \mathscr{L}(\mathscr{D}_{-}, F)$ algebraic and topologically. The reverse inclusion is true if, say, F is a Banach space but not in general. See [**8**], p. 62, where a similar situation is discussed with reference to distributions with compact support.

plane, and its value at a given β is taken as definition of $Y_\beta(\varphi)$ (see [10], loc. cit. for details). For any β, $Y_\beta \in \mathscr{D}'$ and its support is contained in $[0, \infty)$. If $U \in \overline{\mathscr{D}}'_+(F)$, $0 \leqq \alpha < \infty$ we define

$$U^{(\alpha)} = \text{derivative of order } \alpha \text{ of } U = Y_{-\alpha} * U$$

([10], Chapter VI, § 5, p. 174); the definition is justified by the fact that $Y_{-n} = \delta^{(n)}$, $n = 0, 1, \cdots$. If $U = 0$ for $t < a$, the same it true of $U^{(\alpha)}$.

Finally, some notational conventions. If, say, $S \in \mathscr{D}'(\mathscr{L}(E, F))$, $u \in E$, we denote by Su the distribution (in $\mathscr{D}'(F)$) defined by $(Su)(\varphi) = S(\varphi)u$, $\varphi \in \mathscr{D}$. Similar definition for AS, where $A \in \mathscr{L}(F, G)$. Following [8], p. 51, if $T \in \mathscr{D}'$, $u \in F$, $T \otimes u$ is the distribution in $\mathscr{D}'(F)$ given by $(T \otimes u)(\varphi) = T(\varphi)u$. We shall use the same notation for an F-valued function and for the distribution (in $\mathscr{D}'(F)$) that it defines.

DEFINITION 2.1. Let $0 < \alpha < \infty$. The Cauchy problem for the equation

$$U^{(\alpha)} - AU = T \tag{2.1}$$

is well set (in the sense of distributions) if and only if

(a) (Existence) For every $T \in \overline{\mathscr{D}}'_+(E)$ there exists a solution $U \in \overline{\mathscr{D}}'_+(D(A))$ of (2.1).

(b) (Uniqueness) Let $U \in \overline{\mathscr{D}}'_+(D(A))$ be a solution of (2.1) with $T \in \overline{\mathscr{D}}'_+(E)$. Assume $T = 0$ if $t < a$. Then $U = 0$ for $t < a$.

(c) (Continuous dependence) Let $\{T_\gamma\}$ be a generalized sequence of elements of $\overline{\mathscr{D}}'_+(E)$ with $T_\gamma \to 0$ in $\mathscr{D}'(E)$, $T_\gamma = 0$ for $t < a$ $(a > -\infty)$ for all γ. Let $U_\gamma \in \overline{\mathscr{D}}'_+(D(A))$ be the corresponding solutions of (2.1). Then $U_\gamma \to 0$ in $\mathscr{D}'(D(A))$.

A few comments on (b) and (c) will be useful later. Observe first that (b) implies

(b′) Let $u(\cdot)$ be an infinitely differentiable $D(A)$-valued function vanishing for large negative t and such that

$$u^{(\alpha)}(t) - Au(t) = 0$$

for $t \leqq 0$. Then $u(t) = 0$ for $t \leqq 0$.

It is also true that (b') implies (b). To see this, let U, T be the two distributions of (b), $\varphi \in \mathscr{D}$ with support in $(-\infty, a)$. Define

$$\varphi_t(s) = \varphi(s - t),\ (I\varphi)(s) = \varphi(-s)\ .$$

Then if $u(t) = U(\varphi_t) = (U * I\varphi)(t)$, $u(\cdot)$ is a C^∞, $D(A)$-valued function and $u^{(\alpha)}(t) - Au(t) = T(\varphi_t)$. Since $T(\varphi_t) = 0$ for $t < 0$ (T is zero for $t < a$), we have $u(0) = U(\varphi) = 0$, which shows that U itself is zero for $t < a$ as claimed.

By definition of the inductive limit topology ([1], Chapter II, § 4, n^o 6) a generalized sequence $\{T_\gamma\}$ in $\overline{\mathscr{D}}'_+(F)$ converges to zero if and only if $T_\gamma \to 0$ in $\mathscr{D}'(F)$ and all the T_γ's are contained in a fixed $\mathscr{D}'_{[a,\infty)}(F)$ (that is, if $T_\gamma = 0$ for all γ and $t <$ some fixed a). This shows that (c) amounts to the assertion that the map

$$T \to U \tag{2.2}$$

from $\overline{\mathscr{D}}'_+(E)$ to $\overline{\mathscr{D}}'_+(D(A))$ given by the equation (2.1) (which map, by virtue of (a) and (b) is well-defined and linear) is *continuous*. It is also plain that the map (2.2) commutes with translations. We deduce more information about (2.2) by means of the following result.

AUXILIARY LEMMA 2.2 *Let $\mathscr{M}$ be a linear continuous operator from $\overline{\mathscr{D}}'_+(F)$ to $\overline{\mathscr{D}}'_+(G)$ commuting with translations. Assume, moreover that $\mathscr{M}T = 0$ in $t < a$ whenever $T = 0$ in $t < a$. Then there exists $S \in \overline{\mathscr{D}}'_+(\mathscr{L}(F, G))$ with support contained in $t \geqq 0$ such that*

$$\mathscr{M}T = S * T .$$

The proof is identical to that of the "scalar-valued" theorem ([10], Chapter VI, § 3, p. 162; see also [7], p. 150 for the Banach space case). We define a distribution $S \in \mathscr{D}'(\mathscr{L}(F, G))$ by the formula $S(\varphi)u = (\mathscr{M}(\delta \otimes u))(\varphi)$, $\varphi \in \mathscr{D}$, $u \in F$; since $\operatorname{supp} \mathscr{M}(\delta \otimes u) \subseteqq [0, \infty)$, $\operatorname{supp}(S) \subseteqq [0, \infty)$. Then $\mathscr{N}U = S * U$ is a linear continuous operator from $\overline{\mathscr{D}}'_+(F)$ into $\overline{\mathscr{D}}'_+(G)$. (See the previous remarks on convolution.) We only need now to verify the equality $\mathscr{M} = \mathscr{N}$ for distributions U of the form $\tau_a\delta \otimes u$, τ_a the operator of translation by a, $u \in F$[2]. But $\mathscr{M}(\tau_a\delta \otimes u) = \tau_a\mathscr{M}(\delta \otimes u) = \tau_a(Su) = S*(\tau_a\delta \otimes u) = \mathscr{N}(\tau_a\delta \otimes u)$, thus our result is proved.

Return now to the map $\mathscr{M}$ given by (2.2). The distribution $S \in \overline{\mathscr{D}}'_+(\mathscr{L}(E, D(A))$ corresponding to $\mathscr{M}$ will be called the *propagator* of 2.1. It follows from its definition that it satisfies the equation

$$S^{(\alpha)} - AS = \delta \otimes I . \tag{2.3}$$

We prove now a few simple properties of S.

LEMMA 2.3. *The operators $S(\varphi), S(\psi)$, A commute for any*

[2] The subspace of $\mathscr{D}'(F)$ generated by all elements of the form $U \otimes u$, $U \in \mathscr{D}'$ $u \in F$ can be identified with the tensor product $\mathscr{D}' \otimes F$ ([8], p. 50). But $\mathscr{D}' \otimes F$ is dense if $\mathscr{D}'(F)$; on the other hand, the subspace of $\mathscr{D}'$ generated by all elements of the form $\tau_a\delta, a \in R$ is dense in $\mathscr{D}'$ ([10], Chapter II, § 2, p. 75) so that the subspace of $\mathscr{D}'(F)$ generated by all elements of the form $\tau_a\delta \otimes u, a \in R, u \in F$ is dense in $\mathscr{D}'(F)$.

$\varphi, \psi \in \mathscr{D}$.

Proof. Let $u \in D(A)$. Since $U = A(Su) = (Su)^{(\alpha)} - \delta \otimes u$, it is clear that $U \in \overline{\mathscr{D}}'_+(D(A))$; moreover

$$U^{(\alpha)} - AU = A((Su)^{(\alpha)}) - A(A(Su)) = A(\delta \otimes u) = \delta \otimes Au .$$

By uniqueness, $U = S(Au)$, i.e.,

$$A(Su) = S(Au)$$

which shows that $S(\varphi)$ and A commute for any $\varphi \in \mathscr{D}$. As for commutativity of $S(\varphi)$, $S(\psi)$ one only needs to observe that

$$V = (S(\psi)S)*(\delta \otimes u) = S(\psi)(Su)$$

is a solution of $V^{(\alpha)} - AV = \delta \otimes S(\psi)u$ and reason as before.

LEMMA 2.4. *Let $\alpha = n$ be an integer $\geqq 1$, $\varphi, \psi \in \mathscr{D}_0$. Then*

$$S(\varphi * \psi) = \sum_{k=0}^{n-1} S^{(k)}(\varphi) S^{(n-1-k)}(\psi) . \tag{2.4}$$

The proof is a modification of one of Lions ([7], Théorème 5.1, p. 149) for the case $n = 1$. Let $(I\varphi)(t) = \varphi(-t)$ for any $\varphi \in \mathscr{D}$. Take now $\varphi, \psi \in \mathscr{D}_0$. Let $U, W, V_1, \cdots, V_n$ be the solutions in $\overline{\mathscr{D}}'_+(D(A))$ of the equations

$$U^{(n)} - AU = I\psi \otimes u \tag{2.5}$$

$$W^{(n)} - AW = (I\varphi * I\psi) \otimes u \tag{2.6}$$

$$V_k^{(n)} - AV_k = (I\varphi)^{(k)} \otimes U^{(n-1-k)}(0) \qquad (0 \leqq k \leqq n-1) , \tag{2.7}$$

$u \in E$. (Observe that $U, W, V_1, \cdots, V_n$ are obtained by convolution of the propagator with the right-hand members of (2.5), (2.6), (2.7) thus they are all C^∞ functions.)

Let now h be the Heaviside function. A simple computation shows that

$$(hU)^{(n)} = \sum_{k=0}^{n-1} \delta^{(n-1-k)} \otimes U^{(k)}(0) + hU^{(n)} .$$

Taking now into account the fact that U satisfies (2.5) and that $\operatorname{supp}(I\psi) \subset (-\infty, 0)$, $(hU)^{(n)} - A(hU) = \sum_{k=0}^{n-1} \delta^{(n-1-k)} \otimes U^{(k)}(0)$. Then by virtue of (2.7) we get, by uniqueness

$$I\varphi * hU = \sum_{k=0}^{n-1} V_k .$$

Similarly,

$$I\varphi * U = W .$$

Observe, finally, that since $\operatorname{supp}(I\varphi) \subset (-\infty, 0)$

$$(I\varphi * hU)(0) = (I\varphi * U)(0) .$$

Consequently

$$\begin{aligned} S(\varphi*\psi)u &= (S*(I(\varphi*\psi))(0)u = (S*(I\varphi*I\psi))(0)u = W(0) \\ &= (I\varphi*U)(0) = (I\varphi*(hU))(0) = \sum_{k=0}^{n-1} V_k(0) \\ &= \sum_{k=0}^{n-1} (S*((I\varphi)^{(k)}))(0)U^{(n-1-k)}(0) \\ &= \sum_{k=0}^{n-1} S^{(k)}(\varphi)(S^{(n-1-k)}*(I\psi \otimes u))(0) \\ &= \sum_{k=0}^{n-1} S^{(k)}(\varphi)S^{(n-1-k)}(\psi)u \text{ as claimed .} \end{aligned}$$

3. Some regularity results. The results in this section say, roughly speaking, that if u belongs to a set of "smooth elements" of E then Su will actually be a C^∞ function in $t \geqq 0$; moreover, if $u_\gamma \to 0$ "rapidly enough," then Su_γ will converge to zero in a topology considerably stronger than that of $\mathscr{D}'(E)$. We also examine certain smooth solutions of (2.1). As in the last part of the previous section we assume $\alpha = n =$ integer $\geqq 1$.

We introduce at this point a special hypothesis on S, namely

ASSUMPTION 3.1. S is a distribution of finite order locally,(3) which will be assumed to hold throughout the rest of this section (as well as in § 4).

Recall ([8], Proposition 24, p. 86) that, under the preceding hypothesis, if Ω is any open bounded interval in R then there exists a continuous $\mathscr{L}(E, D(A))$-valued function defined in Ω and such that

$$S = f^{(p)} \quad \text{in} \quad \Omega \tag{3.1}$$

(the integer $p \geqq 0$ may depend on Ω).

Let D be the subspace of E consisting of all $u \in E$ such that Su

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3 Assumption 3.1 is unnecessary whenever E is a Banach space-or, more generally, when $\pounds(ED(A))$ is a (DF)-space ([8], § 3). It is also unnecessary, with no special restriction on E, when $n = 1$; for the solution of $U' - AU = \delta \otimes S(\varphi)u$, $\varphi \in \mathscr{D}_0$, $u \in E$ is $U = hV$, V the (C^∞) solution of $V' - AV = I\varphi \otimes u$ ([7], p. 152), which allows one to establish all the following regularity results. We do not know whether Assumption 3.1 can be altogether eliminated in all cases.

coincides with a $D(A)$-valued function $g(t)$, infinitely differentiable in $t > 0$ and such that $g^{(m)}(0+) = \lim_{t\to 0+} g^{(m)}(t)$ exists for all $m \geqq 0$ (all in the topology of $D(A)$). For $u \in D$ we define $G(t)u = g(t)$. For fixed $t \geqq 0$ $G(t)$ is a linear operator in E with domain D.

LEMMA 3.2. $D = D(A^\infty) = \bigcap_{m=1}^\infty D(A^m)$.

Proof. Let $u \in D(A^\infty)$. Consider the identity

$$S^{(n)}u - A(Su) = \delta \otimes u \tag{3.2}$$

(which is a simple consequence of the definition of S). Differentiating (3.2) repeatedly and making use at each step of the commutativity of A and S (Lemma 2.3) we obtain for $m \geqq 1$.

$$\begin{aligned} S^{(mn)}u &= A(S^{(m-1)n}u) + \delta^{(m-1)n} \otimes u = S^{(m-1)n}Au \\ &+ \delta^{(m-1)n} \otimes u = S^{(m-2)n}A^2u + \delta^{(m-1)n} \otimes u \\ &+ \delta^{(m-2)n} \otimes Au = \cdots = SA^m u + \sum_{k=}^{m-1} \delta^{(kn)} \otimes A^{m-1-k}u \, . \end{aligned}$$

Let now Ω be an open interval in R (say $(-a, a)$, $0 < a < \infty$), f the function associated with S in Ω by (3.1). We have

$$S^{(mn)}u = f^{(p)}A^m u + \sum_{k=0}^{m-1} \delta^{(kn)} \otimes A^{m-1-k}u \tag{3.3}$$

in Ω. Choose a m with $mn > p$ and integrate the differential equation (3.3). We obtain

$$S^{(mn-p)}u = fA^m u + \sum_{k=0}^{m-1} Y_{p-kn} \otimes A^{m-1-k}u + P_{m,p} \tag{3.4}$$

in Ω, Y_β the distribution $\in \mathscr{D}'$ defined in § 2, (here we are using the fact that $Y_0 = \delta$, $Y_\alpha^{(\beta)} = Y_{\alpha-\beta}$), $P_{m,p}$ a polynomial of degree $\leqq p-1$ with coefficients in $D(A)$. Since m is arbitrary, it is clear that Su coincides with a C^∞ function in $\{t \in \Omega; t > 0\}$; reasoning in this way for any Ω we see that $u \in D$.

Conversely, assume $u \in D$, and let $g(t) = G(t)u$, for $t \geqq 0$, $U_k(t) = g^{(k)}(t)$ for $t \geqq 0$, $U_k(t) = 0$ for $t < 0$. Call $U = U_0$. Then we have

$$\begin{aligned} U' &= \delta \otimes U(0) + U_1, \cdots \\ U^{(n)} &= \sum_{k=1}^{n-1} \delta^{(n-1-k)} \otimes U^{(k)}(0) + U_n \, . \end{aligned}$$

Observe now that $U = Su$ satisfies $U^{(n)} - AU = \delta \otimes u$; since U is C^∞ in $t > 0$ it satisfies $U^{(n)}(t) = AU(t)$ there. Consequently $\delta \otimes u = U^{(n)} - AU = \sum_{k=0}^{n-1} \delta^{(n-1-k)} \otimes U^{(k)}(0)$; equating coefficients we obtain

$$g(0+) = \cdots = g^{(n-2)}(0+) = 0,\ g^{(n-1)}(0+) = u\ .$$

Observe next that for all $m \geqq 0$ we have

$$g^{(m+n)}(t) = Ag^{(m)}(t),\ t > 0\ .$$

(This is obtained by differentiating the equality for $m = 0$.) Taking $m = n - 1$ and letting $t \to 0$ we obtain $Au = Ag^{(n-1)}(0+) = g^{(2n-1)}(0+)$ which belongs to $D(A)$; then $u \in D(A^2)$ and $A^2u = Ag^{(2n-1)}(0+) = g^{(3n-1)}(0+), \cdots$ etc. An examination of the initial values of g readily shows

COROLLARY 3.3. *Let $u \in D$. Then $G^{(k)}(0+)u = 0$ if $k \neq mn - 1$, $k \geqq 0$ $G^{(mn-1)}(0+)u = A^{m-1}u$, $m \geqq 1$.*

Our next step is to show that $D = D(A^\infty)$ contains "enough" elements. Observe first that if $\varphi \in \mathscr{D}_0$ then $S(\varphi)u \in D(A^\infty)$ for any $u \in E$; for, since $S^{(n)}u - A(Su) = \delta \otimes u$ and $\mathrm{supp}\,(\varphi) \subset (0, \infty)$, $AS(\varphi) = S^{(n)}(\varphi)u$, thus $AS(\varphi)u \in D(A)$ and $A^2S(\varphi)u = S^{(2n)}(\varphi)u$. Repeating the preceding reasoning we see that $S(\varphi)u \in D(A^m)$ for any $m \geqq 0$ and

$$A^mS(\varphi)u = S^{(mn)}(\varphi)u\ . \tag{3.5}$$

LEMMA 3.4. *D is dense in E.*

We shall actually show a stronger result, namely that the subspace generated by $D_0 = \{v \in E;\ v = S(\varphi)u,\ \varphi \in \mathscr{D}_0,\ u \in E\}$ is dense in $D(A)$. Assume this is false. Then there exists $u^* \in D(A^*)$ such that $\langle u^*, S(\varphi)u\rangle = \langle S(\varphi)^*u^*, u\rangle = 0$ for all $u \in E$, $\varphi \in \mathscr{D}_0$, i.e.,

$$S(\varphi)^*u^* = 0 \tag{3.6}$$

for all $\varphi \in \mathscr{D}_0$, where $S(\varphi)^*: D(A)^* \to E^*$ denotes the operator adjoint to $S(\varphi)$. Let now K be any bounded set in E and let $\varphi \in \mathscr{D}$. Since

$$\sup_{u \in K} |\langle S(\varphi)^*u^*, u\rangle| = \sup_{u \in K} |\langle u^*, S(\varphi)u\rangle|$$

it follows from the fact that S is a $\mathscr{L}(E, D(A))$-valued distribution and from the definition of the topology of E^* that $\varphi \to S(\varphi)^*u^* = U(\varphi)$ belongs to $\mathscr{D}'(E^*)$. By applying u^* to elements of E of the form $(-1)^nS(\varphi^{(n)})u - S(\varphi)Au = (-1)^nS(\varphi^{(n)})u - AS(\varphi)u = \varphi(0)u$, $u \in D(A)$, we also see that U satisfies the equation

$$U^{(n)} - A^*U = \delta \otimes u^* \tag{3.7}$$

where $A^*: E^* \to D(A)^*$ is the adjoint of A.[4] Since S has its support

[4] Since $D(A) \subseteqq E$ algebraically and topologically and $D(A)$ is dense in E, we can identify E^* with a subspace of $D(A)^*$, and the inclusion $E^* \subseteqq D(A)^*$ is also topologic.

in $t \geqq 0$, so does U; but, since the vanishing of the expression (3.6) for all $\varphi \in \mathscr{D}_0$ means that U is zero for $t > 0$ we see that $\operatorname{supp}(U)$ reduces to the point 0.

Let now Ω be an interval around the origin, f a $\mathscr{L}(E, D(A))$-valued function satisfying (3.1) in Ω for some $p \geqq 0$. The function $g = f^* u^*$ takes values in E^*, is continuous in Ω and satisfies

$$U = g^{(p)} \tag{3.8}$$

in Ω. Since U is zero both for $t < 0$ and $t > 0$, $g(t) = P(t)$ for $t > 0$, $g(t) = Q(t)$ for $t < 0$, both P and Q being polynomials of degree $\leqq p-1$ with coefficients in E^* and such that $P(0) = Q(0)$.

Consider now the different values of p. If $p = 0$, $g = U = 0$ and there is nothing to prove. If $p = 1$, g is constant in Ω and again $U = 0$. Finally, if $p \geqq 2$ U has to be of form

$$U = \sum_{k=0}^{m} \delta^{(k)} \otimes u_k^* \tag{3.9}$$

where $m = p - 2$, $u_0^*, \cdots, u_m^*$ elements of E^*.[5] Replacing now this expression for U in the equation (3.7) we get

$$\sum_{k=0}^{m} \delta^{(n+k)} \otimes u_k^* = \sum_{k=0}^{m} \delta^{(k)} \otimes A^* u_k^* + \delta \otimes u^* . \tag{3.10}$$

Let now $q \geqq 0$ such that $qn \leqq m < (q+1)n$. By equating coefficients in (3.10) we easily obtain that

$$u^* = -Au_0^*, \; u_0^* = A^* u_n^*, \cdots, u_{(q-1)n}^* = A^* u_{qn}^*, \; u_{qn}^* = 0$$

which shows $u^* = 0$.

LEMMA 3.5. *Let $\{\varphi_\gamma\}$ be a generalized sequence in $\mathscr{D}_0$ convergent to some element $\varphi_0 \in \mathscr{D}_0$ in the topology of $\mathscr{D}$, v any element of E. Then $G(\cdot)S(\varphi_\gamma)v$ converges uniformly to $G(\cdot)S(\varphi_0)v$ on compacts of $t \geqq 0$ together with all its derivatives.*

Proof. Assume—as we may—that $\varphi_0 = 0$. Let Ω be an open set containing the origin and let f be the $\mathscr{L}(E, D(A))$-valued function associated with S in Ω by (3.1). Write formula (3.4) for each $u_\gamma = S(\varphi_\gamma)v$ (the polynomial in the right-hand side is now dependent on γ and will be called $P_{m,p}^\gamma$). It follows from (3.5) applied to u_γ that $A^m u_\gamma \to 0$ for all m; since $S^{(mn-p)} u_\gamma \to 0$ in Ω in the sense of distributions, if $\varphi \in \mathscr{D}$ and $\operatorname{supp}(\varphi) \subset \Omega$,

[5] A vector-valued distribution with support in $\{0\}$ may not necessarily be of the form (3.9).

$$\int P_{m,p}^{\gamma}(t)\varphi(t)dt \to 0 \; . \tag{3.11}$$

But it is not hard to see that (3.11) implies (due to the fact that deg $P_{m,p}^{\gamma}$ is uniformly bounded) that $P_{m,p}^{\gamma} \to 0$ uniformly on compacts of R together with all its derivatives; using this in (3.4) we obtain the desired result.

4. The case $n \geqq 3$.

THEOREM 4.1. *Let n be an integer $\geqq 3$. Assume that the Cauchy problem for* (2.1) *is well set and that Assumption* (3.1) *is satisfied. Then $D(A) = E$, A is continuous and the series*

$$M_n(tA) = \sum_{k=0}^{\infty} \frac{t^k}{(nk)!} A^k \tag{4.1}$$

converges in the topology of $\mathscr{L}(E)$ for all $t > 0$. The propagator S of (2.1) *is actually a $\mathscr{L}(E)$-valued function given by*

$$S(t) = h(t) \sum_{k=0}^{\infty} \frac{t^{nk+n-1}}{(nk+n-1)!} A^k \; . \tag{4.2}$$

Conversely, let the series (4.1) *be convergent for all $t>0$. Then the Cauchy problem for* (2.1) *is well set and the propagator is given by the formula* (4.2).

We shall find it necessary to use in the sequel a few facts about analytic functions with values in a quasi-complete barreled locally convex linear topological space F. A F-valued function f defined in a domain $D \subseteqq C$ is *analytic* in D if the quotient of increments

$$h^{-1}\{f(z+h) - f(z)\}$$

has a limit (in the F-topology) as $h \to 0$ for all $z \in D$. (We shall only consider the cases $F = E$, $F = \mathscr{L}(E)$). All the usual properties of scalar-valued functions (Cauchy's formula etc.) can be extended to F-valued functions; they can be developed in power series in the usual way, the series being convergent for $|z - z_0| < \rho =$ dist $(z_0$, boundary of $D)$. In general, a power series $\sum a_n(z-z_0)^n$ with coefficients in F converges absolutely and uniformly in $|z-z_0| < \rho$, diverges in $|z-z_0| > \rho$, where

$$\rho = \inf \{\liminf |a_n|^{-(1/n)}; \; |\cdot| \in \mathscr{T}\}$$

($\mathscr{T}$ a set of semi-norms defining the topology of F). All these simple facts can be proved essentially like in the Banach space case (see **[11]**, Chapter III, § 2). If $f(\cdot)$ is an F-valued function defined in D and such that $\langle u^*, f\rangle$ is a (usual) analytic function for all $u^* \in F^*$ then f is analytic in the sense outlined above. Likewise, if $B(\cdot)$ is an $\mathscr{L}(F)$-valued function such that $\langle u^*, Bu\rangle$ is an ordinary analytic function for all $u^* \in F^*, u \in F$ then $B(\cdot)$ is analytic as an $\mathscr{L}(F)$-valued function. The proofs of these results also generalize from the Banach space case. ([11], Chapter III, § 2, p. 93); in fact, they are only based in the equicontinuity principle for $\mathscr{L}(F)$ and F^* and in quasi-completeness of these spaces.

The preceding "weak" characterizations of vector-valued analytic functions can be used in combination with "scalar" theorems to obtain generalizations to the F-valued case. We shall make use of two of such extensions:

(a) if f has first continuous partials in D and satisfies the Cauchy-Riemann equations with respect to two independent directions then it is analytic

(b) if f is continuous in a domain D, analytic in D minus a smooth curve Γ, then f is actually analytic in all of D.

We shall also make use of a slight modification of a result of L. Schwartz (Théorème XXIV of Chapter VI in [10], p. 198).

AUXILIARY LEMMA 4.2. *Let* $T \in \mathscr{D}_0'$. *Define for each* $\varphi \in \mathscr{D}$, $\varphi_\xi(t) = \varphi(t - \xi)$.

Assume that for every $\varphi \in \mathscr{D}_0$ *the function* $\xi \to T(\varphi_\xi)$, $\xi > 0$ *can be extended to a function analytic in a fixed region containing* $\xi > 0$. *Then* T *itself coincides with an analytic function in* $\xi > 0$.

The proof is almost identical to the one for the result of Schwartz. Let $a, b, c, d > 0$, $a < b$, $c < d$ but otherwise arbitrary, $\mathscr{D}_{[a,b]} = \{\varphi \in \mathscr{D};$ $\operatorname{supp}(\varphi) \subseteqq [a, b]\}$ with the topology generated by the family of norms $|\varphi|_p = \max_{0\leqq k\leqq p} \max_{a\leqq s\leqq b} |\varphi^{(k)}(s)|$, $p = 0,1,2, \cdots$ $\mathscr{D}^m_{[a,b]}$ the Banach space of all functions φ m times continuously differentiable in R with support in $[a, b]$ (norm: $|\cdot|_m$), $\mathscr{C}_{[c,d]}$ the space of all continuous functions in $[c, d]$. Let $r > 0$, $B_{r,p}\colon \mathscr{D}_{[a,b]} \to \mathscr{C}_{[c,d]}$ defined by

$$(B_{r,p}\varphi)(\xi) = \frac{T(\varphi_\xi)^{(p)}}{p!r^p} = \frac{(T^{(p)} * I\varphi)(\xi)}{p!r^p} \tag{4.3}$$

$p = 0, 1, \cdots$. Reasoning exactly like in [10], p. 198, we see that for some r (depending on a, b, c, d) the family $\mathscr{B} = \{B_{r,p}, p \geqq 0\}$ is equicontinuous. Then there exists an integer $m \geqq 0$ and a real number $\varepsilon > 0$ such that if $\varphi \in \mathscr{D}_{[a,b]}$, $|\varphi|_m \leqq \varepsilon$

$$|B_{r,p}\varphi|_0 \leqq 1$$

in $\mathscr{C}_{[c,d]}$. Consequently, $\mathscr{B}$ is as well equicontinuous as a family of operators from $\mathscr{D}_{[a,b]}$-endowed with the $\mathscr{D}^m$-topology-to $\mathscr{C}_{[c,d]}$. Let now $\varphi \in \mathscr{D}^m_{[a,b]}$, $\{\varphi_k\}$ a sequence in $\mathscr{D}_{[a,b]}$ converging to φ in $\mathscr{D}^m_{[a,b]}$. It is plain from (4.3) that

$$B_{r,p}(\varphi_k) \xrightarrow[k]{} (p!r^p)^{-1}T^{(p)}*I\varphi$$

in the sense of distributions. On the other hand, if k is large enough, $|\varphi_k|_m \leqq |\varphi|_m + 1$, then

$$|B_{r,p}(\varphi_k)|_0 \leqq \varepsilon^{-1}(|\varphi|_m + 1) \tag{4.5}$$

in $\mathscr{C}_{[c,d]}$. But $(d/d\xi)B_{r,p}(\varphi_k) = (p+1)rB_{r,p+1}(\varphi_k)$, thus by virtue of Ascoli's theorem we may assume (passing, if needed, to a subsequence) that for all $p \geqq 1$, $B_{r,p}(\varphi_k)$ is convergent in $\mathscr{D}^0_{[c,d]}$. Then each distribution $(p!r^p)^{-1}T^{(p)}*I\varphi$ coincides in (c, d) with a continuous function and by virtue of the estimates (4.5) the set of all these functions is uniformly bounded in $[c, d]$. This shows, via the definition of $B_{r,p}$ that $T*I\varphi$ is actually analytic in a neighborhood of $[c, d]$ for any $\varphi \in \mathscr{D}^m_{[a,b]}$.

Finally, let Y_{m+2} the distribution in § 2, χ a function in $\mathscr{D}$ such that $\chi = 1$ in $|t| \leqq (b-a)/4$, $\chi = 0$ in $|t| > (b-a)/3$, $k = (a+b)/2$. Plainly $\varphi(t) = (\chi Y_{m+2})(t-k)$ belongs to $\mathscr{D}^m_{[a,b]}$, while $\varphi^{(m+2)} = \tau_k\delta + \eta$, τ_k the operator of translation by k, $\eta \in \mathscr{D}$.

We have

$$(-1)^{m+2}(T*I\varphi)^{(m+2)} - T*I\eta = \tau_{-k}T .$$

Since the left side of the preceding inequality is analytic, so is T in $[c+k, d+k]$; since $k = (a+b)/2$ can be arbitrarily small and c, d are unrestricted, the result follows.

Proof of Theorem 4.1. Let $\omega = \exp(2\pi i/n)$, $C_k = \{\zeta \in C; 2k\pi i/n \leqq \arg\zeta \leqq 2(k+1)\pi i/n\}$, $k = 0, 1, \cdots, n-1$. Let φ be a fixed element in $\mathscr{D}_0$, $u \in E$. Define E-valued functions g_0 (in C_0), g_{n-1} (in C_{n-1}) as follows:

$$g_1(\xi + \eta\omega) = \omega^{-1}\sum_{j=0}^{n-1}\omega^{-j}G^{(j)}(\eta)S^{(n-1-j)}(\varphi_\xi)u , \tag{4.6}$$

$$g_2(\xi + \eta\omega^{-1}) = \omega\sum_{j=0}^{n-1}\omega^{j}G^{(j)}(\eta)S^{(n-1-j)}(\varphi_\xi)u . \tag{4.7}$$

If $\xi, \xi', \eta, \eta' \geqq 0$ we have, for any two integers $p, q \geqq 0$

$$\begin{aligned}G^{(p)}(\eta')S^{(q)}(\varphi_{\xi'})u - G^{(p)}(\eta)S^{(q)}(\varphi_\xi)u = G^{(p)}(\eta')(S^{(q)}(\varphi_{\xi'})\\ - S^{(q)}(\varphi_\xi))u + (G^{(p)}(\eta') - G^{(p)}(\eta))S^{(q)}(\varphi_\xi)u .\end{aligned}$$

This and the regularity results in § 3 show that $g_0(g_{n-1})$ has continuous partials of any order in $C_0(C_{n-1})$. We compute now the first partials. We have

$$\frac{\partial}{\partial\eta} g_0 = \sum_{j=1}^{n} \omega^{-j} G^{(j)}(\eta) S^{(n-j)}(\varphi_\xi)u$$

$$\frac{\partial}{\partial\xi} g_0 = \omega^{-1} \sum_{j=0}^{n-1} \omega^{-j} G^{(j)}(\eta) S^{(n-j)}(\varphi_\xi)u \ .$$

Consequently

$$\left(\frac{\partial}{\partial\eta} - \omega\frac{\partial}{\partial\xi}\right) g_0 = G^{(n)}(\eta)S(\varphi_\xi)u - G(\eta)S^{(n)}(\varphi_\xi)u \ . \tag{4.8}$$

Since $G^{(n)}(\eta)S(\varphi_\xi)u = AG(\eta)S(\varphi_\xi)u = G(\eta)AS(\varphi_\xi)u = G(\eta)S^{(n)}(\varphi_\xi)u$ the right-hand side of (4.4) vanishes. But then (4.8) reduces to the Cauchy-Riemann equation for g_0 (with respect to the directions $1, \omega$) and consequently g_0 is holomorphic in C_0^0, the interior of C_0. Proceeding in exactly the same way with g_{n-1} in C_{n-1} we obtain the equation

$$\left(\frac{\partial}{\partial\eta} - \omega^{-1}\frac{\partial}{\partial\xi}\right) g_2 = 0$$

which likewise implies that g_{n-1} is holomorphic in C_{n-1}. By virtue of Corollary 3.3,

$$g_0(\xi) = g_{n-1}(\xi) = S(\varphi_\xi)u \ . \tag{4.9}$$

This shows that the function g defined as g_0 in C_0, g_{n-1} in C_{n-1} is continuous in $C_0 \cup C_{n-1}$, and thus analytic there.

Denote now $\mathscr{L}_S(E)$ the space of all linear continuous operators from E to E with the topology of simple convergence (or strong topology; see [2], Chapter III, § 3). It follows from the Banach-Steinhaus theorem that $\mathscr{L}_S(E)$ is quasi-complete ([2], Chapter III, § 3). Moreover, any continuous linear functional in $\mathscr{L}_S(E)$ can be written

$$A \longrightarrow \sum_{k=1}^{m} \langle u_k^*, Au_k \rangle$$

where $u_1, \cdots, u_m \in E$, $u_1^*, \cdots, u_m^* \in E$ ([2], Chapter IV, § 2, Proposition 11). Consider the propagator S—as we may—as an element of $\mathscr{D}'(\mathscr{L}_S(E))$ or, rather, as an element of $\mathscr{D}_0'(\mathscr{L}_S(E)) = \mathscr{L}(\mathscr{D}_0; \mathscr{L}_S(E))$. It follows from our results on the function g defined by (4.6), (4.7) and from the equality (4.9) that the distributions in $\mathscr{D}_0'$ obtained from S by applying arbitrary elements of $(\mathscr{L}_S(E))^*$ coincide with functions analytic in $\xi > 0$ in particular, with C^∞ functions. Applying a result in [8], p. 55 (the change of $\mathscr{D}$ by $\mathscr{D}_0$ has no particular significance), we see that S itself coincides in $t > 0$ with a $\mathscr{L}(E)$-valued function

$G(\cdot)$ infinitely differentiable in the $\mathscr{L}_S(E)$-topology, that is $G(\cdot)u$ is a C^∞ function in $\xi > 0$ for all $u \in E$. Clearly if $u \in D = D(A^\infty)$ then Gu coincides with the function defined in § 3. We now extend G to the complex plane as follows; if $\zeta = \xi\omega^k + \eta\omega^{k+1} \in C_k$

$$G(\xi\omega^k + \eta\omega^{k+1})u = \omega^{-(k+1)} \sum_{j=0}^{n-1} \omega^{-j} G^{(j)}(\eta) G^{(n-1-j)}(\xi)u \ , \tag{4.10}$$

$0 \leqq k \leqq n-1$. It follows from the equicontinuity principle (§ 2) that the family $\{G(t); t \in e\}$, e any compact subset of $(0, \infty)$ is equicontinuous in $\mathscr{L}(E)$($\{G(t)u; t \in e\}$ is bounded in E for any $u \in E$). This, and the fact that G is strongly C^∞ in $\xi > 0$ shows that $G(\cdot)$ as defined by (4.10) has any number of continuous partials in C_k^0, the interior of C_k for any $k = 0, \cdots, n-1$. An argument similar to the one for the function g shows that Gu satisfies in each C_k^0 the Cauchy-Riemann equations, and is thus analytic; a fortiori, G itself, as a $\mathscr{L}(E)$-valued function is analytic in $C_0^0 \cup \cdots \cup C_{n-1}^0$.

We now examine more carefully the equality (4.10) when $u \in D$. Let $\{\varphi_n\}$ be a "smoothing kernel" in $\mathscr{D}$, i.e., let $\varphi_n \geqq 0$, $\int \varphi_n dt = 1$, $\operatorname{supp}(\varphi_n) \to 0$ as $n \to \infty$. If $\xi, \eta > 0$ we have, in view of Lemma 2.3, $S((\varphi_n)_\xi)S((\varphi_m)_\eta)u = S((\varphi_m)_\eta)S((\varphi_n)_\xi)u$; letting $m, n \to \infty$ we obtain

$$G(\xi)G(\eta) = G(\eta)G(\xi) \ . \tag{4.11}$$

Differentiating the relation (4.11) and making use of the new equalities thus obtained in (4.10), we easily see (by applying the fact that $G(\xi)u$, $u \in D$ is smooth in $\xi \geqq 0$) that $G(\zeta)u$, $\zeta \in C_k$ is actually continuous even if ξ or η are zero; thus $G(\zeta)u$ is continuous in C_k, except perhaps at the origin. On the other hand, it is not difficult to see by using Corollary 3.3 that the different definitions of G match at the divisory rays $\xi\omega^k$, $\xi \geqq 0$, $k = 0, \cdots, n-1$. Consequently $G(\cdot)u$ is continuous in all of C-except perhaps at the origin; being holomorphic in $C_0^0 \cup \cdots \cup C_{n-1}^0$, it is actually holomorphic in all of C-again, with the possible exception of the origin.

We apply now the same "regularization" method used to obtain (4.11) to the equation (2.4). We obtain

$$G(s+t) = \sum_{j=0}^{n-1} G^{(j)}(s) G^{(n-1-j)}(t) \tag{4.12}$$

for $s, t > 0$. Applying both sides of this equality to a $u \in D$ and using analiticity of Gu, we can extend (4.12) to all complex $z, \zeta \in C\backslash\{0\}$ as long as z, ζ do not both belong to a divisory ray; in particular

$$G(z+\zeta)u = \sum_{j=0}^{n-1} G^{(j)}(z) G^{(n-1-j)}(\zeta)u \tag{4.13}$$

if $z, \zeta \in \bigcup_{k=0}^{n-1} C_k^0$. But for these values of z, ζ the operators in the right-hand side of (4.13) are continuous, and then (4.13) can be extended to all $u \in E$. Observe finally that (4.13) allows us to express Gu near a divisory ray (or near the origin) in a linear and continuous way by means of its values away from them, thus Gu is actually holomorphic in C for any $u \in F$; a fortiori, G is a $\mathscr{L}(E)$-valued entire function.

We compute now the coefficients in the Maclaurin series of G. According to Corollary 3.3 $G^{(2n-1)}(0)u = Au$ for $u \in D$; since A is closed and D dense, $D(A) = E$ and $A = G^{(2n-1)}(0)$ is continuous. The fact that $G^{(k)}(0) = 0$ if $k \neq mn - 1$, $G^{(mn-1)}(0) = A^{m-1}$ for $m \geqq 1$, can be proved by using Corollary 3.3 and the denseness of D. Then

$$G(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^{nk+n-1}}{(nk + n - 1)!} A^k .$$

The convergence of the series for $M_n(tA)$ is clear, as $M_n(tA) = G^{(n-1)}(t^{1/n})$. The final step of the proof of the direct part of Theorem 4.1 (that is, to show that $S = hG$) will be left to the next Remark 4.4 and consists in showing directly that $(hG)^{(n)} - A(hG) = \delta \otimes I$; by uniqueness, it follows that $(hG)u = Su$ for all $u \in E$ and then $hG = S$.

REMARK 4.3. The proof of Theorem 4.1 depends crucially on the fact that the n^{th} roots of unity span C, thought of as a real vector space or, more precisely, on the possibility of writing any $\zeta \in C$ in the form $\zeta = \xi\omega^k + \eta\omega^{k+1}$ for some integer k, $\xi, \eta \geqq 0$, $\omega = \exp(2\pi i/n)$. This is obviously true only if $n \geqq 3$; for if $n = 1$, $\omega = 1$; if $n = 2$, $\omega = -1$.

REMARK 4.4. We end the proof of Theorem 4.1 by establishing the following slightly more general form of its converse part.

LEMMA 4.5. *Let $\alpha > 0$ (not necessarily an integer). Assume $A \in \mathscr{L}(E)$ and that*

$$M_\alpha(tA)u = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} A^k u^{6}$$

converges for all $t>0$ in E for each $u \in E$. Then the Cauchy problem for

(4.14) $$U^{(\alpha)} - AU = T$$

is well set; the propagator of (4.14) is a C^∞, $\mathscr{L}(E)$-valued function in $t \geqq 0$ given by

[1] $M_\alpha(t)$ is the Mittag-Leffler function of classical analysis.

(4.15) $$S(t) = h(t) \sum_{k=0}^{\infty} \frac{t^{\alpha k+\alpha-1}}{\Gamma(\alpha k + \alpha)} A^k ,$$

the series (4.15) *being convergent in* $\mathscr{L}(E)$ *for all* t.

Proof. Since the series defining $M_\alpha(tA)u$ converges for all t, the same is true of the series defining $M_\alpha(\zeta A)u$, $\zeta \in C$. By virtue of the Banach-Steinhaus theorem ([2], Chapter III, § 3 $M_\alpha(\zeta A)$ is a $\mathscr{L}(E)$-valued function; since $M_\alpha(\zeta A)u$ is analytic for each $u \in E$, the same is true of $M_\alpha(\zeta A)$ as a function with values in $\mathscr{L}(E)$. This is easily seen to imply convergence of the series in (4.15) in the topology of $\mathscr{L}(E)$ for all t, uniformly on compact subsets of R (and thus in $\mathscr{D}'(\mathscr{L}(E))$). Recall now that, for $\beta > 0$ the distribution $Y_\beta \in \mathscr{D}'$ used in § 2 to define fractional derivatives coincides with the function $(h(t)t)^{\beta-1}/\Gamma(\beta)$. Then

$$\begin{aligned} S^{(\alpha)} &= Y_{-\alpha} * \Big(\sum_{k=0}^{\infty} Y_{\alpha(k+1)} \otimes A^k \Big) \\ &= \sum_{k=1}^{\infty} Y_{\alpha k} \otimes A^k + Y_0 \otimes I = AS + \delta \otimes I . \end{aligned}$$

Consequently S satisfies the equation (2.3) and this implies that $U = S * T$ satisfies the equation (2.1) for any $T \in \mathscr{D}'_+(E)$. It only remains, then, the question of uniqueness, which we can verify in the form (b'), § 2. Let $u(\cdot)$ be a E-valued C^∞ function, null for $t \leqq 0$ and such that

(4.16) $$u^{(\alpha)}(t) = Au(t)$$

for $t \leqq a$, $a > 0$. Take the convolution product of both sides of (4.16) with the (function) Y_α; we obtain $u(t) = A(Y_\alpha * u)(t)$ for $t \leqq a$. Iterating this equality m times w get $u(t) = A^m(Y_{m\alpha} * u)(t)$ for $t \leqq a$, or

(4.17) $$u(t) = \frac{1}{\Gamma(m\alpha)} \int_0^t (t-s)^{m\alpha-1} A^m u(s) ds$$

for $t \leqq a$. Observe now that $\{u(s), 0 \leqq s \leqq t\}$ is a bounded set in E for every $t > 0$, then as a consequence of the definition of the topology of $\mathscr{L}(E)$ and of the fact that the series for $M_\alpha(tA)$ has infinite radius of convergence in $\mathscr{L}(E)$, if $|\cdot| \in \mathscr{E}$

$$\lim_{m \to \infty} \left| \frac{(t-s)^{m\alpha-1}}{\Gamma(m\alpha)} A^m u(s) \right| = 0$$

uniformly for $0 \leqq s \leqq t$. Applying this estimate in the integral (4.17) we get $u(t) = 0$ for all $t < a$. This result, after a clearly permissible translation is equivalent to (b') of § 2.

REMARK 4.6. In the case E is a Banach space, J. Chazarain ([12] and personal communication) has characterized the operators A for which the Cauchy problem for (4.14) is well set for any α, $0<\alpha<\infty$ not necessarily an integer in terms of the location of $\sigma(A)$, the spectrum of A and the growth of $R(\lambda; A) = (\lambda I - A)^{-1}$. In particular, if $\alpha > 2$ the Cauchy problem for (4.14) is well set if and only if A is everywhere defined and bounded. This result, as well as the one in the next section suggest that Theorem 4.1 is probably true for all $\alpha>2$, i.e., that every time the Cauchy problem for (4.14) is well set for $\alpha > 2$ we have $D(A) = E, A$ is continuous and the series for $M_\alpha(tA)$ is convergent for all $t > 0$. However, the method used here, that is to exploit the simple functional equation (2.4) to extend S to the complex plane breaks down when α is not an integer. Finally, note that Theorem 4.1 generalizes Theorem 3.1 of [3] but apparently only in the case the Cauchy problem for $u^{(n)} = Au$ is, in the terminology of [3] uniformly well posed in $t \geqq 0$ (see [12] for a proof). For if the Cauchy problem for $u^{(n)} = Au$ is only well posed in $t > 0$ the propagator S_{n-1} which plays the role of S in [3] may a priori grow arbitrarily fast as $t \to 0$ and then does not define a distribution in any obvious way. It is not difficult, however, to include also this case in our results. In fact let S be a distribution in $\mathscr{D}_0'(\mathscr{L}(E, D(A))$ satisfying Equation (2.4); if the regularity results of § 2 are *postulated* (they can be easily seen to hold in the situation of [3]) then the proof of Theorem 4.1 can be carried out just in the same way and its conclusion holds. As for Equation (2.4), it is an immediate consequence of Equation (2.8) of [3].

5. **Exponential increase of S.** We relax in this section the requirement that α be an integer, but we are then forced to impose restrictions on the growth of S at ∞.

A few simple properties of vector-valued Laplace transforms will be used in the sequel. Denote, as usual, by $\mathscr{S}$ the space of all infinitely differentiable, complex-valued functions φ that decrease at $|\infty|$ faster than any power of $1/|t|$ together with all their derivatives, endowed with its usual Schwartz topology ([10], Chapter VII, p. 234). The space $\mathscr{S}'(F)$ (of "tempered", F-valued distributions) is $\mathscr{L}(\mathscr{S}; F)$. Given $\omega \in R$, $0 < \omega < \infty$ we write $\Gamma_\omega = (\omega, \infty)$; the space $(\mathscr{S}'(\Gamma_\omega))(F)$ consists of all distributions $T \in \mathscr{D}'(F)$ such that $e_\lambda T \in \mathscr{S}'(F)$ for all $\lambda \in \Gamma_\omega$, e_λ the C^∞ function defined by $e_\lambda(t) = e^{-\lambda t}$. Any distribution $T \in (\mathscr{S}'(\Gamma_\omega))(F)$ has a Laplace transform

$$\mathfrak{L}T = (\mathfrak{L}T)(\lambda), \lambda = \xi + i\eta,$$

a F-valued function holomorphic in $\operatorname{Re} \lambda > \omega$ ([8], p. 74). If, in addition, $T=0$ for $t<0$ then for all $a>\omega$, $u^*\in F^*$ there exists a polynomial p such that

$$|\langle(\mathfrak{L}T)(\lambda), u^*\rangle| \leqq p(|\lambda|) \tag{5.1}$$

for $\operatorname{Re}\lambda \geqq a$ (this is an easy consequence of the "scalar-valued" theorem; see [10], Chapter VIII, p. 310, and [6]) where the polynomial may depend on u^*. Conversely, if L is a F-valued function holomorphic in $\operatorname{Re}\lambda > \omega$ and such that estimates of the form (5.1) hold for it, then $L = \mathfrak{L}T$ where T is a (unique) distribution in $(\mathscr{S}'(\Gamma_\omega))(F)$, $T=0$ for $t<0$ (see again [8], p. 74, and [10], p. 310). If $T=f, f$ an ordinary F-valued function (say, continuous, zero for $t<0$ and such that $\{e_\lambda(t)f(t); t\geqq 0\}$ is bounded in F for any $\lambda\in\Gamma_\omega$) then $\mathfrak{L}T$ coincides with its ordinary Laplace transform, that is

$$(\mathfrak{L}T)(\lambda) = \int_0^\infty e^{-\lambda t}f(t)dt\ .$$

Finally, let $T\in(\mathscr{S}'(\Gamma_\omega))(F)$, $V\in\mathscr{S}'(\Gamma_\omega)$ (and assume, for the sake of simplicity, that both V, T are zero for $t<0$). Then the Laplace transform of the convolution $V*T\in(\mathscr{S}'(\Gamma_\omega))(F)$ is

$$\mathfrak{L}(V*T) = \mathfrak{L}V\mathfrak{L}T$$

([9], Proposition 43, p. 186). We shall only use this result for $V = Y_\beta = (\mathrm{Pf.}\ t^{\beta-1})/\Gamma(\beta)$; $Y_\beta\in\mathscr{S}'$ for all β and its Laplace transform equals $\mathfrak{L}(Y_\beta)(\lambda)=\lambda^{-\beta}$; by virtue of the preceding observation,

$$\mathfrak{L}(T^{(\alpha)})(\lambda) = \mathfrak{L}(Y_{-\alpha}*T)(\lambda) = \lambda^\alpha\mathfrak{L}(T)(\lambda)\ .$$

We shall find it useful to introduce at this point a new space of distributions. We call $(\mathscr{S}_f'(\Gamma_\omega))(F)$ the set of all distributions in $(\mathscr{S}'(\Gamma_\omega))(F)$ such that, for any $\lambda\in\Gamma_\omega$

$$e_\lambda T = f^{(m)} \tag{5.2}$$

for some $m\geqq 0$, where f is a continuous function defined in R, with values in F and such that

$$\{(1+|t|)^{-p}f(t);\ t\in R\} \tag{5.3}$$

is a bounded set in F for some $p\geqq 0$. (Note that any $T\in\mathscr{D}'(F)$ that satisfies the preceding condition belongs to $(\mathscr{S}'(\Gamma_\omega))(F)$). A characterization of some elements in $(\mathscr{S}_f'(\Gamma_\omega))(F)$ is given by the following

AUXILIARY LEMMA 5.1. *Let $T\in(\mathscr{S}'(\Gamma_\omega))(F)$. Then $T\in(\mathscr{S}_f'(\Gamma_\omega))(F)$ and has support in $t\geqq 0$ if and only if for each $a>\omega$ there exists a polynomial $p\geqq 0$ such that the set*

$$\{(1 + p(|\lambda|))^{-1}(\mathfrak{L}T)(\lambda);\ \mathrm{Re}\,\lambda \geqq a\} \tag{5.4}$$

is bounded in F.

Proof. Observe first that the function f in (5.2) can be assumed to be zero for $t < 0$. For if h is the Heaviside function,

$$e_\lambda T = he_\lambda T = hf^{(m)} = (hf)^{(m)} - \sum_{j=1}^{m-1} \delta^{(j)} \otimes f^{(m-1-j)}(0) .$$

Consequently $g = hf - \sum_{j=1}^{m-1} Y_{m-j} f^{(m-1-j)}(0)$ (which is zero in $t < 0$) has the same m-th derivative as f. If g is not continuous, replace m by $m + 1$, g by $\int_0^t g(s)ds$.

We now use (5.2) for $\lambda = a'$, $\omega < a' < a$, and the relation

$$(\mathfrak{L}T)(\lambda) = (\mathfrak{L}(e_{a'}T))(\lambda - a') . \qquad \text{Since}$$

$(\mathfrak{L}(e_{a'}T))(\lambda) = (\mathfrak{L}(f^{(m)}))(\lambda) = \lambda^m(\mathfrak{L}f)(\lambda)$ and, on the other hand

$$|(\mathfrak{L}f)(\lambda)| \leqq \frac{K}{\mathrm{Re}\,\lambda - \varepsilon}, \quad \mathrm{Re}\,\lambda \geqq \varepsilon$$

for any continuous semi-norm $|\cdot|$ in F and any $\varepsilon > 0$ (the constant K may depend on $|\cdot|$, ε) the result follows. Conversely, assume that (5.4) is bounded for all $a > \omega$, p the polynomial corresponding to a, $m =$ degree of p, $\omega < a' < a$. Define

$$g(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}\,\lambda = a'} \lambda^{-(m+2)}(\mathfrak{L}T)(\lambda)e^{\lambda t}dt$$

for $t \in R$. It is not difficult to see that g is a continuous function, zero for $t \leqq 0$, that the set $\{e^{-a't}f(t);\ t \in R\}$ is bounded in F and that

$$(\mathfrak{L}f)(\lambda) = \lambda^{-(m+2)}(\mathfrak{L}T)(\lambda) \tag{5.5}$$

in $\mathrm{Re}\,\lambda > a'$. Equality (5.5) and uniqueness of Laplace transforms plainly imply

$$T = f^{(m+2)} .$$

Observe, finally, that

$$e_aT = e_a(e_{-a}(e_af))^{(m+2)} = \sum_{j=0}^{m+2} \binom{m+2}{j} a^{m-2-j}(e_af)^{(j)}$$

which ends the proof.

THEOREM 5.2. *Let $\alpha > 2$. Assume the Cauchy problem for the equation*

$$U^{(\alpha)} - AU = T \tag{5.6}$$

is well set and that the propagator S *belongs to the space*

$$(\mathscr{S}_f'\Gamma_\omega))(\mathscr{L}(E, D(A))$$

for some $\omega, 0 \leqq \omega < \infty$. *Then* A *is continuous,* $D(A) = E$, $R(\lambda; A) = (\lambda I - A)^{-1}$ *exists for* $|\lambda|$ *large enough and the function* $R(\cdot; A)$ *is analytic at* ∞. *Conversely, the preceding conditions imply that the Cauchy problem for* (5.6) *is well set and that* $S \in (\mathscr{S}_f'(\Gamma_\omega))(\mathscr{L}(E))$ *for some* $\omega < \infty$.

Proof. Taking Laplace transforms of both sides of the equation (2.3) (that is, $Y_{-\alpha}*S - AS = \delta \otimes I$) satisfied by the propagator we obtain

$$(\lambda^\alpha I - A)(\mathfrak{L}S)(\lambda) = I \tag{5.7}$$

for $\operatorname{Re}\lambda > \omega$, where $(\mathfrak{L}S)(\lambda)$ is a $\mathscr{L}(E, D(A))$-valued holomorphic function. By virtue of Lemma 2.3 if $u \in D(A)$ we also have the equality

$$Y_{-\alpha}*(Su) - S(Au) = \delta \otimes u$$

thus

$$(\mathfrak{L}S)(\lambda)(\lambda^\alpha I - A)u = u \tag{5.8}$$

as well. But equalities (5.7), (5.8) plainly imply that $R(\lambda^\alpha; A)$ exists and equals $(\mathfrak{L}S)(\lambda)$ for $\operatorname{Re}\lambda > \omega$. Since $\{\mu \in C; \mu = \lambda^\alpha, \operatorname{Re}\lambda > \omega\}$ contains a neighborhood of ∞ if $\alpha > 2$ (more precisely, the region $|\lambda| > r = (\omega^2 + \tau^2)^{\alpha/2}$, $\tau = \omega tg(\pi/\alpha)$), $R(\lambda; A)$ exists for $|\lambda|$ large. We develop now $R(\cdot; A)$ in Laurent series around ∞,

$$R(\lambda; A) = \sum_{j=1}^{\infty} \lambda^{-j} D_j + D_0 + \sum_{j=1}^{\infty} \lambda^j K_j \tag{5.9}$$

where D_j, K_j are elements of $\mathscr{L}(E, D(A))$. Using the relation

$$(\lambda I - A)R(\lambda; A) = I$$

in (5.9) and equating coefficients in the series so obtained we get the system of equations

$$\begin{aligned} D_{j+1} &= AD_j, \ j \geqq 1 \\ D_1 &= AD_0 + I \\ D_0 &= AK_1 \\ K_j &= AK_{j+1}, \ j \geqq 1 \,. \end{aligned} \tag{5.10}$$

Applying now Lemma 5.1 we see that $R(\lambda; A) = (\mathfrak{L}S)(\lambda^{1/\alpha})$ increases at ∞ less than a polynomial, and then $K_m = 0$ for some $m \geqq 1$; using equations (5.10), we get $K_{m-1} = \cdots = K_1 = 0$, $D_0 = 0$, $D_1 = I$ and consequently $A = D_2$; this shows that A is continuous and that $R(\cdot; A)$ is analytic at ∞ as claimed.

Assume now that A is continuous and that $R(\lambda; A)$ exists in a neighborhood of ∞ and is analytic there. Since the development of $R(\cdot; A)$ at ∞ is

$$R(\lambda; A) = \sum_{j=0}^{\infty} \lambda^{-(j+1)} A^j, \ |\lambda| > r$$

we see that if $|\cdot|$ is a continuous semi-norm in $\mathscr{L}(E)$, $\varepsilon > 0$

$$|A^j| \leqq K(r + \varepsilon)^j, \ j \geqq 1 \tag{5.11}$$

for some $K < \infty$. But then the conditions of Lemma 4.5 are satisfied and consequently the Cauchy problem for (5.6) is well set. We now estimate the propagator S. By virtue of (5.11) and of the formula (4.15),

$$|S(t)| \leqq K \sum_{k=0}^{\infty} \frac{t^{\alpha k + \alpha - 1}(r + \varepsilon)^k}{\Gamma(\alpha k + \alpha)} \ .$$

It follows from results in [5], Chapters IV, V, and VI on asymptotic estimates at ∞ of Maclaurin series that

$$\sum_{k=0}^{\infty} \frac{t^{\alpha(k+1)-1}}{\Gamma(\alpha k + \alpha)} = \frac{1}{\alpha} e^t (1 + o(1))$$

as $t \to \infty$, thus

$$|S(t)| \leqq K' \exp((r + \varepsilon)^{1/\alpha} t)$$

for $t \geqq 0$. This shows that $S \in (\mathscr{S}_f'(\Gamma_\omega))(\mathscr{L}(E))$ for $\omega = r^{1/\alpha}$ and therefore ends the proof.

A number of comments are in order. If F is a Banach space and $T \in (\mathscr{S}'(\Gamma_\omega))(F)$, then $T \in (\mathscr{S}_f'(\Gamma_\omega))(F)$; the reason being that, since F^* is a Banach and then a Baire space, a category argument allows us to pass from the "pointwise" estimates (5.1) for $\mathfrak{L}T$ to the "uniform" estimate in Lemma 5.1. Thus we can change $(\mathscr{S}_f'(\Gamma_\omega))(\mathscr{L}(E, D(A))$ by $(\mathscr{S}'(\Gamma_\omega))(\mathscr{L}(E, D(A))$ in the statement of Theorem 5.2. In the general case these two spaces may be different, and we do not know whether the change is possible, i.e., whether or not S has to be assumed to have "finite order" globally.

Consider the conditions (1_α) A is continuous and the series defining $M_\alpha(tA)$ converges for all t. (2) $R(\lambda; A)$ exists for large $|\lambda|$ and is analytic at ∞. If E is a Banach space, (1_α) for any α, $0 < \alpha < \infty$

and (2) are equivalent to the fact that A is bounded. In the general case we can only say that $1_\alpha \Rightarrow 1_\beta$ if $\alpha \leqq \beta$, $2 \Rightarrow 1_\alpha$, $0 < \alpha < \infty$. The reverse implications are in general false, as we shall now see.

Let E be the space of all functions $x \to u(x)$ defined and continuous in $x \geqq 0$ and such that

$$|u|_n = \sup_{x \geqq 0} |u(x)e^{nx}| < \infty ,$$

$n = 0, 1, \cdots$. If we assign to E the topology generated by the family $\mathscr{E} = \{|\cdot|_0, \cdots\}$ of semi-norms E becomes a Fréchet space. If $0<\beta<\infty$ and we define

$$(A_\beta u)(x) = x^\beta u(x)$$

then A_β is a continuous operator in E. In order to compute $M_\alpha(tA_\beta)$ we use the following asymptotic estimate for the Mittag-Leffler function M_α,

$$M(t^\alpha) = \frac{1}{\alpha} e^t(1 + o(1)) \tag{5.12}$$

(see again [5], Chapter VI). Let now t be fixed, $\alpha \geqq \beta$, $j < k$. The operator $\sum_{p=j}^{k} (tA_\beta)^p/\Gamma(\alpha p + 1)$ coincides with the operator of multiplication by

$$r_{j,k}(x) = \sum_{p=j}^{k} (tx^\beta)^p/\Gamma(\alpha p + 1) .$$

Then if $u \in E$, $n \geqq 0$

$$|r_{j,k}u|_n \leqq \left(\max_{x>0} |r_{j,k}(x)|\, e^{-2x}\right) |u|_{n+2} .$$

Now, by virtue of (5.12)

$$|r_{j,k}(x)e^{-2x}| \leqq M_\alpha(tx^\beta)e^{-2x} \leqq K \exp(x^{\beta/\alpha} - 2x)$$

for $x \geqq 0$, K independent of j, k. Since, on the other hand, $\lim_{j,k\to\infty} r_{j,k}(x) = 0$ uniformly on compacts of $x \geqq 0$, it is clear that the series for $M_\alpha(tA_\beta)$ converges for all t to the operator of multiplication by $M_\alpha(tx^\beta)$. But if we assume that $\alpha < \beta$ and the series for $M_\alpha(tA_\beta)$ is convergent, then the limit has also to be the operator of multiplication by $M_\alpha(tx^\beta)$; but, by virtue of (5.12), this operator is not continuous in E (the operator of multiplication by $\exp(x^\gamma)$, $\gamma > 1$ is not continuous in E). Consequently A_β satisfies 1_α for $\alpha \geqq \beta$ but not for $\alpha<\beta$.

It is not difficult to construct an operator satisfying 1_α for any $\alpha > 0$ but not 2; in fact, let

$$(Au)(x) = \log(1 + x)u(x) .$$

By small modifications of the reasoning above it can be shown that $M_\alpha(tA)$ converges for all $\alpha > 0$, $t \in R$. But $\sigma(A)$ coincides with the positive real axis, then 2 is violated. Applying the results of Sections 4 and 5 we see that the Cauchy problem for

$$U^{(\alpha)} - A_\beta U = T$$

is well set for $\alpha \geqq \beta$ (the propagators increase at ∞ faster than any exponential), is not well set if $2 < \alpha < \beta$, at least if α is an integer. In contrast, the Cauchy problem for

$$U^{(\alpha)} - AU = T$$

is well set for any $\alpha > 0$ but again none of the propagators is of exponential growth at ∞.

6. The case $0 < \alpha \leqq 2$.

THEOREM 6.1. *The Cauchy problem for the equation*

$$U^{(\alpha)} - AU = T \tag{6.1}$$

is well set and the propagator S belongs to the space

$$(\mathscr{S}_f'(\Gamma_\omega))(\mathscr{L}(E, D(A)))$$

if and only if $R(\lambda^\alpha; A)$ exists for $\operatorname{Re}\lambda > \omega$ and for each $a > \omega$ there exists a polynomial $p \geqq 0$ such that

$$\{(1 + p(|\lambda|))^{-1}R(\lambda^\alpha; A); \operatorname{Re}\lambda > a\} \tag{6.2}$$

is equicontinuous in $\mathscr{L}(E, D(A))$ (or in $\mathscr{L}(E)$).

Proof. The necessity of the conditions can be proved as in Theorem (5.2) by showing that $\mathfrak{L}S = R(\lambda^\alpha; A)$ and then using Lemma 5.1. As for the sufficiency, it follows from equicontinuity of (6.2), from the considerations opening § 5 and again from Lemma 5.1 that

$$R(\lambda^\alpha; A) = \mathfrak{L}S$$

where S is a distribution in $(\mathscr{S}_f'(\Gamma_\omega))(\mathscr{L}(E, D(A))$ with support in $t \geqq 0$. Let now $Z = S^{(\alpha)} - AS$. Since $\mathfrak{L}Z = I$, we see that $Z = \delta \otimes I$, which shows that S satisfies (2.3); then $S * T$ satisfies (6.1) for any $T \in \mathscr{D}'(E)$. It only remains then the question of uniqueness of solutions of (6.1), that is to verify (b) (or b') of § 2. Let then $u(\cdot)$ be a C^∞, $D(A)$-valued function, $u(t) = 0$ for $t \leqq a$ and such that

$$u^{(\alpha)}(t) - Au(t) = 0$$

for $t \leqq b$, $a \leqq b$ (we may plainly assume that $b = 0$). Let now $\varphi \in \mathscr{D}$, $\varphi(t) = 1$ in $[a, b]$. Then

$$(6.3) \qquad (\varphi u)^{(\alpha)}(t) - A(\varphi u)(t) = g(t)$$

where $g(t)$ is still zero for $t \leqq 0$ $((\varphi u)^{(\alpha)}(t) = u^{(\alpha)}(t)$ for $t \leqq 0)$ but it also vanishes for large t. We take Laplace transforms of both sides of (6.3) and obtain, after multiplying by $R(\lambda^\alpha, A)$

$$(6.4) \qquad (\mathfrak{L}(\varphi u))(\lambda) = R(\lambda^\alpha; A)(\mathfrak{L}g)(\lambda) .$$

We use now the (easily verifiable) fast that the set $\{(\mathfrak{L}g)(\lambda); \operatorname{Re}\lambda \geqq 0\}$ is bounded in E, the relation (6.4) and equicontinuity of the set (6.2) to deduce that if $\operatorname{Re}\lambda > a$ the set

$$\{(1 + p(|\lambda|))^{-1}(\mathfrak{L}(\varphi u))(\lambda); \operatorname{Re}\lambda \geqq a\}$$

is bounded in E (p the same polynomial in (6.2)). Applying Lemma 5.1 we see that φu (hence u) is zero for $t \leqq 0$. This ends the proof of Theorem 6.1.

Theorem 6.1 reduces for $\alpha = 1$, E a Banach space to a result of Lions (see [7], Théorèmes 6.1, 5.1 and Corollaire 4.1) that gives necessary and sufficient conditions for the Cauchy problem for $U' - AU = T$ to be well set in terms of the theory of distribution semi-groups of exponential increase at ∞.

The author is grateful to Professor J. L. Lions for bringing these problems to his attention and for most valuable suggestions, as well as to the referee for spotting several errors in the first version of the paper.

Results in this article have been announced (under the title "Sur quelques équations différentielles pour les distributions vectorielles) in C. Rendus Acad. Sci. Paris 268 (1969), 707-709.

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Received July 1, 1968, and in revised form October 28, 1968, and April 18, 1969. The preparation of this paper was sponsored by the Office of Naval Research Under Contract NONR 233(76). Reproduction in whole or in part is permitted for any purpose of the United States Government.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

STABILITY THEOREMS FOR LIE ALGEBRAS OF DERIVATIONS

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Let A be a finite dimensional algebra over a field F of characteristic zero and let L be a completely reducible Lie algebra of derivations of A. If A is associative, then there exists an L-invariant Wedderburn factor of A. If A is a Lie algebra, there exists an L-invariant Levi factor of A. If A is a solvable Lie algebra, there exists an L-invariant Cartan subalgebra of A. This paper deals with the uniqueness of such L-invariant subalgebras. For the associative case the assumption of characteristic zero can be dropped if we assume that the radical of A is L-invariant.

2. Preliminaries. If A is a finite dimensional associative algebra over a field F with radical R such that A/R is separable (that is, semisimple and remains so under every field extension of F), then the Wedderburn principal theorem states that there exists a separable subalgebra S such that $A = S + R$, $S \cap R = \{0\}$. S is called a Wedderburn factor of A. Since R is nilpotent, for r in R, $(1 - r)^{-1} = 1 + r + \cdots + r^{n-1}$, where $r^n = 0$. Let C_{1-r} be the inner automorphism of A defined by conjugation by the invertible element $1 - r$. The Malcev Theorem states that if S is any separable subalgebra of A and T is a Wedderburn factor of A, then there exists r in R such that $C_{1-r}(S) \subseteqq T$. Thus, the Wedderburn factors of A are just the maximal separable subalgebras. See [4] for the above information. In § 3 it is shown that if L is completely reducible (every L-invariant subspace of A has a complementary L-invariant subspace), F arbitrary, R L-invariant, and S, T two L-invariant Wedderburn factors of A, then there exists an element r in R such that $C_{1-r}(S) = T$ and $D(r) = 0$ for all D in L. Such an element r is called an L-constant.

If A is a Lie algebra over a field F of characteristic zero and R is the radical (maximal solvable ideal) of A, then the Levi theorem states that $A = S + R$, $S \cap R = \{0\}$, where S is a semisimple subalgebra of A isomorphic to A/R. S is called a Levi factor of A. The Malcev-Hanish-Chandra theorem states that any two Levi factors of A are conjugate by an automorphism $\exp(Adx)$, where x is in N, the nil radical (maximal nilpotent ideal) of A. In § 4 it is shown that for L completely reducible and S, T L-invariant Levi factors of A, then there is an L-constant x in N such that $\exp(Adx)(S) = T$.

If A is a solvable Lie algebra over a field F of characteristic zero, then any two Cartan subalgebras are conjugate by an automorphism of the form $\exp(Adx)$, for $x \in A^\infty = \bigcap_{n=1}^\infty A^n$, see [2]. In § 5, we show that for L completely reducible and S, T L-invariant Cartan subalgebras of A, then there is a L-constant x in A^∞ such that $\exp(Adx)(S) = T$.

In [8] Mostow considered the situation where G, a completely reducible group of algebra automorphisms, acts on a finite dimensional algebra A over a field F of characteristic zero. For each of the three cases for A mentioned above, Mostow shows that there exists the corresponding kind of G-invariant subalgebra. One can use an algebraic group argument, see [1], to conclude the corresponding existence of L-invariant subalgebras. The problem of relating G-invariant subalgebras has been studied by Taft [9], and uniqueness in that case is given via automorphisms defined by fixed points of G. The uniqueness results for L-invariant subalgebras (in terms of L-constants) can be shown directly, and also, for characteristic zero, can be shown to follow from the results of Taft. It should be noted that if x is an L-constant (G-fixed) then C_{1-x} centralizes L (or G) so that if S is an L (or G) invariant subalgebra, so is $C_{1-x}(S)$.

Let F have characteristic zero. The relationship between the situations of L acting on A and that of G acting on A is given by the correspondence between a linear algebraic group and its associated Lie algebra, see Chevalley [3]. In particular, if G is an algebraic group of algebra automorphisms of A, then its associated Lie algebra will consist of derivations of A. Also, complete reducibility is preserved in the algebraic group-Lie algebra correspondence. The following lemma follows easily from the definition of the Lie algebra of an algebraic group. We state it for reference.

LEMMA 2.1. *Let V be a finite dimensional vector space over a field F. Let G be an algebraic group of automorphisms of V and g its associated Lie algebra. If x in V is a fixed point of G, then $X(x) = 0$ for all X in g.*

The author would like to express his appreciation to Professor Earl Taft who suggested these problems and acted as thesis advisor during the preparation of this material.

3. The associative algebra case.

THEOREM 3.1. *Let A be a finite dimensional associative algebra over a field F of characteristic zero and let L be a completely reducible Lie algebra of derivations of A. If S is an L-invariant semisimple subalgebra of A and T an L-invariant maximal semisimple subalgebra of A, then there exists an L-constant r in R, the ra-*

dical of A, such that C_{1-r} carries S into T.

Proof. Given L, let $\bar{L}$ be its algebraic hull, i.e., the smallest algebraic Lie algebra containing L, and let G be the unique connected algebraic group of algebra automorphisms with Lie algebra $\bar{L}$. Then G is also completely reducible. We can apply Theorem 2 of Taft [9] to get r in R such that $C_{1-r}(S) \subseteqq T$ and r is a fixed point of G. By Lemma 2.1 we have that $X(r) = 0$ for all X in $\bar{L}$, and $L \subseteqq \bar{L}$ implies that r is an L-constant.

COROLLARY 1. *Let A and L be as in Theorem 3.1. Then any two L-invariant Wedderburn factors of A are conjugate under an inner automorphism of the form C_{1-r}, where r is an L-constant in R. Also, we may write C_{1-r} in the form $\exp(Ady)$, where y is an L-constant in R.*

Proof. The first statement follows immediately from Theorem 3.1. Let $y = \log(1 - r) = -r - r^2/2 - r^3/3 - \cdots$. Then $X(y) = 0$ for all $x \in L$ and $C_{1-r} = C_{\exp(\log(1-r))} = \exp(Ad(\log(1 - r))) = \exp(Ady)$.

COROLLARY 2. *Let A and L be as in Theorem 3.1. Then any L-invariant semisimple subalgebra S of A is contained in an L-invariant Wedderburn factor.*

Proof. Let T be any L-invariant Wedderburn factor. By Theorem 3.1 there exists an L-constant r in R such that $C_{1-r}(S) \subseteqq T$. Thus, $S \subseteqq (C_{1-r})^{-1}(T) = C_{1-y}(T)$, where $y = -r - r^2 - r^3 - \cdots$. Thus y is an L-constant in R. If $t \in T$, then $C_{1-y}(t) = (1 + y + \cdots + y^n)t(1 - y)$, where $y^{n+1} = 0$. For D in L, $DC_{1-y}(t) = C_{1-y}(D(t))$ since y is an L-constant. Thus, $C_{1-y}(T)$ is L-invariant.

If we drop the assumption of characteristic zero in Theorem 3.1, then the uniqueness result can be proven directly with the additional hypothesis that R be L-invariant. (This is always true for characteristic zero.) The technique used in Theorem 3.1 whereby the situation involving derivations of A is carried over to the situation involving algebra automorphisms of A does not, in general, carry over to the case when F has characteristic $p \neq 0$. It is possible to have an algebraic Lie algebra of derivations of a finite dimensional associative algebra A over a field F of characteristic $p > 0$ which is not the Lie algebra of an algebraic group of algebra automorphisms of A. This cannot occur in characteristic zero. For example, let G be a cyclic group of order p and F an algebraically closed field of characteristic p. Let $A = F(G)$, the group algebra of G over F. Then $\{1, g, \cdots, g^{p-1}\}$ is a basis for A over F and $\{g - 1, \cdots, g^{p-1} - 1\}$ is a basis for the radical R of A. Define a map D of A by $D: g \to 1$ and extend D to a derivation of A. The smallest restricted Lie algebra L of linear transformations of A containing D is algebraic, see [5]. Since the Lie algebra of all derivations of A is restricted, L consists of derivations of A. If G is any algebraic group of automorphisms of A with Lie algebra L, then G cannot consist of algebra automorphisms of A. If so, then R would be G-invariant, and, hence, L-invariant, which is not the case.

THEOREM 3.2. *Let A be a finite dimensional associative algebra over a field F of arbitrary characteristic. Let R be the radical of A and assume A/R is separable. Let L be a completely reducible Lie algebra of derivations of A and assume R is L-invariant. If S is an L-invariant separable subalgebra of A and T is an L-invariant Wedderburn factor of A, then there exists an L-constant x in R such that C_{1-x} carries S into T.*

Proof. We consider two cases:

Case 1. $R^2 = \{0\}$. Let z in R be such that $C_{1-z}(S) \subseteqq T$. z exists by the Malcev theorem. We claim that $D(z) \in R \cap C$, for all $D \in L$, where C is the centralizer of S in A. Given $D \in L$, define $AdD(z)$, a linear map of A, by $AdD(z): a \to D(z)a - aD(z)$, for $a \in A$. Using the facts that $R^2 = \{0\}$ and R is L-invariant, we have that

$$AdD(z) = DC_{1-z} - C_{1-z}D .$$

For $s \in S$, $AdD(z)(s) = DC_{1-z}(s) - C_{1-z}D(s) \in T$ since S and T are L-invariant and $C_{1-z}(S) \subseteqq T$. By assumption, $D(z) \in R$, so $AdD(z)(S) \in R$. Hence, $AdD(z): S \to T \cap R = \{0\}$. Thus, $D(z) \in R \cap C$. $R \cap C$ is an L-invariant subspace of R, so by complete reducibility we have $R = (R \cap C) \oplus U$, where U is an L-invariant subspace of R. Write $z = y + x$, where $y \in R \cap C$ and $x \in U$. Thus $x = z - y$ and for $D \in L$, $D(x) = D(z) - D(y) \in (R \cap C) \cap U = \{0\}$. Hence, x is an L-constant, and $x = z - y$ where $y \in C$ implies that $C_{1-x}(S) = C_{1-z}(S) \subseteqq T$.

If $R^2 \neq \{0\}$, we proceed by induction on the dimension of A. Since R is L-invariant, we have that L is a completely reducible Lie algebra of derivations of R, $T + R^2$, and A/R^2, all of which have dimension less than that of A. Let $a \to \bar{a} = a + R^2$ denote the natural homomorphism of A onto $\bar{A} = A/R^2$. Then $\bar{A}$ has radical $\bar{R}$ and $\bar{S}$ is an L-invariant separable subalgebra of $\bar{A}$ while $\bar{T}$ is an L-invariant Wedderburn factor of $\bar{A}$. By induction, there exists $\bar{v} \in \bar{R}$ such that $C_{\overrightarrow{1-v}}(\bar{S}) \subseteqq \bar{T}$ and $D(v) \in R^2$ for all D in L. R^2 is an I-invariant subspace of R, so by complete reducibility, we have $R = R^2 \oplus U$, where U is L-invariant. Let $v = z + u$, $z \in R^2$, $u \in U$. Then u is an L-constant and $\bar{u} = \bar{v}$. Consider the algebra $T + R^2$. It has dimension less than that of A, has radical R^2, $C_{1-u}(S)$ is an L-invariant separable subalgebra of it (since u is an L-constant and S is L-invariant) and T is an L-invariant Wedderburn factor of $T + R^2$. By induction, there exists r in R^2 such that $D(r) = 0$ for all $D \in L$ and $C_{1-r}C_{1-u}(S) \subseteqq T$. Let $x = u + r - ur$. Then for $D \in L$, $D(x) = D(u) + D(r) - D(u)r - uD(r) = 0$. So x is an L-constant and $C_{1-x}(S) = C_{1-r}C_{1-u}(S) \subseteqq T$.

COROLLARY. *Let A and L be as in Theorem 3.2. Then every L-invariant separable subalgebra of A is contained in an L-invariant Wedderburn factor of A.*

The assumption that R be L-invariant is needed in the above theorem. An example can be given of a semisimple derivation D of an associative algebra A over a field of characteristic 3 such that D leaves invariant more than one Wedderburn factor of A and $D(r) = 0$ for $r \in R$, the radical of A, implies that $r = 0$. Let F be any field of characteristic 3 containing roots of the polynomial $x^3 + x + 1$. Let G be a cyclic group of order 3, $G = \langle g \rangle$, $g^3 = 1$, and form the group algebra $F(G)$ of G over F. Let Q be the quaternion algebra over F, i.e., Q has basis $\{1, i, j, k\}$ over F and $i^2 = j^2 = k^2 = -1$, and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. Let $A = F(G) \otimes_F Q$. Then A is an associative algebra over F of dimension 12. A can also be thought of as the algebra of 2×2 – matrices with entries from $F(G)$. If we write for example, gi for the element $g \otimes i$ of A, then A has basis $\{1, g1, g^21, i, gi, g^2i, j, gj, g^2j, k, gk, g^2k\}$. $\{1, i, j, k\}$ forms a basis for a Wedderburn factor W of A and $\{g1 - 1, g^21 - 1, gi - i, g^2i - i, gj - j, g^2j - j, gk - k, g^2k - k\}$ forms a basis for the radical R of A. Then $R^3 = \{0\}$. Let $r \in R$ where $r = \alpha(g1 - 1) + \beta(g^21 - 1) + \gamma(g^2k - k)$ and $\beta\gamma - \alpha\gamma = \gamma - 1$, $\alpha, \beta, \gamma \in F$. Consider the Wedderburn factors of A obtained by applying C_{1-r} to W. We get the following bases for the resulting Wedderburn factors:

$$\begin{aligned}\{1, (1 + \gamma^2)i &+ \gamma^2 gi + \gamma^2 g^2 i + j + (1 - \gamma)gj \\ &+ (1 + \gamma)g^2 j,\ -i + (\gamma - 1)gi + (-\gamma - 1)g^2 i \\ &+ (1 + \gamma^2)j + \gamma^2 gj + \gamma^2 g^2 j, k\} = \{1, b_1, b_2, k\}\,.\end{aligned}$$

The polynomial $X^3 + X + 1$ has three distinct roots in F and for each distinct root γ we define a distinct Wedderburn factor of A by the above. Define a map D of A as follows:

$$\begin{aligned}&D(1) = 0,\ D(g1) = g1,\ D(g^21) = -g^21,\ D(i) = gj\,,\\ &D(gi) = gi + g^2j,\ D(g^2i) = -g^2i + j,\ D(j) = -gi\,,\\ &D(gj) = -g^2i + gj,\ D(g^2j) = -i - g^2j,\ D(k) = 0\,,\\ &D(gk) = gk,\ D(g^2k) = -g^2k\end{aligned}$$

and extend linearly to all of A. Then D defines a derivation of A, and it is easy to check that for $r \in R$, $D(r) = 0$ implies that $r = 0$. Also, R is not D-invariant since $D(g1 - 1) = g1$ and $(g1)^3 = 1 \notin R$. Also D is semisimple. Consider the Wedderburn factors with bases $\{1, b_1, b_2, k\}$ obtained before, where $\gamma^3 + \gamma + 1 = 0$. Then a direct check shows that $D(b_1) = (\gamma + 1)b_2$ ahd $D(b_2) = -(\gamma + 1)b_1$. So all three Wedderburn factors of A are D-invariant, and they cannot be conjugate by a D-constant in R since the only such constant is 0.

4. The Lie algebra case.

THEOREM 4.1. *Let A be a finite dimensional Lie algebra over a field of characteristic zero and N its nil radical. Let L be a completely reducible Lie algebra of derivations of A. If S is an L-invariant semisimple subalgebra of A and T is an L-invariant Levi factor of A, then there exists an L-constant x in N such that* $\exp(Adx)$ *carries S into T.*

Proof. The proof is similar to that of Theorem 3.2, and the theorem also follows by using Lemma 2.1 and Theorem 4 of [9], where uniqueness is given in this situation in terms of fixed points of a group of automorphisms of A.

5. Solvable Lie algebras.

THEOREM 5.1. *Let A be a finite dimensional solvable Lie algebra over a field of characteristic zero. Let L be a completely reducible Lie algebra of derivations of A. If S and T are L-invariant Cartan subalgebras of A, then there exists x in A^∞ such that x is an L-constant, and* $\exp(Adx)(S) = T$.

Proof. An analogous proof to the theorem for groups in [9] can be given. Also the result follows by Lemma 2.1 and Theorem 6 of [9].

If F has characteristic $p \neq 0$, there are examples of solvable Lie algebras with Cartan subalgebras of different dimensions. For arbitrary characteristic Winter [10] has shown that if G is a completely reducible group of automorphisms of a solvable Lie algebra A and G has no nonzero fixed points, then A has at most one G-invariant Cartan subalgebra. If L is a completely reducible Lie algebra of derivations of a solvable Lie algebra A over a field of arbitrary characteristic, then one can adapt Winter's proof to show that if A has no nonzero L-constants, then A has at most one L-invariant Cartan subalgebra.

6. A counter-example. Let A be a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero and let s be a semisimple automorphism of A. Jacobson [6] shows that there exists an s-invariant Cartan subalgebra in this situation. The question arises as to whether or not a uniqueness result holds in the sense dealt with previously, i.e., given two s-invariant Cartan subalgebras of A, are they conjugate by an automorphism t of A such that t commutes with s? An example will be given to show that uniqueness in this sense need not hold. Let A and s be as above. Recall that s is an invariant automorphism if it is a product

$$\exp(Adx_1) \cdots \exp(Adx_m) ,$$

where each Adx_i is a nilpotent derivation of A. By a result in Borel-Mostow [2] there exists a Cartan subalgebra H of A which is pointwise fixed by s when s is also an invariant automorphism. This follows from the fact that if a regular element is left fixed by S, then the Cartan subalgebra it determines is left pointwise fixed. So let s be an invariant automorphism of A such that H is a Cartan subalgebra of A left pointwise fixed by s. Given any other s-stable Cartan subalgebra T of A, if uniqueness held we would have an automorphism t of A such that $t: H \rightarrow T$ and $st = ts$. Then it follows that T is also pointwise fixed by s. However, the following example shows that a semisimple invariant automorphism s of a semisimple Lie algebra A need not leave every s-stable Cartan subalgebra pointwise fixed. Let A be the simple Lie algebra of $n \times n$-matrices of trace 0 over an algebraically closed field of characteristic zero. Then A has dimension $n^2 - 1$ with Cartan subalgebras of dimension $n - 1$. Let H denote the diagonal matrices of trace 0. Then H has dimension $n - 1$ with basis X_i, $2 \leqq i \leqq n$, where X_i has 1 in the (1, 1)-position and -1 in the (i, i)-position with zeros elsewhere. Let M be the invertible $n \times n$-matrix with 1's in the $(i, i + 1)$-position, $1 \leqq i \leqq n - 1$, 1 in the $(n, 1)$-position, and zero elsewhere. Define an automorphism s of A by $s: N \rightarrow M^{-1}NM$ for $n \in A$. Then s is an invariant automorphism of A, Jacobson [7, p. 283]. Since $M^n = I$, s has order at most n, and so s is semisimple. Thus by the result of Borel-Mostow we know that there exists a Cartan subalgebra of A left pointwise fixed by s. One checks directly that s acts on H as follows: $s(X_i) = X_{i+1} - X_2$ for $2 \leqq i \leqq n - 1$ and $s(X_n) = -X_2$. Thus, H is not pointwise fixed by s, and it also follows that s has order exactly n.

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Received June 24, 1969. The results of this paper are contained in a portion of the author's Doctoral Dissertation, submitted to the Faculty of the Graduate School of Rutgers University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

LOCAL ISOMETRIES OF FLAT TORI

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Let T_1 and T_2 be two flat tori (i.e., provided with a complete Riemannian metric of vanishing curvature). Since they are locally Euclidean each pair of points $P_1, P_2, P_i \in T_i$, has isometric neighborhoods. In general it is not possible, however, to join these separate isometries of neighborhoods to produce a single isometry $T_1 \to T_2$ or $T_2 \to T_1$; indeed there may not even exist a locally isometric map (of the whole surfaces). Necessary and sufficient conditions for the existence of such maps are deduced, making use of a recent conformal classification of maps between tori. As expected "ample" and nonample tori behave differently, and the determination of all local isometries leads to number-theoretic problems. Finally, for two given tori, the local isometries are compared with respect to homotopy by analyzing their effect on the fundamental groups.

Let $\boldsymbol{R}^+$ denote the positive reals, H the upper z-half-plane, and $SL(2, Z)$ the group of all 2×2 unimodular matrices with integral entries acting in the usual way as hyperbolic motions on H. The set of isometry classes of complete flat tori is parametrized by the 3-dimensional manifold $\boldsymbol{R}^+ \times (H/SL(2, Z))$. A point (r^2, τ) of this space represents the isometry class of the torus E^2/Γ, where Γ is the group of Euclidean motions generated by the translations

$$t_1(z) = z + r \quad \text{and} \quad t_2(z) = z + rh ,$$

with $h \in \tau$, (cf. [2]). Instead of "an isometry class of tori" we speak simply of "a torus". A torus $T = (r^2, \tau)$ is called *ample* if there exists $h \in \tau$ such that both $\Re h$ and $|h|^2$ are rational.

2. Riemannian covering maps. The following statements are generalizations of results obtained in [1] which can be similarly proved.

(i) For two tori $T_i = (r_i^2, \tau_i)$ there exist conformal covering maps $T_1 \to T_2$ if and only if two representatives $h_i \in \tau_i$ are equivalent under the action of the group $GL^+(2, Q)$ = group of 2×2 matrices with rational entries and positive determinant.

(ii) Lifting any conformal covering $T_1 \to T_2$ to the universal covering planes we obtain

$$F(z, C, D) = Cz + D , \tag{1}$$

with complex constants $C \neq 0$ and D.

(iii) For nonample T_i only

(2) $$C(\kappa) = \frac{r_2}{r_1}\kappa , \qquad \kappa = \pm 1, \pm 2, \cdots$$

are admissible values in (1).

(iv) For ample $T_i = (r_i^2, \tau_i)$ (2) is replaced by

(3) $$C(\kappa_1, \kappa_2) = \frac{r_2}{r_1}(\kappa_1 + \kappa_2 q'' s'' h_2) ,$$

where $h_2 \in \tau_2$, $h_1 = ah_2$, a an integer, $(\kappa_1, \kappa_2) \neq (0, 0)$ is a pair of arbitrary integers, and the integers q'', s'' are determined via the following relations,

$$2\Re h_2 = \frac{p}{q} , \qquad |h_2|^2 = \frac{r}{s} ,$$

$p, q > 0$, $r > 0$, $s > 0$ integers,

$$\begin{aligned} \text{g.c.d. } (p, q) &= \text{g.c.d. } (r, s) = 1 , \\ g &= \text{g.c.d. } (q, s),\ q' = q/g,\ s' = s/g , \\ g' &= \text{g.c.d. } (a, q),\ a' = a/g',\ q'' = q/g' , \\ g'' &= \text{g.c.d. } (a', s'),\ a'' = a'/g'',\ s'' = s'/g'' . \end{aligned}$$

The following materices are computable from these numbers.

$$\tilde{T}_1 = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix} , \qquad \tilde{T}_2 = \begin{pmatrix} a'ps'', & -a''q'r \\ q''s'', & 0 \end{pmatrix}$$

Our main result is

THEOREM 1. *For the existence of a local isometry $f\colon T_1 \to T_2$ the following conditions are necessary and sufficient:*

(1) *τ_1 and τ_2 are equivalent under $GL^+(2, Q)$;*

(2a) *If T_1 is nonample, then r_1/r_2 must be an integer;*

(2b) *If T_1 is ample, then $(r_1^2/r_2^2)a$ must be an integer N, and N must be representable by the quadratic form*

(4) $$\det (\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2)$$

with suitable integers κ_1 and κ_2.

Proof. Since f is a conformal covering we have necessarily (1) by (i). The following identity is readily verified:

$$\frac{r_1^2}{r_2^2} |C|^2 a = \begin{cases} \det (\kappa \tilde{T}_1) & \text{for } T_1 \text{ nonample} \\ \det (\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2) & \text{for } T_1 \text{ ample} . \end{cases}$$

(The right hand side gives the number N of sheets of the covering f).

Together with the condition $|C| = 1$ for local isometry it leads to (2a) and (2b). The sufficiency follows from (iii) and (iv).

In both cases we have the following consequences. A flat torus can cover a countably infinite set of tori by local isometries. For $T_1 = T_2$ a local isometry is a global isometry, since $|C| = 1$ entails $N = 1$. In general the existence of a local isometry $T_1 \to T_2$ does not imply that there is also a local isometry $T_2 \to T_1$; this occurs if and only if both $r_1 = r_2$ and condition (1) are satisfied. (Then the tori still need not be globally isometric).

3. Homotopy classes. We show how the combination $\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2$ controls also the deformation properties of our maps. If the constant D in (ii) is varied the map stays in the same homotopy class, but maps corresponding to different parameter values κ or (κ_1, κ_2) are not analytically homotopic (i.e., with analytic intermediately stages during the deformation), since the set of admissible values of C is discrete. We show that they are not even homotopic in the ordinary sense.

Since the fundamental group $\pi_1(T)$ of a torus is Abelian the set $\mathscr{H}$ of homotopy classes of continuous maps $T_1 \to T_2$ is in one-to-one correspondence with the set of all homomorphisms $\eta: \pi_1(T_1) \to \pi_1(T_2)$. Denoting by L_i and L_i' $(i = 1, 2)$ the path homotopy classes of two generating loops of $\pi_1(T_i)$, each such η is characterized by the integral matrix

$$\xi = \begin{pmatrix} \xi_4, & \xi_3 \\ \xi_2, & \xi_1 \end{pmatrix}$$

given by

$$\eta(L_1) = L_2^{\xi_1} L_2'^{\xi_2},\ \eta(L_1') = L_2^{\xi_3} L_2'^{\xi_4}\ ;$$

hence $\mathscr{H}$ is parametrized by Z^4. The subset $\{\xi \in Z^4\colon \det \xi \neq 0\}$ contains those points of Z^4 representing monomorphisms, hence it corresponds to the homotopy classes containing covering maps.

THEOREM 2. *The subset of Z^4 corresponding to homotopy classes which contain analytic maps consists of*

(a) 0 *only if τ_1 and τ_2 are nonequivalent under $GL^+(2, Q)$;*

(b) *the 1-dimensional sublattice spanned by $\tilde{T}_1$ if τ_1 and τ_2 are equivalent under $GL^+(2, Q)$ and both are nonample;*

(c) *the 2-dimensional sublattice spanned by $\tilde{T}_1$ and $\tilde{T}_2$ if τ_1 and τ_2 are equivalent under $GL^+(2, Q)$ and both are ample.*

Proof. We prove only (c); (a) and (b) can be handled similarly. The generators L_i, L_i' of $\pi_1(T_i)$ are represented in E_i by the segments S_i, S_i' joining the origin to r_i and $r_i h_i$ respectively. The segments S_1

and S_1' are mapped by $F(z; C, 0)$ (cf. (ii)) into segments from the origin of E_2 to the points

$$\kappa_1 r_2 + \kappa_2 s'' q'' r_2 h_2$$

and

$$-\kappa_2 r a'' q' r_2 + (\kappa_1 a + \kappa_2 s'' p a') r_2 h_2 \ .$$

The former can be deformed into the two sides $\kappa_1 r_2$ and $\kappa_2 s'' q'' r_2 h_2$ of a parallelogram parallel to S_2 and S_2'. The first side represents κ_1 circuits of L_2, the second $\kappa_2 s'' q''$ contours of L_2'. Similarly for S_1'. Hence the homomorphism

$$f_*: \pi_1(T_1) \longrightarrow \pi_1(T_2)$$

induced by f is determined by

$$f_*(L_1) = L_2^{\kappa_1} L_2'^{\kappa_2 s'' q''}$$

and

$$f_*(L_1') = L_2^{-\kappa_2 r a'' q'} L_2'^{\kappa_1 a + \kappa_2 s'' p a'} \ .$$

This is equivalent to $\xi = \kappa_1 \widetilde{T}_1 + \kappa_2 \widetilde{T}_2$.

The determination of *all* local isometries for two given tori is easy for the nonample case. In the ample case it involves the number of ways in which $N = (r_1^2/r_2^2)a$ can be represented by the quadratic form (4). Since this form is positive definite we have, in conjunction with Theorem 2:

THEOREM 3. *The number of homotopy classes of local isometries between two flat tori is finite.*

We obtain an upper bound for this number as follows: From (3) we find

$$\Re C = \frac{r_2}{r_1}\left(\kappa_1 + \kappa_2 s'' \frac{p}{2g'}\right) ,$$

which shows that $\Re C$ has the form $(r_2/r_1)(\gamma/2g')$, with γ an integer. Substituting this in $|\Re C| \leqq |C| = 1$ leads to

$$|\gamma| \leqq 2g' \frac{r_1}{r_2} \ . \tag{5}$$

From $(\Im C)^2 = |C|^2 - (\Re C)^2$ we deduce

$$\kappa_2^2 q''^2 s''^2 (\mathfrak{T} h_2)^2 = \frac{r_1^2}{r_2^2} - \frac{\gamma^2}{4g'^2} \tag{6}$$

and

$$\kappa_1 = \frac{\gamma}{2g'} - \kappa_2 s'' \frac{p}{2g'} . \tag{7}$$

Each of the $2[2g'(r_1/r_2)] + 1$ integers γ compatible with (5) leads to at most two pairs (κ_1, κ_2) compatible with (6) and (7). Thus the number of homotopically different local isometries does not exceed $4[2g'(r_1/r_2)] + 2$.

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Received July 9, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

SOME REMARKS ON CLIFFORD'S THEOREM AND THE SCHUR INDEX

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Some time ago Clifford described the behavior of an irreducible representation of a finite group when it is restricted to a normal subgroup. One interesting case in this description requires that the representation be written in an algebraically closed field. In this note we shall consider this case when the field is "small". We describe conditions under which an irreducible representation decomposes as the tensor product of two projective representations. Our approach uses certain subalgebras of the group algebra and the course of the discussion makes it fairly easy to keep track of the division algebras that appear. Hence we obtain some information about the Schur index. We apply this information to the case where the group is a semi-direct product PA of a p-group P and a normal cyclic group A. If $\mathscr{F}$ is an algebraic number field and χ an absolutely irreducible character of PA, then there normal subgroups $P_1 \supseteqq P_2 \supseteqq P_3$ of P which contain $C_P(A)$ such that the Schur index $m_{\mathscr{F}}(\chi)$ of χ over $\mathscr{F}$ divides $2[P_1: P_2]e$ where e is the exponent of P_2/P_3. The factor 2 can be omitted if $p \neq 2$. Some conditions are available to restrict the P_i further.

1. Preliminaries. In this section we summarize the results about the Schur index and Clifford's theory that will be used later.

Let G be a finite group, $\mathscr{F}$ a field of characteristic zero, M an irreducible $\mathscr{F}(G)$-module with character θ.

(1.1) There are absolutely irreducible (complex-valued) characters $\chi_1, \cdots, \chi_k$ such that $\theta = m(\chi_1 + \cdots + \chi_k)$.

(1.2) Let $\mathscr{F}(\chi_i)$ denote the field generated over $\mathscr{F}$ by the values of χ_i on G. Then $\mathscr{F}(\chi_1)$ is a normal extension of $\mathscr{F}$ and for each $i = 1, \cdots, k$, there is a unique element of the Galois group of $\mathscr{F}(\chi_1)$ over $\mathscr{F}$ which carries χ_1 to χ_i. In particular $(\mathscr{F}(\chi_1): \mathscr{F}) = k$.

(1.3) The integer m is called the *Schur index* of χ_1 over $\mathscr{F}$ and is denoted by $m_{\mathscr{F}}(\chi_1)$. The division algebra $D = \mathrm{End}_{\mathscr{F}(G)}(M)$ has center isomorphic to $\mathscr{F}(\chi_1)$ and the dimension of D over its center is m^2.

One remark on terminology. A matrix ring over D is said to have *index* m if D has dimension m^2 over its center.

The proofs of these statements are available in several places; see for example Curtis and Reiner [2] or Fein [5].

Now let H be a normal subgroup of G. Clifford's theory tells how M behaves as a module over $\mathscr{F}(H)$.

(1.4) $M_H \cong n(V_1 \oplus \cdots \oplus V_s)$ where the V_i are mutually non-isomorphic irreducible $\mathscr{F}(H)$-modules, conjugate under the action of G. Here the coefficient n means direct sum of n copies of $V_1 \oplus \cdots \oplus V_s$.

(1.5) Let $I_1 = \{x \in G \mid xV_1 \cong V_1$ as $\mathscr{F}(H)$-modules$\}$. Then there exists an irredubible $\mathscr{F}(I_1)$-module W_1 such that $(W_1)_H \cong nV_1$ and the induced module $W_1^G \cong M$.

In the case where $s > 1$, I_1 is a proper subgroup of G and the original module is induced from the $\mathscr{F}(I_1)$-module. Hence some questions can be answered by induction. In case $s = 1$ there is no induction but in its place we have the following.

(1.6) Suppose $\mathscr{F}$ is algebraically closed and $s = 1$ in (1.4). Then the representation afforded by M decomposes into the tensor product of two (irreducible) projective representations of G one of which can be viewed as a projective representation of G/H. The one representation has dimension the same as the dimension of V_1, the other, has dimension n.

2. Clifford's Theorem in the general case. We shall continue to use the notation introduced in §1. However we assume $M_H \cong nV$ with $V = V_1$ in (1.4). We shall made one assumption that will simplify the following discussion considerably. Namely we assume that $\mathscr{F}$ contains the values of the character χ_1. Then in view of (1.2) we have $\theta = m\chi$ where $\chi = \chi_1$ in (1.1).

Let V have character γ and suppose

$$\gamma = m_H(\varphi_1 + \cdots + \varphi_t) \tag{2.1}$$

is the decomposition of γ into absolutely irreducible characters of H.

In the group algebra $\mathscr{F}(G)$ let $e(\theta)$ denote the central idempotent which acts as the identity on M and such that $\mathscr{F}(G)e(\theta)$ is a simple algebra; let $e(\gamma)$ denote the centrally primitive idempotent in $\mathscr{F}(H)$ corresponding to V. The condition that $M_H \cong nV$ implies $\gamma(h) = \gamma(x^{-1}hx)$ for all h in H and x in G. It follows that $e(\gamma)$ belongs to the center of $\mathscr{F}(G)$. Also $e(\theta)e(\gamma) \neq 0$ because both act as the identity on V so $e(\theta)e(\gamma)$ is a nonzero central idempotent in $\mathscr{F}(G)e(\theta)$. By simplicity we must have $e(\theta)e(\gamma) = e(\theta)$. Thus multiplication by $e(\theta)$ sends the simple algebra $\mathscr{F}(H)e(\gamma)$ onto the nonzero subalgebra

$\mathscr{F}(H)e(\theta)$ of $\mathscr{F}(G)e(\theta)$ both having the same identity, $e(\theta)$. We note that by (1.3) and (2.1) the center $\mathscr{L}$ of $\mathscr{F}(H)e(\theta)$ is isomorphic to $\mathscr{F}(\varphi_1)$. So we have proved

LEMMA 2.2. *$\mathscr{F}(H)e(\theta)$ is a simple algebra with center $\mathscr{L}$ isomorphic to $\mathscr{F}(\varphi_1)$.*

Each element of G acts by conjugation on $\mathscr{F}(H)e(\theta)$ since $e(\theta)$ is central and H is a normal subgroup. Thus G also acts on the center $\mathscr{L}$ of $\mathscr{F}(H)e(\theta)$ as a group fixing $\mathscr{F}$. Let I denote the kernel of the action of G on $\mathscr{L}$ so that I is normal in G and G/I is a group of $\mathscr{F}$-automorphisms of $\mathscr{L}$. Let $\{y_i\}1 \leqq i \leqq r$ be a set of representatives of the cosets of I in G and let y_i induce the automorphism σ_i oh $\mathscr{L}$.

LEMMA 2.3. *The elements $y_ie(\theta)$ are independent over $\mathscr{F}(I)e(\theta)$ and $\mathscr{F}(G)e(\theta) = \sum \mathscr{F}(I)e(\theta)y_i$.*

Proof. Suppose there exist elements $\{\alpha_j\}$ in $\mathscr{F}(G)e(\theta)$ which centralize $\mathscr{L}$ and with the properties

(a) $\sum_{j=1}^{s} \alpha_j y_j e(\theta) = 0$.

(b) $\alpha_j \neq 0$ for each j.

(c) The integer s is minimal with respect to (a) and (b).

Then for any z in $\mathscr{L}$ we have $\sum \alpha_j z y_j = \sum \alpha_j y_j z = 0$. Then y_j induces σ_j $\mathscr{L}$ so

$$\sum_{j=1}^{s} \alpha_j(z - \sigma_j(z))y_j = 0 . \tag{2.4}$$

There is no loss of generality in assuming that $y_1 = 1$ since a change of coset representatives can always bring this about. Thus $z - \sigma_1(z) = 0$ and the relation (2.4) has fewer than s nonzero terms. By the choice of s it follows $\alpha_j(z - \sigma_j(z)) = 0$ for each j and all z in $\mathscr{L}$. If $j > 1$ then $\sigma_j \neq$ identity so there exists z in $\mathscr{L}$ with $\sigma_j(z) \neq z$. But then $\alpha_j = 0$ contrary to (b). Hence $s = 1$ but this is also contrary to (b). Thus no such relation exists. Since $\mathscr{F}(I)e(\theta)$ centralizes $\mathscr{L}$ we have proved the independence of the $y_ie(\theta)$ over $\mathscr{F}(I)e(\theta)$. The second part of the lemma is clear.

COROLLARY 2.5. *$\mathscr{F}(I)e(\theta)$ is the full centralizer of $\mathscr{L}$ in $\mathscr{F}(G)e(\theta)$ so $\mathscr{F}(I)e(\theta)$ is a simple algebra with center $\mathscr{L}$ and dimension $(\chi(1)/t)^2$ over $\mathscr{L}$.*

Proof. The lemma shows the $y_ie(\theta)$ are independent over the centralizer of $\mathscr{L}$ and so the proper inclusion of $\mathscr{F}(I)e(\theta)$ into the centralizer of $\mathscr{L}$ would make the equality $\mathscr{F}(G)e(\theta) = \sum \mathscr{F}(I)e(\theta)y_i$ impossible. The remaining statements follow from Albert, Theorem 12, page 53 [1] and the facts that $(\mathscr{F}(G)e(\theta): \mathscr{F}) = \chi(1)^2$ and $(\mathscr{L}: \mathscr{F}) = t$ (see(1.2) and (2.2)).

COROLLARY 2.6. *$[G: I] = t$ so $G/I \cong$ Galois group $(\mathscr{L}/\mathscr{F})$.*

Proof. The result follows at once if we use (2.3) and (2.5) to compute the dimension of $\mathscr{F}(G)e(\theta)$ over $\mathscr{F}$ along with the fact that this dimension is $\chi(1)^2$.

Now let $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$. Since $\theta = m\chi$ and $\theta \mid H = n\gamma$, equation (2.1) implies $a = nm_H/m$. Let I_1 denote the inertial group of φ_1; that is

$$I_1 = \{x \in G \mid \varphi_1(h) = \varphi_1(x^{-1}hx) \quad \text{for all} \quad h \in H\} .$$

COROLLARY 2.7. *$I_1 = I$.*

Proof. The irreducible characters of H appearing in $\chi \mid H$ are conjugate under the action of G so that $[G: I_1] = t = [G: I]$. Thus it is sufficient to show $I_1 \subseteqq I$. Let $\mathscr{F}_1 = \mathscr{F}(\varphi_1)$ and notice that in $\mathscr{F}_1(H)$ we have $e(\gamma) = e(\varphi_1) + \cdots + e(\varphi_t)$ where $e(\varphi_i)$ is the central idempotent of $\mathscr{F}_1(H)$ corresponding to φ_i. Recall from above that $e(\gamma)e(\theta) = e(\theta)$ so

$$\mathscr{F}(H)e(\theta) = \mathscr{F}(H)(e(\varphi_1) + \cdots + e(\varphi_t))e(\theta) \xrightarrow{R} \mathscr{F}_1(H)e(\varphi_1)e(\theta)$$

where R denotes right multiplication by $e(\varphi_1)$. The map R is a ring isomorphism. One point requires further comment. The obvious range of R is $\mathscr{F}(H)e(\varphi_1)e(\theta)$ rather than $\mathscr{F}_1(H)e(\varphi_1)e(\theta)$. However we can prove these are equal in the following way. Certainly $\mathscr{F} \subseteqq \mathscr{F}_1$ so $\mathscr{F}(H)e(\varphi_1)e(\theta) \subseteqq \mathscr{F}_1(H)e(\varphi_1)e(\theta)$. We prove equality by computing the $\mathscr{F}$-dimension of both sides. Since multiplication by $e(\theta)$ gives an $\mathscr{F}_1$-algebra isomorphism of $\mathscr{F}_1(H)e(\varphi_1)$ onto $\mathscr{F}_1(H)e(\varphi_1)e(\theta)$ we see the latter algebera has $\mathscr{F}_1$-dimension equal to $\varphi_1(1)^2$ and $\mathscr{F}$-dimension $t\varphi_1(1)^2$. To compute $\mathscr{F}$-dimension of $\mathscr{F}(H)e(\varphi_1)e(\theta)$ we first note $\mathscr{F}$-dimension of $\mathscr{F}(H)e(\gamma)e(\theta)$ equals $\mathscr{F}_1$-dimension of $\mathscr{F}_1(H)e(\gamma)e(\theta)$ because the latter algebra is obtained by extending the scalar field from $\mathscr{F}$ to $\mathscr{F}_1$. Now the algebra $\mathscr{F}_1(H)e(\gamma)e(\theta)$ equals $\sum \mathscr{F}_1(H)e(\varphi_i)e(\theta)$ and this has $\mathscr{F}$-dimension $t\varphi_1(1)^2$ as we wanted.

For an element x in I_1 we have $x^{-1}e(\varphi_1)x = e(\varphi_1)$ so the map R commutes with the action of I_1 on the two algebras in question. It is clear that $\mathscr{F}_1$ is the center of $\mathscr{F}_1(H)e(\varphi_1)e(\theta)$ because $\mathscr{F}_1(H)e(\varphi_1)$ is simple with center $\mathscr{F}_1$. Moreover I_1 fixes $\mathscr{F}_1$ since $\mathscr{F}_1$ is the scalar field. But R maps $\mathscr{L}$ onto $\mathscr{F}_1$ since the center is preserved. Thus I_1 fixes $\mathscr{L}$ and $I_1 \subseteqq I$.

We can now give the analogue of (1.6).

THEOREM 2.8. *The representation of I into $\mathscr{F}(I)e(\theta)$ given by $x \to xe(\theta)$ decomposes into the tensor product of two projective representations over $\mathscr{L}$, U_x and T_x, which map I onto $\mathscr{F}(H)e(\theta)$ and $\mathscr{C}$ respectively where $\mathscr{C}$ is the centralizer of $\mathscr{F}(H)e(\theta)$ in $\mathscr{F}(I)e(\theta)$. The dimensions over $\mathscr{L}$ of these two algebras are $\varphi_1(1)^2$ and a^2 respectively.*

Proof. Each element of I acts by conjugation on $\mathscr{F}(H)e(\theta)$ in such a way that the center is left fixed. Every such automorphism of this simple algebra is inner. Hence for each x in I there is an element U_x in $\mathscr{F}(H)e(\theta)$ such that $x^{-1}wx = U_x^{-1}wU_x$ for every w in $\mathscr{F}(H)e(\theta)$. Clearly the element $\alpha(x, y) = U_xU_yU_{xy}^{-1}$ induces the identity automorphism so $\alpha(x, y)$ is in $\mathscr{L}$ and it follows that $x \to U_x$ is a projective representation of I with factor set α having values in $\mathscr{L}$.

Now let $\mathscr{C}$ denote the centralizer of $\mathscr{F}(H)e(\theta)$ in $\mathscr{F}(I)e(\chi)$. By Theorem 13, page 53 of [1] it follows

$$\mathscr{F}(I)e(\theta) = \mathscr{F}(H)e(\theta)\cdot\mathscr{C} \cong \mathscr{F}(H)e(\theta) \underset{\mathscr{L}}{\otimes} \mathscr{C} \tag{2.9}$$

because $\mathscr{F}(H)e(\theta)$ and $\mathscr{F}(I)e(\theta)$ both have center $\mathscr{L}$. We also know then that $\mathscr{C}$ is simple with center $\mathscr{L}$. Set $T_x = xU_x^{-1}$ so T_x is in $\mathscr{C}$. Then $x \to T_x$ is a projective representation of I with factor set α^{-1} and $xe(\theta) = U_x \otimes T_x$ as required.

We know $(\mathscr{F}(H)e(\theta): \mathscr{L}) = \varphi_1(1)^2$ and from (2.5) that

$$(\mathscr{F}(I)e(\theta): \mathscr{L}) = (\chi(1)/t)^2\,.$$

We also have $\chi(1) = at\varphi_1(1)$ so that we easily obtain from (2.9) the dimension of $\mathscr{C}$ over $\mathscr{L}$ is a^2.

COROLLARY 2.10. *If $\mathscr{F}(\varphi_1) = \mathscr{F}$ then the algebra $\mathscr{F}(G)e(\theta)$ decomposes as the tensor product $\mathscr{F}(H)e(\theta) \otimes \mathscr{C}$ and the representation offered by M decomposes as the tensor product of two projective representations into $\mathscr{F}(H)e(\theta)$ and $\mathscr{C}$ respectively.*

Proof. Since $\mathscr{F}(\varphi_1) = \mathscr{F}$ we have $\mathscr{L} = \mathscr{F}$ and so $G = I$ by (2.6).

3. The Abelian case. We continue with the same situation except that we now suppose G/H is *abelian*.

Since I is the subgroup fixing φ_1, there is an absolutely irreducible character ζ of I such that $\zeta^G = \chi$ and $\zeta \mid H = a\varphi_1$. Now $I \lhd G$ so that in fact we obtain

$$\chi \mid I = \zeta_1 + \cdots + \zeta_t, \zeta_i \mid H = a\varphi_i \tag{3.1}$$

where the ζ_i are irreducible characters of I conjugate under the action of G and $\zeta = \zeta_1$.

LEMMA 3.2. *There exists a subgroup J of I containing H and an irreducible character τ of J such that $\zeta \mid J = a\tau$ and $\tau^I = a\zeta$. Then also $\tau \mid H = \varphi_1$ and $[I: J] = a^2$.*

Proof. Let Λ be the set of linear characters, λ, of I/H such that $\zeta\lambda = \zeta$ and set $J = \cap \ker \lambda$ as λ runs through Λ. We show this J has the required properties.

Let ρ denote the character of the regular representation of I/H. It is a straight-forward computation to verify that $a\varphi_1^I = \zeta\rho$ since both sides are 0 off H and equal to $a[I: H]\varphi_1$ on H. By Frobenius reciprocity ζ has multiplicity a in φ_1^I and so has multiplicity a^2 in $\zeta\rho$. But ρ is the sum of the distinct linear characters of I/H and so there are exactly a^2 linear characters λ such that $\zeta\lambda = \zeta$. Hence $|\Lambda| = a^2$. Note that Λ is also a subgroup of the group of linear characters of I/H so by the duality theory of abelian groups we obtain $[I: J] = a^2 = |\Lambda|$.

Now let τ be an irreducible character of J contained in $\zeta \mid J$ with multiplicity b say. If $[I: J]$ is a prime then either $\tau^I = \zeta$ or $\tau^I = \zeta\Psi_1 + \zeta\Psi_2 + \cdots$ where the Ψ_i are the linear characters of I/J. Because I/J is abelian we use induction to find in the general case that τ^I is a sum of characters $\zeta\Psi$ where Ψ is a linear character of I/J. But every such linear character is in Λ so it follows $\tau^I = b\zeta$. Also there is an integer c such that $\tau \mid H = c\varphi_1$ because $\tau \mid H$ is contained in $\zeta \mid H = a\varphi_1$. Now compute degrees of the characters involved.

$$\tau^I(1) = |I: J|\,\tau(1) = a^2c\varphi_1(1) = b\zeta(1) = ba\varphi_1(1) .$$

So we obtain $ac = b$.

The decomposition of ζ on J has the form

$$\zeta \mid J = b(\tau + \tau_2 + \cdots + \tau_k)$$

so we find

$$\zeta(1) = a\varphi_1(1) = bk\tau(1) = bkc\varphi_1(1) .$$

Thus $a = bkc$ and along with $ac = b$ we find $k = c = 1$ and $a = b$ which proves the lemma.

Notice that $\tau \mid H = \varphi_1$ implies that τ has precisely t conjugates under G, $\tau = \tau_1, \cdots, \tau_t$ and the numbering can be arranged to satisfy $\tau_i \mid H = \varphi_i$. Then also $\chi \mid J = a(\tau_1 + \cdots + \tau_t)$. We shall make use of this in the next result.

LEMMA 3.3. *$\mathscr{F}(J)e(\theta)$ is a simple algebra with center $\mathscr{L}$.*

Proof. The ring $\mathscr{F}(J)e(\theta)$ is semi-simple so simplicity will follow if we show it has only one irreducible module (up to isomorphism). Any irreducible $\mathscr{F}(J)e(\theta)$ module, W, is isomorphic to a direct summand of M_J because M is the unique isomorphism type of $\mathscr{F}(G)e(\theta)$ module. Let μ be the character of W. The character γ must appear in $\mu \mid H$ since γ is the only character of an irreducible $\mathscr{F}(H)e(\theta)$ module in M_H. Thus $\mu \mid H$ contains each φ_i. Moreover the absolutely irreducible characters in μ must appear in $\chi \mid J$. By the remark above the lemma, every τ_i appears in μ. Thus μ is invariant under G and it follows $M_J = k \cdot W$ for some k. Hence $\mathscr{F}(J)e(\theta)$ has only one irreducible module. We find also that $\mathscr{F}(J)e(\theta)$ is isomorphic to $\mathscr{F}(J)e(\mu)$ and its center is isomorphic to $\mathscr{F}(\tau)$. The equations $\zeta \mid J = a\tau$ and $\tau^I = a\zeta$ imply $\mathscr{F}(\tau) = \mathscr{F}(\zeta)$. But then $\mathscr{F}(\zeta)$ is isomorphic to the center of $\mathscr{F}(I)e(\theta)$ so by (2.5) and (2.2) $\mathscr{F}(\zeta) \cong \mathscr{F}(\varphi_1)$. These are isomorphisms over $\mathscr{F}$ so in fact $\mathscr{F}(\zeta) = \mathscr{F}(\varphi_1)$ because both are normal extensions. Hence the center of $\mathscr{F}(J)e(\theta)$ is isomorphic to $\mathscr{L}$. Because of the inclusions

$$\mathscr{L} \subseteqq \mathscr{F}(H)e(\theta) \subseteqq \mathscr{F}(J)e(\theta)$$

and the fact that $\mathscr{L}$ centralizes $\mathscr{F}(J)e(\theta)$ we have $\mathscr{L} =$ center $\mathscr{F}(J)e(\theta)$.

Now that we know $\mathscr{F}(I)e(\theta)$ and $\mathscr{F}(J)e(\theta)$ have the same centers we obtain a decomposition

$$\mathscr{F}(I)e(\theta) = \mathscr{F}(J)e(\theta) \cdot \mathscr{C}_J \cong \mathscr{F}(J)e(\theta) \otimes_{\mathscr{L}} \mathscr{C}_J \tag{3.4}$$

where $\mathscr{C}_J$ is the centralizer of $\mathscr{F}(J)e(\theta)$. If we apply (2.8) with J in place of H we find dimension $\mathscr{C}_J$ over $\mathscr{L}$ is a^2. This is the same a that appears for H because $\chi \mid J = a(\tau_1 + \cdots + \tau_t)$. It is clear that $\mathscr{C}$ (the centralizer of $\mathscr{F}(H)e(\theta)$) contains $\mathscr{C}_J$ and so by dimension count we find $\mathscr{C}_J = \mathscr{C}$. From this it follows that $\mathscr{F}(J)e(\theta) = \mathscr{F}(H)e(\theta)$. This makes it possible to adjust the projective representations U and T so that $U_x = xe(\theta)$ if x is in J and T is constant on the cosets of J.

We are now able to identify the algebra $\mathscr{C}$.

PROPOSITION 3.5. *Let $\mathscr{F}_1$ denote $\mathscr{F}(\varphi_1)$. The algebra $\mathscr{C}$ is iso-*

morphic to a twisted group alegebra $\mathscr{F}_1(I/J)_\beta$ *for some factor set* β *on* I/J *with values in* $\mathscr{F}_1$.

Proof. Recall that the twisted group algebra $\mathscr{F}_1(I/J)_\beta$ has a basis $\{t_x \mid x \in I/J\}$ such that $t_x t_y = \beta(x, y)t_{xy}$. The modification of U above allows us to view T as a projective representation of I/J. Then the correspondence $t_x \longrightarrow T_x$ for x in I/J induces a homomorphism from $\mathscr{F}_1(I/J)_\beta$ into $\mathscr{C}$ provided we have fixed identification of $\mathscr{F}_1$ with $\mathscr{L}$ and $\beta = \alpha^{-1}$ on I/J. If we show this homomorphism is onto $\mathscr{C}$, we will be finished because both algebras have dimension a^2 over $\mathscr{F}_1$. From equation (3.4) it follows

$$\mathscr{F}(I)e(\theta) = \sum_{x \in I/J} \mathscr{F}(J)e(\theta) \otimes T_x$$

becase the right side contains $\mathscr{L}$ along with every $ye(\theta)$ for y in I. It follows that the T_x span $\mathscr{C}$ over $\mathscr{L}$ (because the tensor product is taken over $\mathscr{L}$) and hence the homomorphism above is onto $\mathscr{C}$.

Let A denote I/J. The fact that $\mathscr{F}_1(A)_\beta$ is simple with center $\mathscr{F}_1$ imposes restrictions on A and one can say quite a lot about $\mathscr{F}_1(A)_\beta$. We shall give a brief sketch of the results of DeMeyer [3] which are relevant.

Consider the function $\eta(a, b) = \beta(a, b)/\beta(b, a)$. Because A is abelian and β is a factor set, it follows that η is a (multiplicative) skew bilinear form from $A \times A$ to the multiplicative group of $\mathscr{F}_1$. This is $\eta(ab, c) = \eta(a, c)\eta(b, c)$ and $\eta(a, b) = \eta(b, a)^{-1}$. Because $\mathscr{F}_1$ is the center of $\mathscr{F}_1(A)_\beta$, η is nondegenerate; that is $\eta(a, A) = 1$ holds only for $a = 1$. In a way similar to the method of decomposing a vector space admitting a skew bilinear form, one decomposes A into the direct sum of "hyperbolic planes". The result is the following.

THEOREM (*DeMeyer* [3]). *Let* $\mathscr{F}_1(A)_\beta$ *be central simple over* $\mathscr{F}_1$. *Then* A *decomposes as*

$$A = (C_{11} \times C_{12}) \times (C_{21} \times C_{22}) \times \cdots \times (C_{r1} \times C_{r2})$$

where C_{ij} *is cyclic of prime-power order and* $C_{i1} \cong C_{i2}$. *The function* η *remains nondegenerate on* $C_{i1} \times C_{i2}$ *and the subalgebras* $\mathfrak{A}_i = \mathscr{F}_1(C_{i1} \times C_{i2})_\beta$ *are central simple over* $\mathscr{F}_1$. *Finally we have the decomposition*

$$\mathscr{F}_1(A)_\beta \cong \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_r .$$

This decomposition allows us to get information about the index of the algebra $\mathscr{F}_1(A)_\beta$. We must now restrict $\mathscr{F}$ to be an *algebraic number field*. Now the Brauer-Hasse-Noether theorem can be applied.

It tells us the index of a finite dimensional division algebra over $\mathscr{F}$ is equal to its exponent; that is the order of its class in the Brauer group of the center of the division algebra.

The index of each $\mathfrak{A}_i$ is a divisor of $|C_{i1}|$ since dimension of $\mathfrak{A}_i$ over $\mathscr{F}_1$ is $|C_{i1}|^2$. Thus the index of $\mathscr{F}_1(A)_\beta$ divides the least common multiple of the indices of the $\mathfrak{A}_i$ since the exponent of $\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_r$ divides the l.c.m. of the exponents of the $\mathfrak{A}_i$. This in turn divides the l.c.m. of the numbers $|C_{i1}|\, 1 \leqq i \leqq r$ and this number is precisely the exponent of A. So we have the

PROPOSITION 3.6. *The index of the algebra $\mathscr{C}$ divides the exponent of I/J.*

THEOREM 3.7. *The Schur index $m_{\mathscr{F}}(\chi)$ of χ divides*

$$[G: I]\cdot \text{l.c.m.}\{m_{\mathscr{F}}(\varphi_1),\quad \textit{exponent}\quad (I/J)\}\ .$$

Proof. We have $\mathscr{F}_1 \otimes \mathscr{F}(G)e(\theta)$ equivalent to $\mathscr{F}(I)e(\theta)$ in the Brauer group of $\mathscr{F}_1$ because by Theorem 16, page 56 of [1],

$$\mathscr{F}_1 \otimes \mathscr{F}(G)e(\theta) \cong \mathscr{M}_t(\mathscr{F}) \underset{\mathscr{F}}{\otimes} \mathscr{F}(I)e(\theta)\ .$$

Hence by Theorem 20, page 59 of [1] the factor by which the index has been reduced after extending the field to $\mathscr{F}_1$ must divide $[\mathscr{F}_1: \mathscr{F}]$. By (2.6) this number is $[G: I]$ so the index of $\mathscr{F}(G)e(\theta)$ divides $[G: I]$ times the index of $\mathscr{F}(I)e(\theta)$. By the decomposition of (2.9) we see the index of $\mathscr{F}(I)e(\theta)$ divides the least common multiple of the index of $\mathscr{F}(H)e(\theta)$ and the index of $\mathscr{C}$. The index of $\mathscr{F}(H)e(\theta)$ is $m_{\mathscr{F}}(\varphi_1)$ so the result follows from (3.6).

There is a theorem of Brauer [2, Theorem 70.28] which shows that certain questions about the Schur index of an irreducible character for a finite group can be reduced to questions about $\mathscr{F}$-elementary groups. Recall that among other things an $\mathscr{F}$-elementary group is a semi-direct product PA of a p-group P and a normal cyclic p'-group A. We now consider the case where $G = PA$ is such a semi-direct product (not necessarily $\mathscr{F}$-elementary however). Let $H = C_G(A)$ so that H is normal and G/H is abelian (because A has an abelian automorphism group). Let χ be an absolutely irreducible character of G. In this situation we have the following.

THEOREM 3.8. *Assume $\mathscr{F}$ is an algebraic number field. Then there exists a chain of normal subgroups*

$$G \supseteqq G_1 \supseteqq I \supseteqq J \supseteqq H$$

such that the Schur index of χ over $\mathscr{F}$ divides $2[G_1: I]\cdot$exponent (I/J). The factor 2 can be omitted if $p \neq 2$ or if $p = 2$ but $I \neq J$.

Proof. Since the Schur index will not change, assume $\mathscr{F} = \mathscr{F}(\chi)$. Let M_0 be an irreducible $\mathscr{F}(G)$-module with character $\theta_0 = m\chi$. Suppose $M_0 \mid H = f(V_1 \oplus \cdots \oplus V_r)$ with the V_i distinct irreducible $\mathscr{F}(H)$-modules. Let $G_1 = \{x \in G \mid xV_1 \cong V_1$ as $\mathscr{F}(H)$-modules$\}$. Then there is an irreducible $\mathscr{F}(G_1)$-module W_1 such that $W_1^G \cong M_0$ and $W_1 \mid H \cong f\cdot V_1$. Since G/H is abelian we know $G_1 \lhd G$ so $M_0 \mid G_1 = W_1 \oplus \cdots \oplus W_r$ where the W_i are mutually nonisomorphic irreducible $\mathscr{F}(G_1)$-modules. Now for any nonzero δ in $\mathrm{End}_{\mathscr{F}(G)}(M_0)$ we have $\delta(W_1) \cong W_1$ and since the W_i are nonisomorphic, $\delta(W_1) = W_1$. Hence we imbed $\mathrm{End}_{\mathscr{F}(G)}(M_0)$ into $\mathrm{End}_{\mathscr{F}(G_1)}(W_1)$. Conversely the equation $W_1^G \cong M_0$ provides a natural imbedding of $\mathrm{End}_{\mathscr{F}(G_1)}(W_1)$ into $\mathrm{End}_{\mathscr{F}(G)}(M_0)$. Hence these two division algebras are isomorphic. Let $\chi \mid G_1$ contain the character ζ which also appears in the decomposition of the character for W_1. We have $m_{\mathscr{F}}(\chi) = m_{\mathscr{F}}(\zeta)$ since these numbers represent the indices of the respective endomorphism rings above. Moreover $\mathscr{F}(\zeta)$ is the center of $\mathrm{End}_{\mathscr{F}(G_1)}(W_1)$ so $\mathscr{F} = \mathscr{F}(\zeta)$. We may now apply (3.7) to $\mathscr{F}$, G_1, and W_1 in place of $\mathscr{F}$, G, M and obtain

$$m_{\mathscr{F}}(\chi) \mid [G_1: I]\,\mathrm{l.c.m.}\,\{\text{exponent}\,(I/J),\, m_{\mathscr{F}}(\varphi_1)\}$$

where again φ_1 is an irreducible character of H contained in $\chi \mid H$. But $H = C_P(A) \times A$ is a nilpotent group so by Roquette's theorem [6], $m_{\mathscr{F}}(\varphi_1) = 1$ or possibly 2 in case $p = 2$. Even when $m_{\mathscr{F}}(\varphi_1) = 2$ the l.c.m. of exponent (I/J) and $m_{\mathscr{F}}(\varphi_1)$ will be exponent (I/J) provided $I \neq J$. So the result follows.

COROLLARY 3.9. *If $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$ then $m_{\mathscr{F}}(\chi) \mid 2at$ where the 2 can be omitted if $p \neq 2$.*

Proof. If $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$ then $\zeta \mid H = a(\varphi_1 + \cdots + \varphi_r)$ where $r \mid t$. Thus $|I: J| = a^2$ and exponent (I/J) divides a. The definition of I yields $[G_1: I] = r$ so $[G_1: I] \mid t$. The result now follows from the theorem.

REMARKS. (a) It can happen that $m_{\mathscr{F}}(\chi) = at$. This is the case when χ is an irreducible character of degree 3 for the metacyclic group $\langle x, y\rangle$ where $x^7 = y^9 = 1$ and $y^{-1}xy = x^4$.

(b) The application of DeMeyer's theorem shows the interest in twisted group algebras, $\mathscr{F}(G)_\alpha$, which are simple with center $\mathscr{F}$. A discussion of groups G which admit such a factor set α can be found in [4].

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Received September 9, 1968.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

SYMMETRY AND NONSYMMETRY IN THE GROUP ALGEBRAS OF DISCRETE GROUPS

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A Banach *-algebra $\mathscr{U}$, with identity e, is symmetric if $xx^* + e$ is regular for each x in $\mathscr{U}$. In this paper we generalize certain conditions on a discrete group G that are known to be sufficient to ensure symmetry of $\ell_1(G)$. Also we define semi-symmetry and derive an inequality that must be satisfied if $\ell_1(G)$ is not semi-symmetric. Finally we show that if a group contains a free subsemigroup on two or more generators then $\ell_1(G)$ is not symmetric.

Let G be a discrete group. $\ell_1(G)$ the group algebra of G. $\ell_1(G)$ is a Banach *-algebra with involution defined pointwise by $x^*(g) = \overline{x(g^{-1})}$ and with convolution as multiplication. The mapping $g \rightarrow \delta_g$, where $\delta_g(s) = 0$ if $s \neq g$ and $\delta_g(g) = 1$, is a homomorphism of G into $\ell_1(G)$. In general, we will not distinguish between g and δ_g. Note that if $x \in \ell_1(G)$ then x can be written in the form $x = \sum_{g \in G} x(g)g$.

$\mathscr{H}(G)$ (or $\mathscr{H}$) will denote the real linear subspace of hermitian elements of $\ell_1(G)$. $\mathscr{H}_f(G)$ will be the subspace of $\mathscr{H}(G)$ consisting of all elements x such that

$$N(x) = \{g \mid x(g) \neq 0\}$$

is finite.

Let $\mathscr{K}$ denote the natural cone in $\ell_1(G)$, i.e., $\mathscr{K}$ is the cone generated by all elements of the form xx^* where $x \in \ell_1(G)$. Denote by $\mathscr{F}_0(\mathscr{H})$ the continuous linear functionals defined on $\mathscr{H}$, nonnegative on $\mathscr{K} \cap \mathscr{H}$ and one at the identity.

The right regular representation of $\ell_1(G)$ over $\ell_2(G)$, $x \rightarrow R_x$, is defined by: $R_x(y) = yx$, for each $y \in \ell_2(G)$.

DEFINITION 1.1. $\ell_1(G)$ is semi-symmetric if $x \in \mathscr{H}_f(G)$ and $\mathrm{sp}\,(R_x) \geqq 0$ imply $(x + e)^{-1} \in \ell_1(G)$.

LEMMA 1.2. *If $\ell_1(G)$ is semi-symmetric then*
(i) $\mathrm{sp}\,(xx^*) \geqq 0$ *for each x in $\ell_1(G)$ with $N(x)$ finite, and*
(ii) *if $x \in \mathscr{H}_f(G)$ then $\mathrm{sp}\,(x)$ is real.*

The proof of this lemma is essentially a duplication of the proof of the corresponding results for an arbitrary symmetric Banach *-algebra.

Let $P_f(G)$ be the subset of $\mathscr{H}_f(G)$ consisting of all elements with nonnegative spectra. We observe that $\lambda x \in P_f(G)$ if $\lambda > 0$ and $x \in P_f(G)$, and that, since $\ell_1(G)$ is semi-simple,

$$P_f(G) \cap -P_f(G) = \{0\} .$$

LEMMA 1.3. *If $\ell_1(G)$ is semi-symmetric then $P_f(G)$ is a cone.*

Proof. We need only show that $x + y \in P_f(G)$ if $x \in P_f(G)$ and $y \in P_f(G)$. Let $x \in P_f(G)$ and $y \in P_f(G)$. Then $\mathrm{sp}\,(R_x) \geqq 0$ and $\mathrm{sp}(R_y) \geqq 0$. Thus R_x and R_y are positive definite operators on $\ell_2(G)$ and hence also $R_x + R_y = R_{x+y}$. Therefore $\mathrm{sp}\,(R_{x+y}) \geqq 0$ and thus

$$(x + y + e)^{-1} \in \ell_1(G) .$$

If $\alpha > 0$ then $\alpha^{-1}x$ and $\alpha^{-1}y$ are in $P_f(G)$. Hence

$$(e + \alpha^{-1}x + \alpha^{-1}y)^{-1} = \alpha(\alpha e + x + y)^{-1} \in \ell_1(G) .$$

Therefore $-\alpha \notin \mathrm{sp}\,(x + y)$ and, since $\mathrm{sp}\,(x + y)$ is real, $\mathrm{sp}\,(x + y) \geqq 0$.

If $\ell_1(G)$ is symmetric, then for each $x \in \mathscr{H}(G)$

$$\mathrm{sp}\,(x) \subset \{f(x) \mid f \in \mathscr{F}_0(\mathscr{H})\} .$$

(This result is implicit in the usual proof of Raikov's Theorem, see [7]). If $\ell_1(G)$ is semi-symmetric an abbreviated version of this result can be proven.

LEMMA 1.4. *If $\ell_1(G)$ is semi-symmetric and $x \in \mathscr{H}_f(G)$ then*

$$\mathrm{sp}\,(x) \subset \{f(x) \mid f \in \mathscr{F}_0(\mathscr{H})\} .$$

Proof. Let $x \in \mathscr{H}_f(G)$ be given. Denote by $\mathscr{M}(x)$ a maximal commutative *-subalgebra of $\ell_1(G)$ containing x, and by $\Delta(\mathscr{M}(x))$ the Gelfand representations of $\mathscr{M}(x)$. It is well known that if $y \in \mathscr{M}(x)$ then

$$\mathrm{sp}_{\mathscr{M}(x)}(y) = \mathrm{sp}_{\ell_1(G)}(y)(=\mathrm{sp}\,(y)) .$$

Since $\ell_1(G)$ is semi-symmetric, $P_f(G)$ is a cone. Hence, if we set

$$\mathscr{M}_f(x) = \mathscr{M}(x) \cap \mathscr{H}_f(G)$$

then $P_f(G)$ induces an order on $\mathscr{M}_f(x)$. Furthermore, if for $\delta \in \Delta(\mathscr{M}(x))$ we set $\delta_f = \delta_{|\mathscr{M}(x)}$, then δ_f is positive with respect to this order. By the Monotone Extension Theorem, δ_f has a positive extension to $\mathscr{H}_f(G)$ if

$$(y + \mathscr{M}_f(x)) \cap P_f(G) \neq \varnothing$$

is equivalent with

$$(y + \mathscr{M}_f(x)) \cap -P_f(G) \neq \varnothing$$

for each $y \in \mathscr{H}_f(G)$.

Assume $z \in (y + \mathscr{M}_f(x)) \cap P_f(g)$. Then there is a $z' \in \mathscr{M}_f(x)$ such that $y + z' \in P_f(G)$. Hence

$$\operatorname{sp}(y + z') \subset [0, a]$$

for some $a > 0$. Let $z'' = z' - ae$, then $z'' \in \mathscr{M}_f(x)$ and

$$\operatorname{sp}(y + z'') = \operatorname{sp}(y + z' - ae) = \operatorname{sp}(y + z') - a \subset [-a, 0] .$$

Thus

$$(y + \mathscr{M}_f(x)) \cap -P_f(G) \neq \varnothing .$$

A similar argument establishes the converse.

Let $\bar{\delta}_f$ be an extension of δ_f given by the preceding argument. If $y \in \mathscr{H}_f(G)$ then $y - \nu(y)e \in -P_f(G)$. Hence $\bar{\delta}_f(y - \nu(y)e) \leqq 0$. But

$$\bar{\delta}_f(y - \nu(y)e) = \bar{\delta}_f(y) - \nu(y) .$$

Thus $\bar{\delta}_f(y) \leqq \nu(y)$. Similarly, $\bar{\delta}_f(y) \geqq -\nu(y)$. Therefore

$$| \bar{\delta}_f(y) | \leqq \nu(y) \leqq \| y \|$$

for each $y \in \mathscr{H}_f(G)$. Since $\mathscr{H}_f(G)$ is dense $\mathscr{H}$, $\bar{\delta}_f$ has a continuous extension, f_δ, to $\mathscr{H}$. Since the closure of $P_f(G)$ contains the natural cone $\mathscr{K}$, $f_\delta \in \mathscr{F}_0(\mathscr{H})$.

Now, if $x \in \mathscr{H}_f(G)$ and $\alpha \in \operatorname{sp}(x)$ then there exist an $\mathscr{M}(x)$ and $\delta \in \Delta(\mathscr{M}(x))$ such that $\delta(x) = \alpha$. But then $f_\delta(x) = \delta(x) = \alpha$. Hence

$$\operatorname{sp}(x) \subset \{ f(x) \mid f \in \mathscr{F}_0(\mathscr{H}) \} .$$

It is natural to ask how symmetry of $\mathscr{L}_1(G)$ and semi-symmetry of $\mathscr{L}_1(G)$ are related. The following theorems provide a partial answer.

THEOREM 1.5. *Assume that* $\mathscr{L}_1(G)$ *is semi-symmetric and that whenever* $\lim_n x_n = x$ *for* $\{x_n\} \subset \mathscr{H}_f(G)$, $\lim_n \nu(x_n) = \nu(x)$; *then* $\mathscr{L}_1(G)$ *is symmetric.*

Proof: Let $x \in \mathscr{L}_1(G)$ be given and select $\{x_n\} \subset \mathscr{H}_f(G)$ such that $\lim_n x_n x_n^* = xx^*$. Then, $\lim_n \nu(x_n x_n^*) = \nu(xx^*)$. Hence, if $\varepsilon > 0$ is given, there is a k such that

$$\nu(x_k x_k^*) > \nu(xx^*) - \varepsilon/2$$

and

$$\| x_k x_k^* - xx^* \| < \varepsilon/2 .$$

But, by Lemma 1.4, there is an $f_0 \in \mathscr{F}_0(\mathscr{H})$ such that

$$f_0(x_k x_k^*) = \nu(x_k x_k^*) .$$

Since each $f \in \mathscr{F}_0(\mathscr{H})$ has $\| f \| = 1$,

$$f(x_k x_k^* - xx^*) = f(x_k x_k^*) - f(xx^*) < \varepsilon/2 .$$

Thus

$$f_0(xx^*) > f_0(x_k x_k^*) - \varepsilon/2 = \nu(x_k x_k^*) - \varepsilon/2 > \nu(xx^*) - \varepsilon .$$

Hence

$$\sup_{f \in \mathscr{F}_0(\mathscr{H})} f(xx^*) = \nu(xx^*)$$

for each $x \in \mathscr{L}_1(g)$, and hence $\mathscr{L}_1(G)$ is symmetric.

If $\mathscr{L}_1(G)$ is symmetric then, by Raikov's theorem, (c.f. [8]), the spectral radius of each element of the form xx^* is equal $\| T_{xx^*} \|$ for some *-representation $x \to T_x$. However, this *-representation need not be the right regular representation over $\mathscr{L}_2(G)$. If we assume G is amenable, then this latter representation weakly contains all other *-representations, ([6]), and hence the spectral radius of xx^* is given by $\| R_{xx^*} \|$. Using these facts we can prove

THEOREM 1.6. *If $\mathscr{L}_1(G)$ is symmetric and if G is amenable then $\mathscr{L}_1(G)$ is semi-symmetric.*

Proof. Suppose that $x \in \mathscr{H}_f(G)$, $-1 \in \operatorname{sp}(x)$ and $\operatorname{sp}(R_x)$ is non-negative. Let $y = x - \nu(R_x)e$ then

$$-1 - \nu(R_x) \in \operatorname{sp}(y)$$

and

$$\operatorname{sp}(R_y) = \operatorname{sp}(R_x - \nu(R_x)e) \subset [-\nu(R_x), 0] .$$

Therefore

$$\nu(yy^*) = \nu(y^2) \geqq (1 + \nu(R_x))^2 ,$$

and

$$\nu(R_{yy^*}) = \nu(R_y^2) \leqq \nu(R_x)^2$$

But

$$\nu(R_{yy^*}) = \| R_{yy^*} \| ,$$

and, since G is amenable,

$$\| R_{yy^*} \| \geqq \| T_{yy^*} \|$$

for any $*$-representation $z \to T_z$. Therefore

$$\nu(yy^*) = [1 + \nu(R_x)]^2 > \nu(R_x)^2 \geqq \| R_{yy^*} \| \geqq \| T_{yy^*} \|$$

for any $*$-representation $z \to T_z$. This contradicts Raikov's criteria for symmetry. Hence, if $\mathrm{sp}\,(R_x)$ is nonnegative, then $-1 \notin \mathrm{sp}\,(x)$.

REMARK. The dual hypothesus at Theorem 1.6, namely, that $\ell_1(G)$ was symmetric and that G was amenable, was necessary. Although all known pertinent results tend to indicate that symmetry of $\ell_1(G)$ implies amenability of G, we do not know this to be true.

2. A sufficient condition for semi-symmetry. If H is a subgroup of G then there is a cannonical embedding of $\ell_1(H)$ into $\ell_1(G)$. We will not distinguish between an element of $\ell_1(H)$ and its image in $\ell_1(G)$. Since for each $x \in \ell_1(H)$, $\mathrm{sp}_{\ell_1(H)}(x) = \mathrm{sp}_{\ell_1(G)}(x)$ (cf. [3]), we are assured that this laxity will cause no confusion when making spectral considerations.

Let $m(G)$ be the space of bounded functions defined on G. The mapping $\theta \to \theta^v$, where

$$\theta^v(x) = \sum_{g \in G} \theta(g)x(g)$$

for $x \in \ell_1(G)$, is an isometric isomorphism of $m(G)$ onto $\ell_1(G)^*$.

For $A \subset G$, let $\langle A \rangle$ be the group generated by A.

LEMMA 2.1. *Let $x \in \ell_1(G)$. Then x has no left inverse if, and only if, there is a $\theta \in m(\langle N(x) \rangle)$ such that $\| \theta \| = 1 = \theta(e)$ and the null space of θ^v contains the left ideal generated by x.*

Proof. Assume x has no left inverse in $\ell_1(G)$. The preceding remarks imply that x has no left inverse in $\ell_1(\langle N(x) \rangle)$.

Let L be the left ideal in $\ell_1(\langle N(x) \rangle)$ generated by x. Now, if $y \in \ell_1(\langle N(x) \rangle)$ and $\| y \| < 1$, then

$$(e + y)^{-1} = e + \sum_{n=1}^{\infty} (-1)^n y^n$$

is in $\ell_1(\langle N(x) \rangle)$. Hence, if $\| e - z \| < 1$, set $y = -e + z$, and then $z^{-1} = (e + y)^{-1}$ is in $\ell_1(\langle N(x) \rangle)$. Thus

$$L \cap \{y \in \ell_1(\langle N(x) \rangle) \mid \| e - y \| < 1\} = \emptyset \ ,$$

and the distance of L from e is at least one. Hence the desired θ^v exist.

The converse is obvious.

Let $x \in \mathscr{H}_f(G)$ such that $x + e$ is singular and $\operatorname{sp}(R_x)$ is non-negative. Let $A = N(x) \cup \{e\}$, $H = \langle A \rangle$ and $s(n) = c(A^n)$, the cardinality of A^n. Enumerate the elements of H in the following manner:

$$\{g_1, g_2, \cdots, g_{s(1)}\} = A$$

and

$$\{g_{s(n)+1}, \cdots, g_{s(n+1)}\} = A^{n+1} \sim A^n$$

for $n = 1, 2, \cdots$.

Since $x + e$ is singular, and $x + e$ is hermitian, $x + e$ has neither a right nor a left inverse. Hence there is a $\theta \in m(H)$ such that $\theta(e) = 1 = \|\theta\|$ and the null space of θ^v contains L, the left ideal generated by $x + e$. For θ^v to vanish on all L, it is necessary and sufficient that, in particular, $\theta^v(g_i x) = 0$ for each $g_i \in H$.

Let $\theta_i = \theta(g_i)$ for $i = 1, 2, \cdots$, and for each positive integer n, define $\theta(n + 1)$ in $\ell_2(H)$ by

$$\theta(n + 1)(g_i) = \begin{cases} \theta_i, & \text{if } 1 \leqq i \leqq s(n + 1) \\ 0, & \text{if } i > s(n + 1) \end{cases} .$$

Then

$$\| R_{x+e}(\theta(n + 1)) \|_2^2 \geqq \| \theta(n + 1) \|_2^2 + 2(R_x(\theta(n + 1)), \theta(n + 1)) .$$

But $\operatorname{sp}(R_x) \geqq 0$, hence

$$(R_x(\theta(n + 1)), \theta(n + 1)) \geqq 0 .$$

Therefore

2.2 $$\| R_{x+e}(\theta(n + 1)) \|_3^2 \geqq \| \theta(n + 1) \|_2^2$$

for $n = 1, 2, \cdots$.

Now, if $g_i \in A^n$ then

$$N(g_i(x + e)) \subset A^n A = A^{n+1} .$$

Thus

$$\begin{aligned} R_{x+e}(\theta(n + 1))(g_i) &= [\theta(n + 1)(x + e)](g_i) \\ &= \sum_{j=1}^{\infty} [\theta(n + 1)(g_i)][(x + e)(g_j^{-1} g_i)] \\ &= \sum_{j=1}^{s(n+1)} \theta_j [(x + e)(g_i^{-1} g_j)] \\ &= \sum_{j=1}^{s(n+1)} \theta_j [g_i(x + e)(g_j)] \\ &= \theta^v(g_i(x + e)) = 0 . \end{aligned}$$

If $g_i \in H \sim A^{n+2}$ then

$$N(g(x+e)) \subset H \sim A^{n+1} .$$

Hence again

$$R_{x+e}(\theta(n+1))(g_i) = 0 .$$

Therefore

$$\| R_{x+e}(\theta(n+1)) \|_2^2 = \sum_{j=s(n)+1}^{s(n+2)} | R_{x+e}(\theta(n+1))(g_j) |^2 .$$

But for $g_j \in A^{n+2} \sim A^n$,

$$N(g_j(x+e)) \subset A^{n+3} \sim A^{n-1} .$$

Hence

$$\begin{aligned} & | R_{x+e}(\theta(n+1))(g_j) |^2 \\ & \leqq \Big(\sum_{i=s(n-1)+1}^{s(n+1)} | \theta_i [g_j(x+e)(g_i)] | \Big)^2 \\ & \leqq \Big(\sum_{i=s(n-1)+1}^{s(n+1)} | g_j(x+e)(g_i) |^2 \Big) \Big(\sum_{i=s(n-1)+1}^{s(n+1)} | d_{ji}\theta_i |^2 \Big) \\ & \leqq \| x+e \|_2^2 \sum_{i=s(n-1)+1}^{s(n+1)} | d_{ji}\theta_i |^2 \end{aligned}$$

where $d_{ji} = 0$ if $g_j(x+e)(g_i) = 0$ and one otherwise. Note that for fixed j, $d_{ji} \neq 0$ for at most $c(A)$ i's. Therefore

$$\begin{aligned} \| R_{x+e}(\theta(n+1)) \|_2^2 & \leqq \sum_{j=s(n)+1}^{s(n+2)} \Big(\| x+e \|_2^2 \sum_{i=s(n-1)+1}^{s(n+1)} | d_{ji}\theta_i |^2 \Big) \\ & \leqq c(A) \| x+e \|_2^2 \sum_{i=s(n-1)+1}^{s(n+1)} | \theta_i |^2 . \end{aligned}$$

We also have

$$\| \theta(n+1) \|_2^2 = \sum_{j=1}^{s(n+1)} | \theta_j |^2 .$$

Combining these results by 2.2. we have

$$c(A) \| x+e \|_2^2 \sum_{j=s(n-1)+1}^{s(n+1)} | \theta_j |^2 \geqq \sum_{j=1}^{s(n+1)} | \theta_j |^2 \tag{2.3}$$

for each $n = 2, 3, \cdots$.

We compile the above argument in

THEOREM 2.4. *If $\mathscr{L}_1(G)$ is not semi-symmetric then for some $x \in \mathscr{H}_f(G)$ there is a $\theta = (\theta_i) \in m(\langle N(x) \rangle)$ such that $\| \theta \| = 1 = \theta(e)$ and (θ_i) satisfies* 2.3.

3. **Condition** SS. For a given G let $\mathscr{S}(G)$ denote the family of finite symmetric subsets of G containing the identity. Adel 'son-Vel' skiĭ and Šreider, [1], considered the following condition on a group G:

$(A-S)$ for each $A \in \mathscr{S}(G)$

$$c(A^n) = o(d^n) \text{ for any } d > 1 .$$

They proved that if G satisfies $(A-S)$ then G is amenable. Hulanicki, [5], later showed that if a group satisfies $(A-S)$ then $xx^* + e$ is regular for each x in $\ell_1(G)$ with finite support.

We now define a condition which is weaker than $(A-S)$:

(SS) for each $A \in \mathscr{S}(G)$

$$\liminf_n c(A^{n+1} \sim A^n)^{1/n} \leqq 1 .$$

It is not difficult to show that if G satisfies $(A-S)$ then G satisfies (SS) and that if G satisfies (SS), G is amenable. We also have

THEOREM 3.1. *If G satisfies SS then $\ell_1(G)$ is semi-symmetric.*

Proof. If $\ell_1(G)$ is not semi-symmetric then by Theorem 2.4 there is a $\theta = (\theta_i) \in m(\langle A \rangle)$, where $A = N(x) \cup \{e\}$, such that $\|\theta\| = 1 = \theta(e)$ and

$$c(A)\,\|x+e\|_2^2 \sum_{j=s(n-1)+1}^{s(n+1)} |\theta_j|^2 \geqq \sum_{j=1}^{s(n+1)} |\theta_j|^2 \geqq \sum_{j=1}^{s(n-1)} |\theta_j|^2$$

for each $n = 2, 3, \cdots$. Let $a' = c(A)\,\|x+e\|_2^2$, $a = (a'+1)/a'$, and

$$b = \sum_{j=1}^{s(2)} |\theta_j|^2 .$$

Then, since $\theta(e) = 1$ and $e \in A$, $b > 0$. We have

$$\sum_{j=1}^{s(4)} |\theta_j|^2 = \sum_{j=1}^{s(2)} |\theta_j|^2 + \sum_{j=s(2)+1}^{s(4)} |\theta_j|^2 \geqq ab ;$$

and if

$$\sum_{i=1}^{s(2n)} |\theta_j|^2 \geqq (a)^{n-1} b$$

then

$$\sum_{j=1}^{s(2n+2)} |\theta_j|^2 = \sum_{j=1}^{s(2n)} |\theta_j|^2 + \sum_{j=s(2n)+1}^{s(2n+2)} |\theta_j|^2 \geqq (a)^{n-1}b + (1/a')[(a)^{n-1}b] \geqq (a)^n b .$$

Therefore

$$\sum_{j=1}^{s(2n+2)} |\theta_j|^2 \geqq (a)^n b$$

for $n = 2, 3, \cdots$.

Since $\|\theta\| = 1$; $|\theta_j| \leqq 1$ for each j. Hence

$$\sum_{j=s(2n)+1}^{s(2n+2)} |\theta_j|^2 \leqq c(A^{2n+2} \sim A^{2n}) .$$

Consequently

$$c(A^{2n+2} \sim A^{2n}) \geqq (1/a')a^n b .$$

for $n = 2, 3, \cdots$. If $B = A^2$ then $B \in \mathscr{S}(G)$ and $\liminf_n C(B^{n+1} \sim B^n) > 1$. This contradiction implies $\mathscr{L}_1(G)$ is semi-symmetric

4. Condition (C'). Hulanicki [5] proves that $\mathscr{L}_1(G)$ is symmetric for any group G satisfying:

(C) there is a k such that for any finite set $A \subset G$

$$\sup c(At_1At_2 \cdots At_n) \leqq k^m f_A(m, n) ,$$

where the least upper bound on the left is taken over all sequences $(t_1, t_2, \cdots, t_n) \in G^n$ where at most m of the t_i's are different from the identity, and the function $f_A(m, n)$ satisfies the condition $f_A(m, n) = o(c^n)$ for any $c > 1$, uniformly with respect to $m \leqq n$. We will obtain the same result for any group G satisfying the condition

(C') there is a k such that for each $A \in \mathscr{S}(G)$

$$\liminf_n \sup_{(s_i) \in G^n} [c(As_1As_2 \cdots As_n)]^{1/n} < k .$$

LEMMA 4.1. *If G satisfies (C') then G also satisfies (SS).*

Proof. If G satisfies (C') then for each $A \in \mathscr{S}(G)$

$$\liminf_n c(A^n)^{1/n} < k .$$

If for some $B \in \mathscr{S}(G)$,

$$\liminf_n c(B^n)^{1/n} \geqq \delta > 1 ,$$

then choose a positive integer p so that $\delta^p \geqq k$. Then

$$\liminf_n c((B^p)^n)^{1/n} = [\liminf_n c(B^{pn})^{1/pn}]^p = \delta^p \geqq k .$$

Thus, for each $A \in \mathscr{S}(G)$,

$$\liminf_n c(A^n)^{1/n} \leqq 1 .$$

Now,

$$c(A^n \sim A^{n-1}) \leqq c(A^n)$$

for each $n \geqq 2$. Hence

$$\liminf_n c(A^n \sim A^{n-1})^{1/n} \leqq 1$$

for each $A \in \mathscr{S}(G)$.

LEMMA 4.2. *Assume G satisfies (C') with constant k. Let x and y be elements of $\ell_1(G)$ such that $N(x)$ is finite. Then*

$$\nu(xy) \leqq k \,\|y\|\,\|R_x\| \,.$$

Proof.

$$\begin{aligned} \|(xy)^n\| &= \|(x \sum_{s \in G} y(s)s)^n\| = \|\sum_{s_1,\cdots,s_n \in G} y(s_1) \cdots y(s_n) xs_1 \cdots xs_n\| \\ &\leqq \sum_{s_1,\cdots,s_n \in G} |y(s_1)| \cdots |y(s_n)|\, \|xs_1 \cdots xs_n\| \\ &\leqq \sup_{s_1,\cdots,s_n \in G} \|xs_1 \cdots xs_n\| \sum_{s_1,\cdots,sn \in G} |y(s_1)| \cdots |y(s_n)| \,. \end{aligned}$$

An application of Schwarz inequality gives

$$\|xs_1 \cdots xs_n\| \leqq c(N(xs_1 \cdots xs_n))^{1/2} \|xs_1 \cdots xs_n\|_2 \,.$$

For any $z \in \ell_1(G)$

$$\|R_z\| = \sup_{\|y\|_2 = 1} \|R_z(y)\|_2 \geqq \|R_z(e)\|_2 = \|z\|_2 \,.$$

Therefore

$$\|xs_1 \cdots xs_n\|_2 \leqq \|R_{xs_1 \cdots xs_n}\| = \|R_{xs_1} R_{xs_2} \cdots R_{xs_n}\| \leqq \|R_x\|^n \,.$$

Also,

$$N(xs_1 \cdots xs_n) \subset N(xs_1) \cdots N(xs_n) = N(x)s_1 \cdots N(x)s_n \,.$$

Therefore, if we set $A = N(x) \cup \{e\} \cup N(x)^{-1}$,

$$\|xs_1 \cdots xs_n\| \leqq c(As_1 \cdots As_n)^{1/2} \|R_x\|^n \,.$$

Finally

$$\sum_{s_1,\cdots,s_n \in G} |y(s_1)| \cdots |y(s_n)| = \|y\|^n \,.$$

Consequently

$$\|(xy)^n\| \leqq \|y\|^n \|R_x\|^n \sup_{s_1,\cdots,s_n \in G} c(As_1 \cdots As_n)^{1/2} \,,$$

and hence

$$\nu(xy) = \liminf_n \| (xy)^n \|^{1/n} \leqq \| y \| \| R_x \| \liminf_n \sup_{s_1, \cdots, s_n} c(As_1 \cdots As_n)^{1/2n}$$
$$\leqq k \| y \| \| R_x \| .$$

We are now ready to prove.

THEOREM 4.3. *If G satisfies (C') then $\mathscr{L}_1(G)$ is symmetric.*

Proof. If $\mathscr{L}_1(G)$ is not symmetric, then by Raikov's Theorem (c.f. [8], p. 307) there is a $yy^* \in \mathscr{L}_1(G)$ such that

$$r = \sup_{f \in \mathscr{F}_0(\mathscr{H})} f(yy^*) < \nu(yy^*) .$$

We may assume that;

$$\inf_{f \in \mathscr{F}_0(\mathscr{H})} f(yy^*) = s > 0:$$

if not we consider the element $yy^* + \alpha e$ for some $\alpha > 0$.

Let $x = yy^*$ and choose u and v so that $0 < u < s$ and $r < v < \nu(x)$. Then

$$0 < u < f(x) < v < \nu(x)$$

for each $f \in \mathscr{F}_0(\mathscr{H})$.

Let k be the constant of (C'), $t > 1$ and p be a positive integer such that $\nu(x)^p > ktv^p$. Pick $A \in \mathscr{S}(G)$ so that, if z is x restricted to A then

(i) $0 < f(z) < v$, for each $f \in \mathscr{F}_0(\mathscr{H})$ and

(ii) $\| z^{-p}x^p \| < t$.

To see that (i) is possible merely note that by taking A sufficiently large, $\| z - x \|$ is less than both $v - r$ and u. Then, since each $f \in \mathscr{F}_0(\mathscr{H})$ is of norm one, the condition is satisfied.

For (ii) we first observe that by Lemma 1.4 and 4.1

$$\operatorname{sp}(z) \subset \{ f(z) \mid f \in \mathscr{F}_0(\mathscr{H}) \} \subset (0, v) .$$

Thus z is regular, and for fixed p, $z^{-p}x^p$ converges to the identity as A increases.

We now apply Lemma 4.2.

$$\nu(x^p) = \nu[z^p(z^{-p}x^p)] \leqq k \| z^{-p}x^p \| \| R_{z^p} \| .$$

But

$$\| R_{z^p} \| \leqq \| R_z \|^p$$

and

$$\| R_z \| = \nu(R_z) \leqq \nu(z) < v .$$

Therefore

$$\nu(x^p) = \nu(x)^p \leqq ktv^p .$$

But p was chosen so that

$$\nu(x^p) > ktv^p .$$

This contradiction implies that $\mathscr{L}_1(G)$ is symmetric.

5. **Nonsymmetric group algebras.** In [3], Frey asked if there are amenable groups with nonamenable subsemigroups. He proved that if such groups exist they must contain a free nonabelian subsemigroup on two generators. Hochster [4], has recently presented an example of such a group. In [7], a similar example is presented, and it is shown that the algebra of this group is nonsymmetric. The following theorem shows that all such groups have nonsymmetric algebras. The proof employs the well known fact that in a symmetric Banach *-algebra the hermitian elements have real spectra (c.f. [8]).

THEOREM 5.1. *Let G be a group generated by a and b such that S, the semigroup generated by a and b is free. Then, $\mathscr{L}_1(G)$ is nonsymmetric.*

Proof. We will show that $\delta i \in \mathrm{sp}\,(x)$, where

$$x = \alpha a + \beta b + \lambda ab + \bar{\lambda} b^{-1}a^{-1} + \bar{\beta} b^{-1} + \bar{\alpha} a^{-1} ,$$

if $\delta i = \alpha\beta/\lambda$ and $|\lambda| \geqq \max\{3|\alpha|, 3|\beta|\}$. To accomplish this we will construct a nonzero $\theta \in m(G)$ such that θ^v vanishes on the left ideal generated by $y = x - \delta ie$.

Let $S_0 = S \cup \{e\}$ and $S_1 = aS_0 \cup \{e\} \cup b^{-1}S_0^{-1}$. Define $\theta(g) = 0$ if $g \notin S_1$. Let $A = N(y)$ and $S' = S_0 \cup b^{-1}S_0 \cup S_0^{-} \cup aS_0^{-1}$. Direct computations yield:

$$Ag \cap S_1 \neq \varnothing \Leftrightarrow g \in S' .$$

Enumerate the elements of S^1 as follows: $s_1 = e$ and for $n = 1, 2, \cdots, s_{2n} = as_n$; $s_{2n+1} = bs_n$; $t_n = b^{-1}s_{2n}$: $s_{-1} = s_1$ and for $n = 1, 2, \cdots$, $s_{-2n} = a^{-1}s_{-n}$; $s_{-(2n+1)} = b^{-1}s_{-n}$; $t_{-n} = as_{-(2n+1)}$.

One can easily verify that the homogeneous equations $\theta^v(ys_i) = 0$, $-3 \leqq i \leqq 2$, $i \neq 0$, and $\theta^v(yt_j) = 0$, $-2 \leqq j \leqq 2$, $j \neq 0$, have a nontrivial simultaneous solution.

For n a positive integer,

$$N(yt_n) = \{s_{2n}, s_{4n}, at_n, a^{-1}t_n, b^{-1}a^{-1}t_n, b^{-1}t_n, t_n\} .$$

If $at_n = ab^{-1}s_{2n} \in aS_0$ then $b^{-1}s_{2n} \in S_0$ which is impossible. Similarly, if $at_n \in b^{-1}S_0^{-1}$ then $b^{-1} = a^{-1}s^{-1}$ for some $s \in S$. Since S is free in G, this also is impossible. Certainly $at_n \neq e$. Thus $at_n \notin S_1$. Similar arguments show that

$$\{a^{-1}t_n, b^{-1}a^{-1}t_n, a^{-1}a^{-1}t_n, t_n\} \cap S_1 = \varnothing .$$

We also have

$$N(ys_n) = \{s_{2n}, s_{2n+1}, s_{4n+2}, s_n, a^{-1}s_n, b^{-1}s_n, b^{-1}a^{-1}s_n\} .$$

For $n \geqq 3$,

$$N(ys_n) \cap S_1 = N(ys_n) \cap (aS_0 \cup \{e\}) .$$

If n is odd then

$$N(ys_n) \cap S_1 \subset \{s_{2n}, s_{4n+2}, b^{-1}s_n\}$$

while for n even

$$N(ys_n) \cap S_1 \subset \{s_{2n}, s_{4n+2}, a^{-1}s_n, b^{-1}a^{-1}s_n, s_n\} .$$

Note that not both $a^{-1}s_n$ and $b^{-1}a^{-1}s_n$ are elements of S_1. Also, $a^{-1}s_n = s_m \in S_1$ implies $m < n$ and $b^{-1}a^{-1}s_n = s_m \in S_1$ implies $m < n$.

Assume now that $n \geqq 3$ is given, and that $\theta(s_k)$ has been defined for $1 \leqq k < 4n$ such that $|\theta(s_k)| \leqq 1$ and

$$\theta^v(yt_m) = 0 = \theta^v(ys_m)$$

for $m < n$. Now

$$\theta^v(yt_n) = \beta\theta(s_{2n}) + \lambda\theta(s_{4n})$$

and

$$\theta^v(ys_n) = \alpha\theta(s_{2n}) + \lambda\theta(s_{4n+2}) + \bar{\alpha}\theta(a^{-1}s_n) + \bar{\beta}\theta(b^{-1}s_n) + \bar{\lambda}\theta(b^{-1}a^{-1}s_n) + \delta i\theta(s_n) .$$

Let

$$\theta(s_{4n}) = (-\beta/\lambda)\theta(s_{2n})$$

and

$$\theta(s_{4n+2}) = (-\bar{\lambda}/\lambda)\theta(b^{-1}a^{-1}s_n) - 1/\lambda[\alpha\theta(s_{2n}) + \bar{\alpha}\theta(a^{-1}s_n) + \bar{\beta}\theta(b^{-1}s_n) + \delta s\theta(s_n)] .$$

We consider the two possibilities:

(i) If $b^{-1}a^{-1}s_n \in S_1$ then $b^{-1}a^{-1}s_n \in aS_0 \cup \{e\}$ and hence

$$\{a^{-1}s_n, b^{-1}s_n\} \cap S_1 = \varnothing .$$

Also, there is an $n > m \geqq 1$ such that $s_n = s_{2m}$. Since $\theta^v(yt_k) = 0$

for $1 < k < n$ and since $\theta^v(yt_k) = \beta\theta(s_{2k}) + \lambda\theta(s_{4k}) = 0$ if, and only if, $\theta(s_{4k}) = (-\beta/\lambda)\theta(s_{2k})$; $\theta(s_{2n}) = \theta(s_{4m}) = (-\beta/\lambda)\theta(s_{2m})$. Thus

$$\begin{aligned}\theta(s_{4n+2}) &= (-\bar{\lambda}/\lambda)\theta(b^{-1}a^{-1}s_n) - (1/\lambda)(\delta i - \alpha\beta/\lambda)\theta(s_{2m}) \\ &= (-\bar{\lambda}/\lambda)\theta(b^{-1}a^{-1}s_n) .\end{aligned}$$

(ii) If $b^{-1}a^{-1}s_n \notin S$, then

$$\theta(s_{4n+2}) = (-1/\lambda)[\alpha\theta(s_{2n}) + \bar{\alpha}\theta(a^{-1}s_n) + \bar{\beta}\theta(b^{-1}s_n) + \delta i\theta(s_n)]$$

and at most three terms within the parenthesis on the right are nonzero.

In either case $|\theta(s_{4n+2})| \leqq 1$. Certainly $|\theta(s_{4n})| \leqq 1$. Thus, by induction, we can define $\theta(s_{2n})$ such that $|\theta(s_{2n})| \leqq 1$ and such that

$$\theta^v(yt_n) = 0 = \theta^v(ys_n)$$

for $n = 1, 2, \cdots$.

Similarly, we can define $\theta(s_{-(2n+1)})$ so that $|\theta(s_{-(2n+1)})| \leqq 1$ and

$$\theta^v(yt_{-n}) = 0 = \theta^v(ys_{-n})$$

for $n = 1, 2, \cdots$.

REMARK. If G is an amenable group with a nonamenable subsemigroup then G has a subgroup H that satisfies the hypothesis of Theorem 5.1. Hence $\ell_1(H)$ is nonsymmetric, and since for each $x \in \ell_1(H)$,

$$sp_{\ell_1(H)}(x) = \mathrm{sp}_{\ell_1(G)}(x) ,$$

$\ell_1(G)$ is also nonsymmetric.

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Received February 3, 1969. Some of the results stated in this paper are found in the author's doctoral dissertation, written at the University of Illinois under the direction of Professor M. M. Day. Partial support was received from National Science Foundation Research Grant GP-6990.

STATE UNIVERSITY OF NEW YORK AT ALBANY

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

OUTER GALOIS THEORY FOR SEPARABLE ALGEBRAS

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Let G be a finite group of automorphisms of a ring Λ which has identity element. Let C be the center of Λ, let Γ be the subring of G-invariant elements of Λ, and assume that C is a separable extension of $C \cap \Gamma$. In the first section of this paper, it is shown that every finite group of automorphisms of Λ over Γ is faithfully represented as a group of automorphisms of C by restriction if, and only if, $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$. Moreover, suppose that $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$ and Ω is a subring of Λ such that $\Gamma \subseteqq \Omega \subseteqq \Lambda$. Then there exists a finite group H of automorphisms of Λ such that Ω is the subring of H-invariant elements of Λ if, and only if, $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$.

Let R be a commutative ring with identity element; and assume now that Λ is a separable algebra over R and G is a finite group of automorphisms of the R-algebra Λ. In the second section of this paper, it is shown that C is the centralizer of Γ in Λ if, and only if, $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$. Moreover, suppose that $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$ and Ω is a subalgebra of Λ such that $\Gamma \subseteqq \Omega \subseteqq \Lambda$. Then there exists a finite group H of automorphisms of Λ such that Ω is the subalgebra of H-invariant elements of Λ if, and only if, Ω is a separable algebra over R.

These results are obtained without the assumption of no nontrivial idempotent elements of C, which is required for the Kanzaki-DeMeyer Galois theory of separable algebras. Moreover, these results extend the Villamayor-Zelinsky Galois theory of commutative rings in the same way that the results of Kanzaki and DeMeyer extend the Chase-Harrison-Rosenberg Galois theory of commutative rings.

1. Galois theory. Throughout this paper, ring will mean ring with identity element and subring of a ring will mean subring which contains the identity element of the ring. Let Γ be a subring of a ring Λ. Call Λ a projective Frobenius extension of Γ if Λ is a finitely generated, projective right Γ-module and there is a (Γ, Λ)-bimodule isomorphism of Λ onto $\mathrm{Hom}_\Gamma(\Lambda, \Gamma)$. Call Λ a separable extension of Γ if the (Λ, Λ)-bimodule epimorphism of $\Lambda \otimes_\Gamma \Lambda$ onto Λ, which is determined by the ring multiplication in Λ, splits. Equivalently, Λ is a separable extension of Γ if there exist a positive integer n and elements x_i, y_i of Λ, for $1 \leqq i \leqq n$, such that $\sum_{i=1}^n x_i y_i = 1$ and $\sum_{i=1}^n a x_i \otimes y_i = \sum_{i=1}^n x_i \otimes y_i a$ in $\Lambda \otimes_\Gamma \Lambda$ for every $a \in \Lambda$. Also, let M be a left Λ-module and let N be a Γ-submodule of M. A canonical Λ-module homomorphism ϕ of $\Lambda \otimes_\Gamma N$ into M is determined by the

correspondence of $a \cdot x$ to $a \otimes x$ for $a \in \Lambda$ and $x \in N$. It will be convenient of write $M = \Lambda \otimes_{\Gamma} N$ when ϕ is an isomorphism.

Let G be a finite group of automorphisms of a ring Λ, and let Γ be the subring of G-invariant elements of Λ. Call Λ a Galois extension of Γ relative to G if there exist a positive integer n and elements x_i, y_i of Λ, $1 \leqq i \leqq n$, such that $\sum_{i=1}^{n} x_i \cdot \sigma(y_i) = \delta_{1,\sigma}$ for all $\sigma \in G$. If Λ is a Galois extension of Γ relative to G, then Λ is a separable extension of Γ by [8, Proposition 1.3]. Let C be the center of Λ. If Λ is a Galois extension of Γ relative to G and C is the centralizer of Γ in Λ, call Λ an outer Galois extension of Γ relative to G. A generalization of the concept of outer Galois extension is that of outer semi-Galois extension given in [7, Definition 2.4]. Λ will be called an outer semi-Galois extension of Γ if, in addition to the assumptions stated at the beginning of this paragraph, Λ is a separable extension of Γ and C is the centralizer of Γ in Λ. Finally, we note that, if S is a G-stable subring of Λ; then a homomorphism of G onto a finite group $\bar{G}$ of automorphisms of S is obtained by restricting each element of G to S, and $S \cap \Gamma$ is the subring of $\bar{G}$-invariant elements of S.

For the remainder of the paper, let G be a finite group of automorphisms of a ring Λ, let Γ be the subring of G-invariant elements of Λ, and let C be the center of Λ.

THEOREM 1.1. *If S is a G-stable subring of C such that S is a separable extension of $S \cap \Gamma$, then the following statements are equivalent.*

(i) $C = S \otimes_{S \cap \Gamma} (C \cap \Gamma)$ *and* $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$.

(ii) $\Lambda = S \otimes_{S \cap \Gamma} \Gamma$.

(iii) *An isomorphism of the group of all automorphisms of Λ over Γ for which S is stable onto the group of all automorphisms of S over $S \cap \Gamma$ is obtained by restricting each automorphism of Λ to S.*

(iv) *Every finite group of automorphisms of Λ over Γ for which S is stable is faithfully represented as a group of automorphisms of S by restriction.*

Proof. It is evident that statement (i) implies statement (ii). If $\Lambda = S \otimes_{S \cap \Gamma} \Gamma$, then every automorphism of S over $S \cap \Gamma$ may be extended to an automorphism of Λ over Γ and the identity map on Λ is the only automorphism of Λ over Γ which restricts to the identity map on S. With these observations it is easily verified that statement (ii) implies statement (iii). Clearly statement (iii) implies statement (iv), and it only remains to verify that statement (iv) implies statement (i). Since S is a commutative ring and a separable extension of $S \cap \Gamma$, S is an outer semi-Galois extension of $S \cap \Gamma$. Let e be an idempotent element of S and σ be an element of G such that $\sigma(e \cdot a) = e \cdot a$ for all $a \in S$. An automorphism $\bar{\sigma}$ of Λ over Γ is defined by the rule $\bar{\sigma}(a) = \sigma(e \cdot a) + (1 - e) \cdot a$ for $a \in \Lambda$. Let H be the group of automorphisms of Λ over Γ which is generated by $\bar{\sigma}$. Since G is a finite group, σ has finite order. Therefore $\bar{\sigma}$ has finite order and H is a finite group. Moreover, each element of S is H-invariant; and H is faithfully represented as a group of automorphisms of S by restriction, only if $\bar{\sigma}(a) = a$ and, hence, $\sigma(e \cdot a) = e \cdot a$ for all $a \in \Lambda$. Since C is stable for any group of automorphisms of Λ, the following lemma may be applied with $T = C$ to establish that statement (iv) implies statement (i).

LEMMA 1.2. *Let S, T be G-stable subrings of Λ such that $S \subseteqq T$ and S is an outer semi-Galois extension of $S \cap \Gamma$. Assume that whenever e is a central idempotent of S and σ is an element of G such that $\sigma(e \cdot a) = e \cdot a$ for all $a \in S$, then $\sigma(e \cdot a) = e \cdot a$ for all $a \in \Lambda$. Then $T = S \otimes_{S \cap \Gamma} (T \cap \Gamma)$ and $\Lambda = T \otimes_{T \cap \Gamma} \Gamma$.*

Proof. By hypothesis, S is a separable extension of $S \cap \Gamma$. Let n be a positive integer and let x_i, y_i be elements of S for $1 \leqq i \leqq n$, such that $\sum_{i=1}^{n} x_i y_i = 1$ and $\sum_{i=1}^{n} a x_i \otimes y_i = \sum_{i=1}^{n} x_i \otimes y_i a$ in $S \otimes_{S \cap \Gamma} S$ for every $a \in S$. Setting $e_\sigma = \sum_{i=1}^{n} x_i \cdot \sigma(y_i)$, $a \cdot e_\sigma = e_\sigma \cdot \sigma(a)$ for $a \in S$ and $\sigma \in G$. Therefore e_σ is an element of the centralizer of $S \cap \Gamma$ in S, which is the center of S, for $\sigma \in G$. Moreover

$$e_\sigma^2 = \sum_{i=1}^{n} e_\sigma \cdot x_i \cdot \sigma(y_i) = \sum_{i=1}^{n} x_i \cdot e_\sigma \cdot \sigma(y_i) = \sum_{i=1}^{n} x_i \cdot y_i \cdot e_\sigma = e_\sigma$$

for $\sigma \in G$. Thus $\{\sigma(e_\tau) \mid \sigma, \tau \in G\}$ is a finite set of central idempotents in S, and it generates a finite, G-stable subalgebra E of the Boolean algebra of all central idempotents in S. Letting M be the set of minimal elements in E; M is a finite, G-stable set of pairwise orthogonal idempotents such that $\sum_{e \in M} e = 1$. For $e \in M$ and $\sigma \in G$, let $e\sigma$ denote the mapping $a \rightsquigarrow e \cdot \sigma(a)$, $a \in \Lambda$; and let

$$N = \{e\sigma \mid e \in M, \sigma \in G\} .$$

The mapping $\alpha \rightsquigarrow \sigma \circ \alpha$, $\alpha \in N$, is a permutation on N for each $\sigma \in G$. Consequently, letting γ be the sum of the distinct elements of N; γ is a left $S \cap \Gamma$-module endomorphism of Λ, the image of which must be contained in Γ. Since S and T are G-stable, γ must map S into $S \cap \Gamma$ and T into $T \cap \Gamma$. If, for $e \in M$ and $\sigma \in G$,

$$\sum_{i=1}^{n} x_i \cdot e \cdot \sigma(y_i) = e \cdot e_\sigma = e_\sigma \cdot \sigma(e)$$

is not zero; then $e = e\cdot e_\sigma = e_\sigma\cdot\sigma(e) = \sigma(e)$ since e and $\sigma(e)$ are minimal elements of E, $\sigma(e\cdot a) = e_\sigma\cdot\sigma(e\cdot a) = e\cdot a\cdot e_\sigma = e\cdot a$ for all $a \in S$, and by hypothesis $\sigma(e\cdot a) = e\cdot a$ for all $a \in \Lambda$. Therefore $\sum_{i=1}^{n} x_i\cdot\gamma(y_i a) = \sum_{e\in M} e\cdot a = a$ for all $a \in \Lambda$. It is now readily verified that the canonical left S-module homomorphism of $S \otimes_{S\cap\Gamma} \Gamma$ into Λ has an inverse which is the mapping $a \rightsquigarrow \sum_{i=1}^{n} x_i \otimes \gamma(y_i a)$, $a \in \Lambda$; and the canonical left S-module homomorphism of $S \otimes_{S\cap\Gamma} (T \cap \Gamma)$ into T has an inverse which is the mapping $a \rightsquigarrow \sum_{i=1}^{n} x_i \otimes \gamma(y_i a)$, $a \in T$. Thus $\Lambda = S \otimes_{S\cap\Gamma} \Gamma$ and $T = S \otimes_{S\cap\Gamma} (T \cap \Gamma)$. Since $S \otimes_{S\cap\Gamma} \Gamma$ is naturally isomorphic to $S \otimes_{S\cap\Gamma} (T \cap \Gamma) \otimes_{T\cap\Gamma} \Gamma$, $\Lambda = T \otimes_{T\cap\Gamma} \Gamma$.

If S has no central idempotents other than 0 and 1, then the hypotheses of Lemma 1.2 are equivalent to the requirements that S and T be G-stable subrings of Λ such that $S \subseteqq T$, S be an outer semi-Galois extension of $S \cap \Gamma$, and G be faithfully represented as a group of automorphisms of S by restriction. The following example, however, shows that in general the conclusion of Lemma 1.2 cannot be obtained if only these latter conditions are assumed.

EXAMPLE 1.3. Let Λ be the ring of all complex 3×3 matrices of the form $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix}$, and set $\sigma\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} a & -b & 0 \\ -c & d & 0 \\ 0 & 0 & \bar{g} \end{pmatrix}$ and $\tau\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ \bar{c} & \bar{d} & 0 \\ 0 & 0 & \bar{g} \end{pmatrix}$. σ and τ are automorphisms of Λ, and they generate a subgroup G of order four in the group of all automorphisms of Λ. The subring Γ of G-invariant elements of Λ consists of all real, diagonal 3×3 matrices, and the center C of Λ consists of all complex, diagonal 3×3 matrices of the form diag $\{a, a, b\}$. Take $S = T = C$. C is a commutative G-stable subring of Λ and G is faithfully represented as a group of automorphisms of C by restriction. Moreover it may be verified that C is a Galois extension of $C \cap \Gamma$ with respect to the group H of automorphisms of C generated by the restriction of τ to C, but $\Lambda \neq C \otimes_{C\cap\Gamma} \Gamma$. In fact, setting $\phi\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} a & -b & 0 \\ -c & d & 0 \\ 0 & 0 & g \end{pmatrix}$, ϕ is a nontrivial automorphism of Λ over Γ which restricts to the identity map on C.

The remaining results of this section are directed toward developing a Galois theory for a ring Λ which satisfies any of the four equivalent statements of Theorem 1.1.

LEMMA 1.4. *If S is a G-stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $\Lambda = S \otimes_{S\cap\Gamma}\Gamma$, then Λ is an outer semi-Galois extension of Γ and Γ is a (Γ, Γ)-bimodule direct summand of Λ.*

Proof. Let S be a G-stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $\Lambda = S \otimes_{S\cap\Gamma}\Gamma$. Then one may readily verify that Λ is a separable extension of Γ and the centralizer of Γ in Λ is C. Therefore Λ is an outer semi-Galois extension of Γ. Furthermore, since S is a commutative ring, S is an outer semi-Galois extension of $S \cap \Gamma$; and, by [7, Th. 3.2], S is a projective Frobenius extension of $S \cap \Gamma$. In particular, S is a finitely generated, projective module over $S \cap \Gamma$; and it follows from [1, Proposition A. 3] and [9, Proposition 1] that $S \cap \Gamma$ is an $S \cap \Gamma$-module direct summand of S. Therefore Γ is a (Γ, Γ)-bimodule direct summand of $\Lambda = S \otimes_{S\cap\Gamma}\Gamma$.

LEMMA 1.5. *Let $\bar{G}$ be a finite group of automorphisms of a commutative ring S, let R be the subring of $\bar{G}$-invariant elements of S, and assume that S is a separable extension of R. For an intermediate ring $T, R \subseteqq T \subseteqq S$, the following statements are equivalent.*

(i) *There exists a finite group H of automorphisms of S such that T is the subring of H-invariant elements of S.*

(ii) *S is a projective Frobenius extension of T.*

(iii) *T is a separable extension of R.*

Proof. Apply Lemma 1.4 with $\Lambda = S = C$ and $\Gamma = R$ to establish that S is an outer semi-Galois extension of R and R is an R-module direct summand of S. The equivalence of statements (i) and (ii) follows from [7, Th. 3.3]. But it is a consequence of [7, Th. 2.3] and [10, 3.15] that S is a weakly Galois R-algebra, and the equivalence of statements (i) and (iii) follows from [10, Th. 3.8].

THEOREM 1.6. *Let S be a G-stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $\Lambda = S \otimes_{S\cap\Gamma}\Gamma$; and let Ω be a subring of Λ such that $\Gamma \subseteqq \Omega \subseteqq \Lambda$. There exists a finite group H of automorphisms of Λ such that S is H-stable and Ω is the subring of H-invariant elements in Λ if, and only if, $S \cap \Omega$ is a separable extension of $S \cap \Gamma$ and $\Omega = (S \cap \Omega) \otimes_{S\cap\Gamma}\Gamma$.*

Proof. Suppose H is a finite group of automorphisms of Λ such that S is H-stable and Ω is the subring of H-invariant elements in Λ. Then $S \cap \Omega$ is the subring of H-invariant elements in S, and

$S \cap \Omega$ is a separable extension of $S \cap \Gamma$ by Lemma 1.5. Also by Lemma 1.5, S is a finitely generated, projective module over $S \cap \Omega$; and it follows from [1, Proposition A. 3] and [7, Lemma 1.6] that S is a faithfully flat module over $S \cap \Omega$. Since $\Lambda = S \otimes_{S \cap \Gamma} \Gamma$ and $\Gamma \subset \Omega$, every group of automorphisms of Λ over Ω for which S is stable is faithfully represented as a group of automorphisms of S by restriction. Therefore $\Lambda = S \otimes_{S \cap \Omega} \Omega$ by Theorem 1.1. But $\Lambda = S \otimes_{S \cap \Omega} (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$ also. Since S is a faithfully flat module over $S \cap \Omega$, $\Omega = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$.

Conversely, suppose $S \cap \Omega$ is a separable extension of $S \cap \Gamma$ and $\Omega = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$. By Lemma 1.5, there exists a finite group H of automorphisms of S such that $S \cap \Omega$ is the subring of H-invariant elements in S. Since $\Lambda = S \otimes_{S \cap \Gamma} \Gamma$, there is a unique extension of H to a group of automorphisms of Λ over Γ. Let Ω' be the subring of H-invariant elements in Λ. $\Gamma \subseteqq \Omega'$; and, by the first part of this proof, $\Omega' = (S \cap \Omega') \otimes_{S \cap \Gamma} \Gamma$. But $S \cap \Omega'$ is the subring of H-invariant elements in S, so $S \cap \Omega' = S \cap \Omega$ and

$$\Omega' = (S \cap \Omega') \otimes_{S \cap \Gamma} \Gamma = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma = \Omega .$$

If S is a G-stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $\Lambda = S \otimes_{S \cap \Gamma} \Gamma$; then $C = S \otimes_{S \cap \Gamma} (C \cap \Gamma)$ and $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$ by Theorem 1.1, and C is a separable extension of $C \cap \Gamma$ by [2, Corollary 1.6]. Since C is stable for any group of automorphisms of Λ, S may be replaced by C in the preceding considerations. The following corollary is stated for comparison with Lemma 1.5.

COROLLARY 1.7. *Assume that C is a separable extension of $C \cap \Gamma$ and every finite group of automorphisms of Λ over Γ is faithfully represented as a group of automorphisms of C by restriction. For a subring Ω of Λ such that $\Gamma \subseteqq \Omega \subseteqq \Lambda$, the following statements are equivalent.*

(i) *There exists a finite group H of automorphisms of Λ such that Ω is the subring of H-invariant elements of Λ.*

(ii) *Λ is a projective Frobenius extension of Ω.*

(iii) *$C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and*

$$\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma .$$

Proof. Since every finite group of automorphisms of Λ over Γ is faithfully represented as a group of automorphisms of C by restriction, $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$ by Theorem 1.1. Therefore Λ is an outer semi-Galois extension of Γ and Γ is a (Γ, Γ)-bimodule direct summand of Λ by Lemma 1.4. Statements (i) and (ii) are equivalent by [7, Th. 3.3], and statements (i) and (iii) are equivalent by Theorem 1.6.

2. Separable algebras. In this section, let Λ be an algebra over a commutative ring R and let G be a finite group of automorphisms of the R-algebra Λ. Let Γ be the subalgebra of G-invariant elements of Λ and let C be the center of Λ. The results of the preceding section may be sharpened if Λ is a separable algebra over R. Thus let Λ be a separable algebra over R. Then Λ is a separable extension of C and C is a separable algebra over R by [2, Th. 2.3]. Clearly $R\cdot 1 \subseteqq C \cap \Gamma \subseteqq \Gamma$; and, consequently, Λ is a separable extension of Γ and C is a separable extension of $C \cap \Gamma$.

PROPOSITION 2.1. *If Λ is a separable extension of C and C is the centralizer of Γ in Λ, then the group of all automorphisms of Λ over Γ is faithfully represented as a group of automorphisms of C by restriction.*

Proof. Assume that Λ is a separable extension of C and C is the centralizer of Γ in Λ; but suppose that the group of all automorphisms of Λ over Γ is not faithfully represented as a group of automorphisms of C by restriction, and let η be a nontrivial automorphism of Λ over Γ which restricts to the identity map on C. Let a be an element of Λ such that $\eta(a) \neq a$, let m be a maximal ideal of C which contains the set $\{x \in C \mid x\cdot(\eta(a) - a) = 0\}$, and let C_m be the quotient ring of C with respect to the multiplicative system $C - m$. $\Lambda \otimes_C C_m$ is a central separable algebra over $C \otimes_C C_m = C_m$ by [2, Corollary 1.6], and $\eta \otimes 1$ is an automorphism of $\Lambda \otimes_C C_m$ over C_m. Since C_m is a local ring, $\eta \otimes 1$ is an inner automorphism by [2, Th. 3.6 and the remark which follows it]. Let $w \otimes 1/s$, $w \in \Lambda$ and $s \in C - m$, be a unit in $\Lambda \otimes_C C_m$ such that $w\cdot\eta(x) \otimes 1/s = xw \otimes 1/s$ for all $x \in \Lambda$. Λ is a finitely generated module over C by [2, Th. 2.1]; so let n be a positive integer and $\{b_i \in \Lambda \mid 1 \leqq i \leqq n\}$ be a set of generators for the C-module Λ. Since $w\cdot\eta(b_i) \otimes 1/s = b_i\cdot w \otimes 1/s$; there exists $t_i \in C - m$ such that $t_i\cdot(w\cdot(b_i) - b_i\cdot w) = 0$, $1 \leqq i \leqq n$, by [3, §2, No. 2, Proposition 4]. Letting $t = \prod_{i=1}^n t_i$, it is easily verified that $t \in C - m$ and $tw\cdot\eta(x) = xtw$ for all $x \in \Lambda$. Therefore tw is an element of the centralizer of Γ in Λ, which is C; $w \otimes 1/s = tw \otimes 1/(ts)$ is a unit in the center of $\Lambda \otimes_C C_m$; and, consequently, $\eta(x) \otimes 1 = x \otimes 1$ for all $x \in \Lambda$. In particular $\eta(a) \otimes 1 = a \otimes 1$; and, by [3, §2, No. 2, Proposition 4], there exists $u \in C - m$ such that $u\cdot(\eta(a) - a) = 0$. But such an element u cannot exist by the choice of m, and the proposition follow from this contradiction.

COROLLARY 2.2. *If Λ is a separable algebra over R, then the following statements are equivalent.*

(i) *C is the centralizer of Γ in Λ.*

(ii) *Λ is an outer semi-Galois extension of Γ.*
(iii) $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$.

Proof. Assume that Λ is a separable algebra over R. Then Λ is a separable extension of Γ, and therefore statements (i) and (ii) are equivalent. Moreover, Λ is a separable extension of C and C is a separable extension of $C \cap \Gamma$. It follows from Proposition 2.1 and Theorem 1.1 that statement (i) implies statement (iii). Clearly statement (iii) implies statement (i).

THEOREM 2.3. *Let Λ be a separable algebra over R such that $\Lambda = C \otimes_{C \cap \Gamma} \Gamma$; and let Ω be a subalgebra of Λ such that $\Gamma \subseteqq \Omega \subseteqq \Lambda$. There exists a finite group H of automorphisms of Λ such that Ω is the subalgebra of H-invariant elements of Λ if, and only if, Ω is a separable algebra over R.*

Proof. Λ is an outer semi-Galois extension of Γ and Γ is a (Γ, Γ)-bimodule direct summand of Λ by Lemma 1.4. Since Λ is a separable algebra over R, Λ is a (Λ, Λ)-bimodule direct summand of $\Lambda \otimes_R \Lambda$; and thus Γ is a (Γ, Γ)-bimodule direct summand of $\Lambda \otimes_R \Lambda$. As a (Γ, Γ)-bimodule, $\Lambda \otimes_R \Lambda$ is a left module over the enveloping algebra $\Gamma^e = \Gamma \otimes_R \Gamma^0$ of Γ; and for any left Γ^e-module X there is a natural isomorphism of $\mathrm{Hom}_{\Gamma^e}(\Lambda \otimes_R \Lambda, X)$ onto $\mathrm{Hom}\,({}_\Gamma\Lambda, \mathrm{Hom}\,(\Lambda_\Gamma, X))$. But Λ is a projective Frobenius extension of Γ by [7, Th. 3.2]; and, therefore, Λ is projective as either a left or right Γ-module. Consequently, $\Lambda \otimes_R \Lambda$ must be a projective left Γ^e-module. Therefore Γ is a projective left Γ^e-module, and it follows that Γ is a separable algebra over R.

By Theorem 1.6, there exists a finite group H of automorphisms of Λ such that Ω is the subring of H-invariant elements of Λ if, and only if, $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$. But if $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$, then one may readily verify that Ω is a separable algebra over R. Conversely, suppose Ω is a separable algebra over R. Since $R \cdot 1 \subseteqq C \cap \Gamma$, Γ and Ω are separable extensions of $C \cap \Gamma$; and, since C is the centralizer of Γ in Λ, $C \cap \Gamma$ is the center of Γ while $C \cap \Omega$ is both the centralizer of Γ in Ω and the center of Ω. But then $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ by [2, Th. 2.3], and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$ by [2, Th. 3.1].

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Received February 18, 1969. The author gratefully acknowledges receiving partial support in his research from the Florida State University Research Council and the National Science Foundation under grant GP-8424.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

ON VISUAL HULLS

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The concept of visual hull has been introduced by G. H. Meisters and S. Ulam. In the following article we study a few of the problems arising from this notion and, in particular, establish (Theorem 3) a conjecture of W. A. Beyer and S. Ulam.

Let C be a set in R^n and $1 \leqq j \leqq n-1$. Then the j^{th} visual hull $H_j(C)$ of C is defined to be the largest set whose j^{th} projections are contained in those of C. Alternatively, $H_j(C)$ is the set of points x in R^n such that each $(n-j)$-flat through x contains a point of C. Let G_j^n denote the Grassmannian of j-subspaces in R^n with $\mu_j(G_j^n) = 1$ for the usual measure μ_j associated with G_j^n regarded as a metric 0_n-factorspace. (For further information about μ_j compare, for example, [3]). The j^{th} virtual hull $V_j(C)$ of C is defined to be the set of points $x \in R^n$ such that almost all (with respect to μ_{n-j}) $(n-j)$-flats through x contain a point of C. Thus, if $n = 3, j = 2, H_2(C)(V_2(C))$ corresponds to those points in R^3 which are photographically indistinguishable (with probability one) from C. A j^{th} minimal hull of C in R^n is a minimal set in R^n whose j^{th} projections coincide with those of C. In [2] the announced purpose of the paper was to disprove the conjecture that $H_j(C) - C$ is connected to C, i.e., $\not\ni$ disjoint open sets U, V such that $U \supset H_j(C) - C \neq \emptyset$ and $V \supset C \neq \emptyset$. To this we remark that a simple counterexample can be obtained by considering the closed set C formed by removing the relative interiors of alternate sides of a regular hexagon inscribed in a plane circle with centre a. The first visual hull $H_1(C)$ is then $C \cup \{a\}$.

2. Visual hulls of unions of polytopes.

THEOREM 1. *Let $A_1, \cdots, A_{j+1}$ be spherically convex, closed subsets (not necessarily nonempty) of the sphere S^{n-1}, such that each $(n-j-1)$-subsphere of S^{n-1} has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. Then $A_1 \cap \cdots \cap A_{j+1} \neq \emptyset$. (so, that, in particular, each set A_i is nonempty).*

REMARK. S^{n-1} is the unit sphere of R^n and an $(n-j-1)$-subsphere of S^{n-1} is the intersection of an $n-j$ subspace with S^{n-1}. A set $C \subset S^{n-1}$ is spherically convex if C is contained in an open hemisphere of S^{n-1} and, if $x, y \in C$ then C contains the minor arc on the 1-subsphere determined by x, y and 0 (the centre of S^{n-1}).

Proof. The case $n = 1$ is trivial. We assume inductively that the result is true for all $n' < n$ and it remains to prove the result for $j+1$ sets on S^{n-1}. Assume on the contrary that there exist spherically convex closed subsets $A_1, \cdots, A_{j+1} \subset S^{n-1}$ such that

$$T \cap (A_1 \cup \cdots \cup A_{j+1}) \neq \emptyset$$

for each $(n-j-1)$-subsphere T of S^{n-1}, and $A_1 \cap \cdots \cap A_{j+1} = \emptyset$. Let $A = A_1 \cap \cdots \cap A_j$. Then A, A_{j+1} are disjoint spherically convex closed subsets of S^{n-1}, and there exists an $(n-2)$-subsphere S' of S^{n-1} which separates A and A_{j+1} and such that $S' \cap A = \emptyset$, $S' \cap A_{j+1} = \emptyset$. Set $A_i' = A_i \cap S'$ $(1 \leqq i \leqq j)$. Then each A_i' is a spherically convex closed subset of S' and, since $A_{j+1} \cap S' = \emptyset$, each $(n-j-1)$-subsphere of S' has a nonempty intersection with $A_1' \cup \cdots \cup A_j'$. Hence by the inductive assumption $A_1' \cap \cdots \cap A_j' = A \cap S' \neq \emptyset$; contradiction.

THEOREM 2. *In R^n let $C_1, \cdots, C_{j+1}$ be $j+1$ compact convex sets. If $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then either $x \in \bigcup_{i=1}^{j+1} C_i$ or there exists a halfline l emanating from x such that $l \cap C_i \neq \emptyset, 1 \leqq i \leqq j+1$.*

COROLLARY. *In R^n let $C_1, \cdots, C_{j+1}$ be compact convex sets. Then asufficient condition for $H_j(\bigcup_{i=1}^{j+1} C_i) = \bigcup_{i=1}^{j+1} C_i$ is that the sets do not have a common transversal.*

Proof. On S^{n-1} define $j+1$ spherically convex closed subsets $A_1, \cdots, A_{j+1}$ so that $u \in A_i$ if $u \in S^{n-1}$ and the half line $\{x + \lambda u \mid \lambda \geqq 0\}$ meets C_i. Then, as $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ each $(n-j-1)$-subsphere of S^{n-1} has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. And so, by Theorem 1, there exists $u \in \bigcap_{i=1}^{j+1} A_i$, i.e., the halfline $\{x + \lambda u \mid \lambda \geqq 0\}$ meets each of $C_1, \cdots, C_{j+1}$.

THEOREM 3. *In R^n let $C_1, \cdots, C_{j+1}$ be nonempty compact convex sets. Then the number of components of $H_j(\bigcup_{i=1}^{j+1} C_i)$ is at most $j+1$ with equality if and only if $C_1, \cdots, C_{j+1}$ are pairwise disjoint.*

Proof. By Theorem 2, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i) - \bigcup_{i=1}^{j+1} C_i$, then there exists a halfline $l = \{x + \lambda u \mid \lambda \geqq 0\}$ such that l meets each of

$$C_1, \cdots, C_{j+1}\,.$$

Then $x + \alpha_k u \in C_k$ for some $\alpha_k > 0$. We set $\alpha = \min\{\alpha_k \mid 1 \leqq k \leqq j+1\}$ and want to show that $x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for all λ with $0 \leqq \lambda \leqq \alpha$. Set $y = x + \lambda u$ and let P be an $(n-j)$-subspace. As $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ there exists i such that the $(n-j)$-flat $x + P$ meets C_i at v, say. Set $z = x + \alpha_i u \in C_i$. Then, as y lies between x and z on l, there exists μ, $0 \leqq \mu \leqq 1$, such that $y = \mu x + (1-\mu)z$. Then the $(n-j)$-flat $y + P$ through y contains the point $\mu v + (1-\mu)z$ of C_i. As P

was arbitrary we conclude that $y \in H_j(\bigcup_{i=1}^{j+1} C_i)$ and hence that $x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for $0 \leqq \lambda \leqq \alpha$. Hence, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then x is connected, via a line segment in $H_j(\bigcup_{i=1}^{j+1} C_i)$, to at least one of the sets C_i. Hence $H_j(\bigcup_{i=1}^{j+1} C_i)$ has at most $j + 1$ components with equality only if the C_i's are disjoint. If the sets $C_1, \cdots, C_{j+1}$ are pairwise disjoint then in order to show that $H_j(\bigcup_{i=1}^{j+1} C_i)$ has exactly $j + 1$ components it is enough to show that for each k, $1 \leqq k \leqq j + 1$, there exist disjoint open sets U_k, V_k such that $U_k \cup V_k \supset H_j(\bigcup_{i=1}^{j+1} C_i)$ and $U_k \supset C_k$, $V_k \supset \{C_1 \cup \cdots \cup C_{k-1} \cup C_{k+1} \cup \cdots \cup C_{j+1}\}$. We suppose, without loss of generality, that $k = 1$. For $i = 2, \cdots, j + 1$ let H_i denote a hyperplane which strictly separates C_1 from C_i, and let H_i^0 be the open halfspace bounded by H_i and containing C_1. We can assume that the H_i's are in general position. Set $U_1 = \bigcap_{i=2}^{j+1} H_i^0$, $V_1 = R^n - \bar{U}_1$. Then U_1 and V_1 are disjoint open sets, $C_1 \subset U_1$, $\bigcup_{i=2}^{j+1} C_i \subset V_1$. It remains to show that $H_j(\bigcup_{i=1}^{j+1} C_i) \subset U_1 \cup V_1$, and it is enough to show that $(\bar{U}_1 \cap \bar{V}_1) \cap H_j(\bigcup_{i=1}^{j+1} C_i) = \varnothing$. Since the H_i's are in general position, their intersection $\bigcap_{i=2}^{j+1} H_i$ is an $(n - j)$-dimensional flat L. Let I be the j-dimensional subspace orthogonal to L. If M is any subset of R^n we denote by $\operatorname{proj}_I M$ the set of all points $x \in I$ for which the flat L_x, which is parallel to L and contains x, has a nonempty intersection with M. $\operatorname{proj}_I U_1$ and $\operatorname{proj}_I V_1$ are two open sets in I with common boundary $\operatorname{proj}_I (\bar{U}_1 \cap \bar{V}_1)$. As $\operatorname{proj}_I C_1 \subset \operatorname{proj}_I U_1$, $\operatorname{proj}_I \bigcup_{i=2}^{j+1} C_i \subset \operatorname{proj}_I V_1$ it follows that $(\operatorname{proj}_I (\bar{U}_1 \cap \bar{V}_1)) \cap (\operatorname{proj}_I \bigcup_{i=1}^{j+1} C_i) = \varnothing$. Now, if z is an arbitrary point in $\bar{U}_1 \cap \bar{V}_1$ it follows that $L_z \cap (\bigcup_{i=1}^{j+1} C_i) = \varnothing$, and since $\dim L_z = n - j$, we find, by the definition of H_j, that z does not belong to $H_j(\bigcup_{i=1}^{j+1} C_i)$. Therefore $(\bar{U}_1 \cap \bar{V}_1) \cap H_j(\bigcup_{i=1}^{j+1} C_i) = \varnothing$.

REMARKS. The proof of Theorem 3 also shows that any component of $H_j(\bigcup_{i=1}^{j+1} C_i)$ has the property that any two points of it can be joined by a broken line in it, consisting of at most 3 segments. Hence it is natural to ask: When are these components convex? (supposing now that the C_i's are disjoint). In [1] W. A. Beyer has shown an example of three (nondisjoint) polytopes C_i in R^3 such that $H_2(C_1 \cup C_2 \cup C_3)$ is not a polyhedron. We don't know whether a similar construction would be possible with disjoint polytopes. Let us mention here a few more technical terms. If M is any subset of R^n, we denote by $\operatorname{aff} M$ the affine hull of M and by $\operatorname{conv} M$ the convex hull of M. $\operatorname{relint} M$ means the interior of M with respect to the natural topology in $\operatorname{aff} M$. By the dimension $\dim M$ of M we understand the algebraic dimension of the flat $\operatorname{aff} M$. A polytope is the convex hull of some finite set. If $P \subset E^n$ is a convex set we denote by $\operatorname{ext} P$ the set of extreme points of P and by $\exp P$ the set of its exposed points. For an exact definition of these terms the reader may compare, for example, the introductory chapters of [4].

THEOREM 4. (i) *In R^n let C_1, C_2 be compact convex sets. Then $H_1(C_1 \cup C_2)$ is the union of at most two convex components which are polytopes whenever C_1 and C_2 are polytopes.*

(ii) *There exist in R^3 three disjoint polytopes such that one of the components of the second visual hull of their union is not convex.*

LEMMA 1. *Let C_1, C_2 be n-dimensional polytopes in R^n. If $a \notin H_1(C_1 \cup C_2)$ there exists a hyperplane H such that*

(1) $a \notin H$, *H separates a from C_1*

(2) $H \cap C_i = \emptyset$ *or H supports C_i ($i = 1, 2$)*

(3) $\operatorname{aff}(H \cap (C_1 \cup C_2)) = H$.

Proof of Lemma 1. The case $n = 1$ is trivial, and we assume $n \geqq 2$. If there exists a hyperplane P through a which does not meet $C_1 \cup C_2$ and does not separate C_1 and C_2 then $\operatorname{conv}(C_1 \cup C_2)$ is an n-dimensional polytope not containing a, and the lemma follows from standard results on polytopes. Hence it can be supposed that there is a hyperplane H for which (1) and also (2'): H separates C_1 and C_2 holds. We choose H in the set $\mathfrak{H}$ of hyperplanes for which (1) and (2') holds. We assume that $h = \dim \operatorname{aff} T$ is maximal, where $T = H \cap (C_1 \cup C_2)$. Obviously $h \geqq 0$. If $h < n - 1$, let $F \subset H$ be an $(n - 2)$-dimensional hyperplane in H containing T, and denote by π: $R^n \rightarrow E$ the projection along F onto a 2-dimensional flat E orthogonal to F. It is easy to see that there is a line L in E such that: (α): the singleton $\pi(T)$ is contained in L. (β): $\pi(a) \notin L$, L separates $\pi(a)$ from the polygon $\pi(C_1)(\gamma)$: L separates $\pi(C_1)$ and $\pi(C_2)$.

$$(\delta) \operatorname{aff}(L \cap (\pi(C_1) \cup \pi(C_2)) = L \ .$$

(Notice that the conditions $(\alpha) - (\gamma)$ are fulfilled by $\pi(H)$). The hyperplane $\pi^{-1}(L)$ of E^n intersects $C_1 \cup C_2$ in a set S with $\dim \operatorname{aff} S = h + 1$. Since $S \in \mathfrak{H}$ this contradicts the maximality of h. Hence the lemma is established.

Proof of Theorem 4. (i) We first prove the result when C_1, C_2 are n-dimensional polytopes. If $C_1 \cap C_2 \neq \emptyset$ then

$$H_1(C_1 \cup C_2) = \operatorname{conv}(C_1 \cup C_2) \ ,$$

which is a polytope. We suppose therefore that $C_1 \cap C_2 = \emptyset$. Let $\{H_i\}_{i=1}^m$ be the finite set of those hyperplanes which do not contain an interior of C_j $(j = 1, 2)$ and for which $\dim(H_i \cap (C_1 \cup C_2)) = n - 1$. By C_j^* we denote the (finite) intersection of those closed half spaces which contain C_j and whose bounding hyperplane is amongst $\{H_i\}_{i=1}^m$, $j = 1, 2$. Then C_j^* is polyhedral and, since C_1, C_2 are compact, C_j^* is a polytope,

$j = 1, 2$. We show that $H_1(C_1 \cup C_2) = C_1^* \cup C_2^*$. Suppose that $x^* \notin C_1^* \cup C_2^*$. Then there exist closed halfspaces H_1^*, H_2^* with bounding hyperplanes H_1, H_2 amongst $\{H_i\}_{i=1}^m$ such that $x^* \notin H_1^* \supset C_1$, $x^* \notin H_2^* \supset C_2$. If

$$x^* \in H_1(C_1 \cup C_2),\ H_1 \quad \text{and} \quad H_2$$

must separate C_1 and C_2. Consider H_1 and the two disjoint compact sets $H_1 \cap C_1, H_1 \cap C_2$ in H_1. There exists an $n - 2$ dimensional flat L in H_1 which strictly separates $H_1 \cap C_1$ and $H_1 \cap C_2$. By slightly rotating H_1 about L in the appropriate direction we obtain a hyperplane H_1' which strictly separates C_1 and C_2 as well as x^* and C_1. Similarly we can obtain a hyperplane H_2' which strictly separates C_1 and C_2, and x^* and C_2. We may suppose that H_1', H_2' are not parallel and so $H_1' \cap H_2'$ is an $n - 2$ flat. Suppose, without loss of generality, that $H_1' = \{x \mid \langle x, \xi\rangle = \alpha > 0\}$, $H_2 = \{x \mid \langle x, \eta\rangle = \beta > 0\}$. Then

$$C_1 \subset \{x \mid \langle x, \xi\rangle > \alpha\} \cap \{x \mid \langle x, \eta\rangle > \beta\}$$
$$C_2 \subset \{x \mid \langle x, \eta\rangle < \alpha\} \cap \{x \mid \langle x, \eta\rangle < \beta\} .$$

Consider the hyperplane H: $\{x \mid \langle x, \lambda\xi + (1 - \lambda)\eta\rangle = 0\}$, where $\lambda\alpha + (1 - \lambda)\beta = 0$ and $0 < \lambda < 1$. Then $x^* \in H$ and, using the above inequalities, $C_i \cap H = \emptyset$, $i = 1, 2$. Hence x^* is not in $H_1(C_1 \cup C_2)$, and we have $H_1(C_1 \cup C_2) \subset C_1^* \cup C_2^*$. Conversely, if $x^* \in C_1^* \cup C_2^* - H_1(C_1 \cup C_2)$, suppose without loss of generality that $x^* \in C_1^*$. Then, by Lemma 1, there exists a hyperplane H amongst $\{H_i\}_{i=1}^m$ which does not contain x^* and which separates x^* from C_1. Then, if H^* donotes the closed halfspace containing C_1 whose bounding hyperplane is H, $x^* \notin H^*$ and so $x^* \in C_1^*$; contradiction. And so $H_1(C_1 \cup C_2) = C_1^* \cup C_2^*$, which is the union of two polytopes. If C_1, C_2 are compact convex sets we choose decreasing sequences $\{P_1^n\}_{n=1}^\infty$, $\{P_2^n\}_{n=1}^\infty$ of polytopes such that $C_i = \bigcap_{n=1}^\infty P_i^n$, $i = 1, 2$. Then, using the above notation,

$$H_1(C_1 \cup C_2) = \bigcap_{n=1}^{\infty} P_1^{n*} \cap \bigcap_{n=1}^{\infty} P_2^{n*} .$$

(ii) Let W be the cube $\{x = (x_1, x_2, x_3) \mid -1 \leqq x_i \leqq 1,\ i = 1, 2, 3\}$ in R^3, and denote by W_i the facet of W defined by $x_i = 1$. Set $C_1 = W_1$, $C_2 = 2W_2$, $C_3 = 3W_3$. Let $B_i (1 \leqq i \leqq 3)$ be the components of $H_2(\bigcup_{i=1}^3 C_i)$, where the indices are chosen such that, for all i, $C_i \subset B_i$. Clearly $(0, 0, 0) \in B_1$ as does, of course, the point $(1, -1, -1) \in B_1 \cap C_1$. However we show that the line segment m: $\{x = \lambda(1, -1, -1) \mid 0 < \lambda < 1\}$ is not in B_1. Now $C_1 \cup C_2$ is contained in the halfspace $\{x \mid \langle x, (0, 1, 1)\rangle \geqq 0\}$ whose bounding hyperplane P passes through the points $(0, 0, 0)$, $(1, -1, 1)$ and $(-1, -1, 1)$; $P \cap \operatorname{aff} W_1$ is a line in direction $(0, -1, 1)$. If $y \in m$, then $y = \mu(1, -1, -1)$ for some μ, $0 < \mu < 1$. Consider the line $l = y + \{\lambda(0, -1, 1) \mid \lambda \text{ real}\}$. If $z = (z_1, z_2, z_3) \in l$ then $z_1 = \mu < 1$, i.e., $z \notin C_1$. Also $\langle z, (0, 1, 1)\rangle = -2\mu < 0$ which means that $z \notin C_1 \cup C_2$. Therefore l does not meet $C_1 \cup C_2 \cup C_3$, m does not belong to B_1, and B_1 is not convex.

In [6] V. L. Klee proved that if all j^{th} projections of a compact convex body C in R^n (j fixed $\geqq 2$) are polytopes, then C is a polytope. As a partial analogue to this for unions of two convex bodies we prove

THEOREM 5. *Let C_1, C_2 be two disjoint compact convex bodies in R^n such that each j^{th} projection of $C_1 \cup C_2$ (j fixed $\geqq 2$) is the union of two polytopes. Then* (i) $\operatorname{ext}(C_i) = \exp(C_i)$ *and* $\operatorname{ext}(C_i)$ *is countable* $(i = 1, 2)$ *but* (ii) $\operatorname{ext}(C_i)$ *is not necessarily finite.*

Proof. Let a be an extreme point of C_1 and we suppose, without loss of generality, that $a = 0$, the origin of R^n. Then, to prove (i) it is enough to prove that the convex cone K of outward normals to C_1 at 0 is n-dimensional. We assume that $\dim K \leqq n - 1$ so that K is contained in an $(n - 1)$-subspace P_1, and seek a contradiction. Let P_2 be an $(n - 1)$-subspace which supports C_1 at 0. Of course $P_1 \neq P_2$. We can choose an $(n - 1)$-subspace P_3 so that there exists a translate of P_3 which strictly separates C_1 and C_2 and such that the normal to P_3 at 0 intersects P_1 only at 0. Then $P_2 \cap P_3$ is a subspace of dimension at least $n - 2$ and we choose an $n - j$ subspace Q in $P_2 \cap P_3$. The orthogonal complement S of Q in R^n is a j-dimensional subspace which meets P_1 in a $(j - 1)$-subspace. The projection of $C_1 \cup C_2$ onto S is the union of two polytopes. Further, as $P_3 \cap C_2 = \emptyset$, 0 is at positive distance from $\operatorname{proj} C_2$. As 0 is an extreme point of $\operatorname{proj} C_1$, it follows that 0 is a locally polyhedral extreme point for $\operatorname{proj} C_1$. Hence, in S, the cone of outward normals to $\operatorname{proj} C_1$ at 0 is j-dimensional. Further, any $(j - 1)$-plane H of support in S to $\operatorname{proj} C_1$ at 0 can be extended to an $(n - 1)$-plane of support $H + Q$ in R^n to C_1 at 0. Also, the outward normals to these planes form a j-dimensional convex cone lying in S. Hence $j = \dim(K \cap S) = \dim(P_1 \cap S) = j - 1$; contradiction. And so (i) is proved.

To prove (ii) we construct an example in R^3 of two convex bodies C_1, C_2, both of which have a countable infinity of extreme points but, nevertheless, each 2-projection of $C_1 \cup C_2$ is the union of two convex polygons. Let $l = \{x \mid x_1 = x_2 = 0, -1 \leqq x_2 \leqq 1\}$ be a line segment and $S = \{x \mid (x_1 - 1)^2 + x_2^2 = 1, x_3 = 0\}$ a plane circle. By T we denote the set of those points on S with x_2-coordinate $^{\pm}(1/n)$ for $n = 1, 2, \cdots$. We take $C_1 = \operatorname{conv}\{l \cup T\}$, which is a compact convex body in R^3 with extreme points $T \cup \{(0, 0, -1), (0, 0, 1)\}$. It is easily seen that there is precisely one 2-projection of C_1 which is not a convex polygon, and that is in the direction $(0, 0, 1)$. Further the only limit point of extreme points of this projection is $(0, 0, 0)$. Define C_2 as a disjoint copy of

C_1 formed by placing C_2 above C_1 in such a way that their respective major lines pierce the centres of their respective circles. From above, every 2-projection of $C_1 \cup C_2$ is the union of two convex polygons and and both C_1 and C_2 are compact bodies with a countable infinity of extreme points.

3. Visual hulls of more general sets. The following problem can be formulated.

Is the visual (virtual) (minimal) hull of a borel (analytic) set in R^n necessarily borel (analytic)?

The answer is affirmative (Theorem 6) for virtual hulls and negative (Theorem 7) for minimal hulls. Whilst it is not true (Theorem 8) that the j^{th} visual hull of a borel set is necessarily borel, we have been unable to decide whether or not the j^{th} visual hull of a borel or of an analytic set is always analytic, except in the cases covered by Theorem 9. It is possible also that the j^{th} visual hull of a convex borel (analytic) set is a borel (analytic) set, and we include some partial results (Theorem 9) in this direction. As before we denote by G_j^n the Grassmannian of j-subspaces of R^n and by μ_j the invariant (with respect to 0_n acting in the usual way on G_j^n) measure normalised so that $\mu_j(G_j^n) = 1$.

LEMMA 2. *Let A be an analytic set in R^n and denote by A^* the set of those j-subspaces in G_j^n which meet A. Then*

(i) *A^* is an analytic set in G_j^n and hence A^* is μ_j measurable.*

(ii) *If $\mu_j(A^*) > a$ then there exists a compact subset A' of A such that $\mu_j(A'^*) > a$.*

(iii) *If $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of analytic sets in R^n then $\mu_j(\bigcup_{i=1}^{\infty} A_i)^* = \lim_{i\to\infty} \mu_j(A_i^*)$.*

(iv) *If $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of analytic sets in R^n then $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* = \lim_{i\to\infty} \mu_j(A_i^*)$.*

Proof. (i) Let I be the set of irrational numbers in $[0, 1]$ and, if $i = (i_1, \cdots, i_n, \cdots)$ is a typical member of I expressed as a continued fraction, set $i \mid n = (i_1, \cdots, i_n)$. Then, as A is analytic, it can be represented as $A = \sum_{i\in I} \bigcap_{n=1}^{\infty} A(i \mid n)$ where the sets $A(i \mid n)$ form, for each fixed i, a decreasing sequence of compact subsets of R^n. Then $A^* = \sum_{i\in I} \bigcap_{n=1}^{\infty} A^*(i \mid n)$. As each $A^*(i \mid n)$ is a compact subset of G_j^n, we conclude that A^* is an analytic set.

(ii) If $\mu_j(A^*) > a + \delta$ with $\delta > 0$, then we can choose m_1, $1 \leqq m_1 < \infty$, such that if I_1 denotes the set of irrational numbers

$$i = (i_1 \cdots i_n \cdots)$$

with $1 \leqq i_1 \leqq m_1$ and $A_1^* = \sum_{i\in I_1} \bigcap_{n=1}^{\infty} A^*(i \mid n)$ then $\mu_j(A_1^*) > a + \delta$.

Proceeding by induction we may define natural numbers m_p, $1 \leqq p < \infty$, such that if I_q denotes the subset of those irrationals i with $1 \leqq i_p \leqq m_p$ for $p = 1, \cdots q$, and $A_q^* = \sum_{i \in I_q} \bigcap_{n=1}^{\infty} A^*(i \mid n)$ then $\mu_j(A_q^*) > a + \delta$. Let I' be the compact subset of $[0, 1]$ defined as the set of those irrational numbers i for which $1 \leqq i_p \leqq m_p$ for $p = 1, 2, \cdots$, and

$$A'^* = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A^*(i \mid n) .$$

Then $\bigcap_{q=1}^{\infty} A_q^* = A'^*$ and so $\mu_j(A'^*) \geqq a + \delta > a$. Also

$$A' = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A(i \mid n)$$

is a compact subset of A, as I' is a compact subset of I.

(iii) $\mu_j(\bigcup_{i=1}^{\infty} A_i)^* = \mu_j(\bigcup_{i=1}^{\infty} A_i^*) = \lim_{i \to \infty} \mu_j(A_i^*)$.

(iv) Clearly $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* \leqq \lim_{i \to \infty} \mu_j(A_i^*)$. Now set $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* = a$ and suppose $\lim_{i \to \infty} \mu_j(A_i^*) > a + \varepsilon$, for some positive number ε. By (ii) we find a compact set $B_1 \subset A_1$ such that $\mu_j(B_1^*) \geqq \mu_j(A_1^*) - \varepsilon/2$. Now we have $A_2^* = (B_1 \cap A_2)^* \cup (A_2^* - B_1^*)$, where

$$A_2^* - B_1^* = \{F \in G_j^n \mid F \cap A_2 \neq \varnothing \ , \quad \text{but} \quad F \cap B_1 = \varnothing\} .$$

Since $A_2^* \subset A_1^*$ we derive further $A_2^* \subset (B_1 \cap A_2)^* \cup (A_1^* - B_1^*)$, or $\mu_j(A_2^*) \leqq \mu_j(B_1 \cap A_2)^* + \varepsilon/2$. Since $B_1 \cap A_2$ is analytic there exists, again by (ii), a compact set $B_2 \subset (B_1 \cap A_2)$ such that

$$\mu_j(B_2)^* \geqq \mu_j(B_1 \cap A_2)^* - \varepsilon/4$$

and consequently $\mu_j(B_2)^* \geqq \mu_j(A_2)^* - (\varepsilon/2 + \varepsilon/4)$. Continuing this process we obtain a decreasing sequence $\{B_i\}_{i=1}^{\infty}$ of compact subsets of R^n such that $B_i \subset A_i$, $i = 1, 2, \cdots$, and $\mu_j(B_i^*) \geqq \mu_j(A_i^*) - \sum_{p=1}^{i} \varepsilon/(2^p)$. Then $\bigcap_{i=1}^{\infty} B_i^* = (\bigcap_{i=1}^{\infty} B_i)^* \subset (\bigcap_{i=1}^{\infty} A_i)^*$, and $\mu_j(\bigcap_{i=1}^{\infty} B_i^*) = \lim_{i \to \infty} \mu_j(B_i^*) \leqq a$; but also $\lim_{i \to \infty} \mu_j(B_i^*) \geqq \lim_{i \to \infty} \mu_j(A_i^*) - \varepsilon$. Combining the last two inequalities we find $\lim_{i \to \infty} \mu_j(A_i) \leqq a + \varepsilon$, a contradiction.

THEOREM 6. *Let C be a borel (analytic) set in R^n. Then the j^{th} virtual hull $V_j(C)$ is a borel (analytic) set.*

Proof. Suppose first that C is a borel set in R^n, and we need to show that $V_j(C)$ is a borel set. If D is a subset of R^n and $x \in R^n$, let $D[x, n-j]$ denote the set of those $n - j$ subspaces F in G_{n-j}^n such that $(x + F) \cap D \neq \varnothing$. If $0 < \lambda < 1$ let $D(n - j, \lambda)$ be the set of all x in R^n such that $\mu_{n-j}(D[x, n - j]) > \lambda$. Let B denote the largest family of subsets of R^n such that $D \in B$ if (i) D is a borel set in R^n. (ii) $D(n - j, \lambda)$ is a borel set for all λ, $0 < \lambda < 1$. We shall prove that B coincides with the family of borel subsets of R^n, and it is enough to show that B contains the open sets and is closed under the operations of increasing union and decreasing intersection. If D is an open subset of R^n, then it is easy to see that $D(n-j, \lambda)$ is open for all λ, $0 < \lambda < 1$, and so B contains all the open sets. Now suppose that $\{E_i\}_{i=1}^{\infty}$ is an increasing sequence of sets in B and set $E = \bigcup_{i=1}^{\infty} E_i$. We want to show that for each λ, $0 < \lambda < 1$, the equality $E(n-j, \lambda) = \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$ holds. In order to do this we observe the following equivalences: $x \in E(n-j, \lambda) \leftrightarrow \mu_{n-j}(E[x, n-j]) > \lambda \leftrightarrow \lim_{i\to\infty} \lambda_{n-j}(E_i[x, n-j]) > \lambda \leftrightarrow x \in \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$. Here the first equivalence holds by definition, the second one follows directly from Lemma 2, (iii), if we observe that this lemma remains true if M^* denotes, for each $M \subset R^n$, the set $M[x, n-j]$ ($x \in R^n$ fixed). (The lemma itself is stated for the special case where x is the origin of R^n.) The last equivalence again follows immediately from the definitions, we only have to observe that the sequence $\{E_i\}_{i=1}^{\infty}$ is increasing. Now suppose that $\{H_i\}_{i=1}^{\infty}$ is a decreasing sequence of subsets of B and set $H = \bigcap_{i=1}^{\infty} H_i$. Suppose λ fixed, $0 < \lambda < 1$, and let m be a natural number such that $\lambda + 1/m < 1$. Then, using (iv) of Lemma 2, we find by an argument analogous to the one above, $H(n-j, \lambda) = \bigcup_{p=m}^{\infty} \bigcap_{i=1}^{\infty} H_i(n-j, \lambda + 1/p)$. Hence $H(n-j, \lambda)$ is a borel set, and $H \in B$. Therefore, B is the family of borel subsets of R^n and so, in particular, $C \in B$. Further $V_j(C) = \bigcap_{p=2}^{\infty} C(n-j, 1-(1/p))$ and so $V_j(C)$ is a borel set.

To show that $V_j(A)$ is analytic whenever A is analytic, we use the well known result that there exists an $F_{\sigma\delta}$ set K in R^{n+1} such that A is the orthogonal projection $\operatorname{proj} K$ of K into R^n (see, for example, [8]). Call an $(n-j+1)$-subspace H of R^{n+1} upright if H has the form $\{\hat{H} + \lambda(0, \cdots, 0, 1) \mid -\infty < \lambda < \infty\}$ where $\hat{H} \in G_{n-j}^n$. Let U_{j+1} be the set of upright $(n-j+1)$-subspaces in R^{n+1} with the measure μ' induced by μ_{n-j} in the obvious manner. We can define $U_{j+1}(C)$ of a set C in R^{n+1} as the set of all those points x in R^{n+1} such that almost all (with respect to μ') upright $(n-j+1)$-flats through x meet C. As above, it can been shown that $U_{j+1}(C)$ is a borel set whenever C is a borel set. Clearly $\operatorname{proj} U_{j+1}(K) = V_j(A)$ and, since the projection of a borel set is analytic, we conclude that $V_j(A)$ is an analytic subset of R^n.

THEOREM 7. *Let C be an open convex subset of R^n. Then assuming the continuum hypothesis, C contains a minimal j^{th} hull D such that every analytic subset of D is countable.*[1]

Proof. We assume the continuum hypothesis and let Ω be the

[1] As the referee pointed out, Theorem 7 may be a special case of a much more general theorem on effective constructions.

first uncountable ordinal. Let $\{A_\xi\}_{\xi<\Omega}$ be an enumeration of the analytic subsets of R^n of $(n-j)$-dimensional measure zero; let $\{H_\xi\}_{\xi<\Omega}$ be an enumeration of the $(n-j)$-flats which meet C. Let F be a fixed $(n-j)$-subspace of R^n and denote by α a fixed set, which is not a point of R^n. We now choose a set $E=\{M_\xi\}_{\xi<\Omega}$ and a collection of translates $\{F_\xi\}_{\xi<\Omega}$ of F inductively as follows. Take $M_1 \in (H_1 - A_1) \cap C$ and let F_1 be a translate of F through M_1. Suppose now that $M_{\xi'}$, $F_{\xi'}$ have been defined for all $\xi' < \xi$, where ξ is some ordinal proceeding Ω. If H_ξ is a translate of F we take $F_\xi = H_\xi$ and consider two possibilities:

(a) If $\exists \xi' < \xi$ such that $M_{\xi'} \in H_\xi$ then we take $M_\xi = \alpha$.

(b) If $\exists \xi' < \xi$ such that $M_{\xi'} \in H_\xi$ we choose M_ξ in the set $(H_\xi - (\bigcup_{\xi'<\xi} H_{\xi'} \cup \bigcup_{\xi'<\xi} A_{\xi'})) \cap C$. Such a choice is possible as $H_\xi \cap C$ has positive $(n-j)$-dimensional measure whereas $H_\xi \cap (\bigcup_{\xi'<\xi} H_{\xi'} \cup \bigcup_{\xi'<\xi} A_{\xi'})$ has zero $(n-j)$-dimensional measure, being a countable union of sets of measure zero. If H_ξ is not a translate of F we find, by similar arguments, that the set $(H_\xi - (\bigcup_{\xi'<\xi} H_{\xi'} \cup \bigcup_{\xi'<\xi} A_{\xi'} \cup \bigcup_{\xi'<\xi} F_{\xi'})) \cap C$ is not empty. We choose M_ξ in this set and let F_ξ be the translate of F through M_ξ. We claim that the set $D = E - \alpha$ is a j^{th} minimal hull for C which meets each analytic subset in at most a countable number of points. To show that all j^{th} projections of D coincide with those of C, it is enough to show that the j^{th} visual hull of D contains C. Let x be a point of C and let P be an $(n-j)$-flat through x. Then P is amongst $\{H_\xi\}_{\xi<\Omega}$, say $P = H_{\xi'}$. If $M_{\xi'} \neq \alpha$ then $M_{\xi'} \in D \cap H_{\xi'}$. If $M_{\xi'} = \alpha$ then $\exists M_{\xi''}$, $\xi'' < \xi'$, such that $M_{\xi''} \in D \cap H_{\xi'}$. In either case P meets D and so $x \in H_j(D)$.

If D is not minimal then there exists M_ξ, $\xi < \Omega$, such that

$$H_j(D - M_\xi) = C .$$

But, projecting C and $D - M_\xi$ onto the orthogonal complement of F we see that by construction $\operatorname{proj} C \cap \operatorname{proj} F_\xi \neq \varnothing$, but $\operatorname{proj}(D - M_\xi) \cap \operatorname{proj} F_\xi = \varnothing$. Hence D is a j^{th} minimal hull for C. Finally, suppose that B is an uncountable analytic subset of D. If B has positive j-dimensional measure then it is possible to find an uncountable analytic subset of B of zero j-dimensional measure. Hence it can be supposed that B has zero j-dimensional measure and so $B = A_\xi$ for some $\xi < \Omega$. But $A_\xi = A_\xi \cap D \subset \bigcup_{\xi'<\xi} M_{\xi'}$, which is countable; contradiction.

Of course, if G is an open or compact set in R^n then $H_j(G)$ will accordingly be an open or compact set. Apart from these cases it does not seem entirely trivial to determine the nature of $H_j(G)$ for a given subset G of R^n. Here we prove the following

THEOREM 8. (i) *There exists, in the plane R^2, a borel set C such that $H_1(C)$ is analytic but not borel.*

(ii) *If D is an F_σ-subset of R^n then $H_j(D)$ is the complement of an analytic set.*

REMARKS. We note that by (i) if C is analytic then $H_1(C)$ is not necessarily the complement of an analytic set. To disprove the statement that whenever A is analytic then $H_j(A)$ is analytic, it would be enough, using (ii), to find an F_σ-subset D of R^n such that $H_j(D)$ is not borel. (Notice that, a subset, M of R^n is borel if and only if M and $R^n - M$ are both analytic. Compare, for example, [5]).

Proof. (i) As already observed, every analytic set in R^1 can be represented as the projection into R^1 of some $F_{\sigma\delta}$ set in R^2. Let A be an analytic subset of R^1 such that A is not a borel set and let B be an $F_{\sigma\delta}$ set in R^2 such that $\operatorname{proj} B = A$. Take C to be the union of B and the "y-axis" $(R^1)^\perp$. Then it is easily seen that $H_1(C)$ is the union of all lines which are parallel to $(R^1)^\perp$ and contain a point of C. However this is not a borel set as $H_1(C) \cap R^1 = A \cup \{(0, 0)\}$ is not a borel set.

(ii) We define a complete separable metric space Ω, whose points are the $(n-j)$-flats of R^n, as follows. For each $(n-j)$-flat F in R^n let y be the nearest point of F to 0 and set $F \cap (S^{n-1} + y) = \hat{F}$. Then the distance $\rho(F, F')$ of two $(n-j)$-flats in Ω is defined as the Hausdorff distance of $\hat{F}, \hat{F}'$ in R^n. Let $D \subset R^n$ be an F_σ set, say $D = \bigcup_{i=1}^\infty D_i$ with $D_i \subset D_{i+1}$, each D_i compact, $i = 1, 2, \cdots$. Let D_i^*, $i = 1, 2 \cdots$ denote the closed subsets of Ω such that $F \in D_i^*$ if F meets D_i in R^n. Similarly defined, relative to D, is D^*. Then $D^* = \bigcup_{i=1}^\infty D_i^*$ and so D^* is an F_σ subset of Ω. Hence $\Omega - D^*$ is a G_δ set and so, in particular, $\Omega - D^*$ is an analytic subset of Ω. Set

$$\Omega - D^* = \sum_{i \in I} \bigcap_{p=1}^{\infty} A(i \mid p) ,$$

where the $A(i \mid p)$, $p = 1, 2, \cdots$, form a decreasing sequence of compact subsets of Ω, for each $i \in I$. Set

$$B_m = \{x \mid x \in R^n, -m \leqq x_i \leqq m, i = 1, \cdots, n\} .$$

Let $K_m(i \mid p)$ be the closed subset of B_m such that $x \in K_m(i \mid p)$ if x is contained in an $(n-j)$-flat F with $F \in A(i \mid p)$. Similarly, we define $K_m \subset B_m$ relative to $\Omega - D^*$. Then $K_m = \sum_{i \in I} \bigcap_{p=1}^\infty K_m(i \mid p)$ is an analytic subset of R^n and so, therefore, is $K = \bigcup_{m=1}^\infty K_m$. We claim that $H_j(D) = R^n - K$. If $x \in K$ then $x \in K_m$ for some m and so x is contained in some $(n-j)$-flat F which is contained (in Ω) in some set $\bigcap_{p=1}^\infty A(i \mid p)$. Hence $F \in \Omega - D^*$ which means that F does not meet D; i.e., $x \notin H_j(D)$. Therefore $R^n - K \supset H_j(D)$. Conversely if $x \notin H_j(D)$ then there exists an $(n-j)$-flat F through x such that F does not meet D. Hence $F \in \Omega -$

D^* and so $F \in \bigcap_{p=1}^{\infty} A(i \mid p)$ for some $i \in I$. Hence $x \in \bigcap_{p=1}^{\infty} K_m(i \mid p)$ for some positive integer m, i.e., $x \in K$. Therefore $R^n - K \subset H_j(D)$ and so $H_j(D) = R^n - K$ is the complement of the analytic set K.

DEFINITION. An irregular point x of some closed convex set C in R^3 is an extreme point x of C such that x lies in two distinct 1-faces l_1, l_2 of C, with neither of l_1, l_2 being contained in a 2-face of C. Let C be a closed subset of a simple closed curve in the plane OXY. We say that a set $B \subset C \times (-\infty, \infty)$ is vertically convex if every line which is perpendicular to OXY meets B in a (possibly empty) line segment. We shall make use of the following immediate corollary to a theorem of K. Kunugui [7].

LEMMA 3. (*Kunugui*) *Let B be a vertically convex borel set in $C \times (-\infty, \infty)$. Then the projection of B into C is a borel set.*

As an immediate consequence of Lemma 3, we have

LEMMA 4. *Let B be a vertically convex borel subset of some vertically convex closed subset D in $C \times (-\infty, \infty)$. Then the set $D \cap \{(\text{proj.}\, B) \times (-\infty, \infty)\}$ is a vertically convex borel set.*

In [9] the authors have derived properties of visual hulls for the class of convex sets. Our contribution in this direction is

THEOREM 9. (i) *If C is a convex borel (analytic) set in R^3 then $H_2(C)$ is a borel (analytic) set.*

(ii) *If C is a convex borel (analytic) set in R^3 and $\bar{C}$ does not have irregular points then $H_1(C)$ is a borel (analytic) set.*

Proof. (i) We first show that if C is a convex borel (analytic) set in R^2 then $H_1(C)$ is a borel (analytic) set. If $\dim C = 1$ then the result is trivial and so it can be supposed that $\dim C = 2$. Note that $C^0 \subset H_1(C) \subset \bar{C}$. Let the 1-faces of $\bar{C}$ be $\{F_i\}_{i=1}^{\infty}$. Then

$$H_1(C) \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) = C - \bigcup_{i=1}^{\infty} F_i ,$$

which is a borel set. Let $\{F_{i_\nu}\}_{\nu=1}^{\infty}$ be the 1-faces of $\bar{C}$ which meet C. Then relint $F_{i_\nu} \subset H_1(C) \cap F_{i_\nu}$, $\nu = 1, 2, \cdots$. The two endpoints of F_{i_ν} may, or may not, be in $H_1(C)$. Nevertheless, $H_1(C)$ differs from the borel set $(C - \bigcup_{i=1}^{\infty} F_i) \cup \bigcup_{\nu=1}^{\infty}$ relint F_{i_ν} by at most a countable number of points. And so $H_1(C)$ is a borel set. Similarly, if C is a convex analytic set in R^2, then $H_1(C)$ is an analytic set. Suppose now that C is a convex borel set in R^3. If $\dim C \leqq 2$ then $H_2(C) = C$, and so it can be supposed that $\dim C = 3$. Let $\{F_i\}_{i=1}^{\infty}$ be an enumeration of the 2-faces of $\bar{C}$. Then each F_i is closed and $H_2(C) \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) = C \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i)$, which is a borel set. As $H_2(C) \subset \bar{C}$, it is now enough to show that $H_2(C) \cap F_i$ is a borel set for $i = 1, 2, \cdots$. Let $H_1'(C \cap F_i)$ denote the first visual hull of $C \cap F_i$ relative to aff F_i. Then, from above, $H_1'(C \cap F_i)$ is a borel set. Let $\{F_{i_j}\}_{j=1}^{\infty}$ be an enumeration of the 1-faces of F_i. Then $H_2(C) \cap (F_i - \bigcup_{j=1}^{\infty} F_{i_j}) = H_1'(C \cap F_i) - \bigcup_{j=1}^{\infty} F_{i_j}$ which is a borel set K_i, say. Let $\{F_{i_{j_\nu}}\}_{\nu=1}^{\infty}$ be the 1-faces of F_i which meet C and have the property that the only plane of support to $\bar{C}$ which contains $F_{i_{j_\nu}}$ is aff F_i. Then relint $F_{i_{j_\nu}} \subset H_2(C)$ and the end points of $F_{i_{j_\nu}}$ may or may not be in $H_2(C)$. Hence $H_2(C) \cap F_i$ differs from the borel set $K_i \cup (\bigcup_{\nu=1}^{\infty} \text{relint } F_{i_{j_\nu}}) \cup (\bigcup_{j=1}^{\infty} (F_{i_j} \cap C))$ by at most a countable number of points. Therefore $H_2(C) \cap F_i$ is a borel set, and so, therefore, is $H_2(C)$. Similarly, it can be shown that if C is a convex analytic set in R^3 then $H_2(C)$ is an analytic set.

(ii) Again we shall prove the result for convex borel sets, and indicate at the end the modifications required for convex analytic sets. Let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers and let P_{ik} denote the 2-flat $\{x \mid x_k = r_i\}$ $k = 1, 2, 3$; $i = 1, 2, \cdots$. For each i, j, k, let $B(i, j, k)$ denote the closed set formed by the point set union of all maximal line segments in $\bar{C} - C^0$ which meet both both P_{ik} and P_{jk}. Let $\{G_m\}_{m=1}^{\infty}$ be the 2-faces of $\bar{C}$. If *a* 2-face G_m of $\bar{C}$ meets $B(i, j, k)$ then G_m meets $C_i (C_i = (\bar{C} - C^0) \cap P_{ik})$ and $C_j (C_j = (\bar{C} - C^0) \cap P_{jk})$ in line segments 1_{im} and 1_{jm} respectively. Let $1_m^1, 1_m^2$ denote the (at most) two maximal line segments in G_m such that each segment contains an endpoint of 1_{im} and 1_{jm} but 1_m^1 and 1_m^2 do not intersect except possibly at end points. Set $C^* = (\bar{C} - C^0) \cap P$, where P is a plane parallel to P_{ik} and lying strictly between P_{ik} and P_{jk}. Then G_m cuts C^* in an interval I_m. Let 1_m denote the subinterval of I_m with endpoints $1_m^1 \cap C^*, 1_m^2 \cap C^*$, and let 1_m^0 be the relative interior of 1_m. Then

$$C' = B(i, j, k) \cap \left(C^* - \bigcup_{m=1}^{\infty} 1_m^0\right)$$

is a closed subset of C^*. If $x \in C'$, let $\hat{x}$ denote the unique maximal line segment in $B(i, j, k)$ which passes through x and meets C_1 and C_2. Let X denote the closed set formed by the point set union of the line segments $\hat{x}$, $x \in C'$, and set $Q(i, j, k) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C \neq \emptyset, y \in \hat{x}\}$. We now show that $Q(i, j, k)$ is a borel set. Every point y of X can be given a coordinate vector $y = \langle x, h \rangle$, where $y \in \hat{x}$ and h is the height, relative to the j^{th} coordinate, of y above C^*. Because $\bar{C}$ does not have irregular points, the number of points y in X which receive two different coordinate vectors is countable. Let Φ be the mapping $X \to C^* \times (-\infty, \infty)$ defined by taking $\Phi\langle x, h \rangle = (x, h)$, $x \in C'$. Then K is a borel subset of X if and only if $\Phi(K)$ is a borel subset of the closed set $\Phi(X)$. Hence $\Phi(C \cap X)$ is a vertically convex borel subset of $C' \times (-\infty, \infty)$. Hence the set $D = X \cap \{\text{proj}\, \Phi(C \cap X) \times (-\infty, \infty)\}$ is a convex borel set and so $Q(i, j, k) = \Phi^{-1}(D)$ is a borel set. Hence the set $R(i, j, k) = Q(i, j, k) - \bigcup_{m=1}^{\infty} G_m$ is a borel set. Consider now the set $S = \bigcup_{i,j,k} R(i, j, k)$ and consider the borel set T defined as the point set union of all 1-faces of $\bar{C}$ which are not contained in some 2-face of $\bar{C}$. We assert that the set $H_1^1(C) = H_1(C) \cap (T - \bigcup_{m=1}^{\infty} G_m)$ equals S. For if $y \in H_1^1(C)$ then, because $\bar{C}$ does not have any irregular points, there exists a unique 1-face l, not contained in $\bigcup_{m=1}^{\infty} G_m$, such that $y \in l$. Then $y \in H_1(C)$ if and only if $l \cap C = \emptyset$, which happens if and only if $l \subset Q(i, j, k)$ or in other words $y \in R(i, j, k)$ for some i, j, k. Hence $H_1^1(C) = S$. Let V denote the borel set of exposed points of $\bar{C}$ and $H_1^2(C) = V \cap H_1(C)$, $H_1^3(C) = \bigcup_{m=1}^{\infty} (H_1(C) \cap (G_m - V))$. Now $H_1(C) = H_1^1(C) \cup H_1^2(C) \cup H_1^3(C)$. $H_1^1(C) = S$ is a borel set and, since $H_1^2(C) = V \cap C$, $H_1^2(C)$ is a borel set. Hence it is enough to show that $H_1(C) \cap (G_m - V)$ is a borel set for all m. Now let $\{G_{m_\nu}\}_{\nu=1}^{\infty}$ be those 2-faces of $\bar{C}$ which meet C. Then relint $G_{m_\nu} \subset H_1^3(C)$ for all ν. Let $\{G_{m_\nu n}\}_{n=1}^{\infty}$ be the 1-faces of G_{m_ν}. Then either relint $G_{m_\nu n} \subset H_1^3(C)$ or relint $G_{m_\nu n} \cap H_1^3(C) = \emptyset$. Then the endpoints of $G_{m_\nu n}$ may or may not be in $H_1^3(C)$. Let H_{m_ν} be the countable set of those endpoints of $\{G_{m_\nu n}\}_{\nu=1}^{\infty}$ which lie in $H_1^3(C)$ and let $\{G_{m_\nu n_\mu}\}_{\mu=1}^{\infty}$ be the 1-faces of G_{m_ν} whose relative interiors are contained in $H_1^3(C)$. We have $G_{m_\nu} \cap H_1^3(C) =$ relint $G_{m_\nu} \cup (\bigcup_{\mu=1}^{\infty}$ relint $G_{m_\nu n_\mu}) \cup H_{m_\nu}$, which is a borel set. If, on the other hand, a 2-face of $\bar{C}$ does not meet C, its intersection with $H_1^3(C)$ is empty. Therefore $H_1^3(C) \cap G_m$ is a borel set for all m, and $H_1(C)$ is a borel set.

For the case when C is an analytic set, say $C = \sum_{i \in I} \bigcap_{n=1}^{\infty} C(i \mid n)$ in the usual representation, the only modification required to the above proof is to show that the set $Q(i, j, k)$ is an analytic set. With the previous notation, $Q(i \mid n) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C(i \mid n) \neq \emptyset, y \in \hat{x}\}$. Then $Q(i \mid n)$ is a closed set and $Q(i, j, k) = \sum_{i \in I} \bigcap_{n=1}^{\infty} Q(i \mid n)$. Therefore $Q(i, j, k)$ is an analytic set.

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Received March 12, 1969, and in revised form May 15, 1969. The first author was supported by a Harkness Fellowship of the Commonwealth Fund and the second author by a Fellowship from Swiss National Foundation.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

ON GROUPS OF LINEAR RECURRENCES II. ELEMENTS OF FINITE ORDER

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For each quadratic polynomial $f(x) \in Z[x]$, whose ratio of roots is not ± 1, a group $G(f)$ of equivalence classes of certain linear recurrences with companion polynomial $f(x)$ has been constructed by the author. Its structure was shown to be connected with the structure of the sets of prime divisors of the linear recurrences. The group $G(f)$ is infinite but its torsion subgroup is finite and usually, but not always, consists of just two elements; the class of the Lucas sequence $\mathscr{F} = [0, 1]$ of $f(x)$ and the class of the recurrence $\mathscr{E} = [2, P]$ associated with $f(x)$. This subgroup is completely determined here for each polynomial $f(x)$. In 1961 M. Ward raised the question whether $(\mathscr{F})$ and $(\mathscr{E})$ are the only classes whose sets of prime divisors can be characterized globally. It is shown in this article that there are groups $G(f)$ with elements of finite order, other than $(\mathscr{F})$ and $(\mathscr{E})$, whose prime divisors can be similarly characterized.

We shall use the notation and results of [1]. Part of the object of defining the group structure $G(f)$, where $f(x) = x^2 - Px + Q \in \boldsymbol{Z}[x]$ and $(P, Q) = 1$, is to determine those recurrences among the set of all recurrences with companion polynomial $f(x)$ which are in some sense special. Probably the true sense of special would mean those recurrences which have peculiar arithmetical properties not shared by the remaining ones.[1] For example, the Lucas sequence $\mathscr{F} = [0, 1]$ of $f(x)$ is such a recurrence and so to all intents and purposes is the sequence $\mathscr{E} = [2, P]$ of $f(x)$. Both $(\mathscr{F})$ and $(\mathscr{E})$ are of finite order in $G(f)$; here we interpret 'special' as meaning of finite order in $G(f)$. There are only a finite number of such elements in $G(f)$; furthermore if $(\mathscr{W}) \in G(f)$ and $(\mathscr{W})^k = (\mathscr{F})$ it would seem that the arithmetical properties of $\mathscr{W}$ are fairly closely related to those of $\mathscr{F}$. This is so for $(\mathscr{E})$ and for the other elements $(\mathscr{A})$ and $(\mathscr{B})$ of order two in $G(f)$ (when they exist); for example see Theorem 4.6 of [1] and the final paragraph of that paper. Also some properties of recurrences of finite order are readily deducible which, although they may be true for most or even all recurrences, are not so easily proved in full generality (see for example [1], 3.9.1). Here we determine the structure of the subgroup of elements of finite order. Then we shall show by means of examples that the prime divisors of

[1] An article "On groups of Linear Recurrences III. Arithmetic properties" is in preparation.

some elements of finite order (other than $\mathscr{I}$ and $\mathscr{E}$) can be characterized globally—thus some elements of finite order are even special in the arithmetical sense. Finally we add a few words concerning elements of the group which are locally finite everywhere.

1. The elements of finite order in $G(f)$. We shall carry out the computations only when $f(x)$ is irreducible over $\boldsymbol{Q}$; The results remain valid when $f(x)$ is reducible but involve slightly longer calculations. Let $\mathscr{W} \in F(f)$ be given by

$$w_n = (A\theta_1^n - B\theta_2^n)/(\theta_1 - \theta_2) , \tag{1.1}$$

for all $n \in \boldsymbol{Z}$, with $A = w_1 - w_0\theta_2$, $B = w_1 - w_0\theta_1$, $w_0, w_1 \in \boldsymbol{Z}$ and $(w_0, w_1) = (Q, w_1) = 1$. Thus $\mathscr{W}$ is a reduced recurrence (see beginning of § 3 of [1]).

We denote the subgroup of elements of finite order in $G(f)$ by $H(f)$. Thus $(\mathscr{W}) \in H(f)$ if and only if $\mathscr{W}^m \equiv \mathscr{I}$ in $F(f)$ for some positive integer m. But this holds exactly when

$$(w_1 - w_0\theta_2)^m = d\theta_1^n \tag{1.2}$$

for some $n, d \in \boldsymbol{Z}$. We conclude that $(AB)^m = d^2Q^n$ and as $\mathscr{W}$ is reduced that $(d, Q) = 1$. Thus both d^2 and Q^n are m-th powers in $\boldsymbol{Z}$; put $d^2 = g^m$, $g \in \boldsymbol{Z}$, and on squaring both sides of (1.2) obtain

$$(w_1 - w_0\theta_2)^{2m} = g^m\theta_1^{2n} . \tag{1.3}$$

It follows that if $\mathscr{W}^m \equiv \mathscr{I}$, then some m-th root of θ_1^{2n} lies in $\boldsymbol{Q}(\theta_1)$ and if we denote this root by $\sqrt[m]{\theta_1^{2n}}$ we get

$$(w_1 - w_0\theta_2)^2 = g\zeta\sqrt[m]{\theta_1^{2n}} \tag{1.4}$$

for some $n, g \in \boldsymbol{Z}$ and some m-th root of unity $\zeta \in \boldsymbol{Q}(\theta_1)$.

We remark that if ζ, m, n are fixed, the solutions $(\mathscr{W}) \in G(f)$ obtained from (1.4) are independent of g.

Now we do have a solution to (1.4) when $n = m$ and $\zeta = 1$, namely $(\mathscr{I})$ and $(\mathscr{E})$ (see § 4 of [1]). So we may take n/m to be the least positive number for which (1.4) has a solution with $w_0, w_1, g \in \boldsymbol{Z}$ and some root of unity $\zeta \in \boldsymbol{Q}(\theta_1)$. It follows that $0 < n \leqq m$ and that n divides m. The latter is clear since if we put $t - 1 \leqq m/n < t$, then $m = tn - s$, $0 \leqq s < n$ and on substituting in (1.4) with both sides raised to the power t we obtain $((w_1 - w_0\theta_2)^t)^2 = g^t\zeta^t\theta_1^2\sqrt[m]{\theta_1^{2s}}$. Hence if (1.4) has a solution so does this equation, but $s/m < n/m$ so that $s = 0$. If we put $kn = m$, $k \in \boldsymbol{Z}$, then $1/k$ is the least positive number for which there exists a solution w_0, w_1, g, ζ to (1.4); if another solution exists for some $m = m'$ and $n = n'$, then $n'/m' = t/k$ for some $t \in \boldsymbol{Z}$, $0 < t \leqq k$. Therefore all elements of $H(f)$

are obtained by solving the k equations

$$(w_1 - w_0\theta_2)^2 = g\zeta \sqrt[k]{\theta_1^{2t}}\,, \qquad t = 1, 2, \cdots, k \tag{1.5.t}$$

for some w_0, w_1, $g \in \boldsymbol{Z}$ and some root of unity $\zeta \in \boldsymbol{Q}(\theta_1)$.

The solutions obtained from the equation (1.5.k) are $(\mathscr{I})$ and $(\mathscr{E})$ and if $(\mathscr{W})$ is one solution of (1.5.t), the other is $(\mathscr{W}\mathscr{E})$. Since $\zeta \in \boldsymbol{Q}(\theta_1)$ it can only take the value ± 1, $\pm i$ and $\pm w$, $\pm w^2$, where w is a complex cube root of unity. So we have three cases.

Case 1. Here we assume that $\boldsymbol{Q}(\theta_1)$ contains no complex root of unity. We are left with solving (1.5.t) with $\zeta = 1$. Let $(\mathscr{W})$, $(\mathscr{W}\mathscr{E}) \in H(f)$ be the solutions derived from (1.5.1); then $(\mathscr{W})^2 = (\mathscr{W}\mathscr{E})^2$ and so provide one solution of (1.5.2), $(\mathscr{W})^3 \neq (\mathscr{W}\mathscr{E})^3$ are the two solutions derived from (1.5.3), and similarly up to the solutions $(\mathscr{W})^k$ and $(\mathscr{W}\mathscr{E})^k$ obtained from (1.5.k) the solutions of which are, as mentioned above, $(\mathscr{I})$ and $(\mathscr{E})$. If k is odd then $(\mathscr{W})^k \neq (\mathscr{W}\mathscr{E})^k$ and so one of them is $(\mathscr{E})$, say $(\mathscr{W})^k = (\mathscr{E})$. Then clearly $H(f) = \langle(\mathscr{W})\rangle \cong \boldsymbol{Z}_{2k}$, a cyclic group of order $2k$. If k is even and $(\mathscr{X})^2 = (\mathscr{E})$ has a solution in $G(f)$ (see § 4 of [1]), then necessarily $(\mathscr{W})^k = (\mathscr{W}\mathscr{E})^k = (\mathscr{E})$ and again $H(f) = \langle(\mathscr{W})\rangle \cong \boldsymbol{Z}_{2k}$. On the other hand, if $(\mathscr{X})^2 = (\mathscr{E})$ has no solution in $G(f)$, then $(\mathscr{W})^k = (\mathscr{W}\mathscr{E})^k = (\mathscr{I})$ and it follows that $H(f) = \langle(\mathscr{W})\rangle \times \langle(\mathscr{E})\rangle \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_2$ (direct product).

If $f(x)$ is reducible over $\boldsymbol{Q}$, then we have to solve simultaneously $(w_1 - w_0\theta_2)^{2m} = g^m\theta_1^{2n}$ and $(w_1 - w_0\theta_1)^{2m} = g^m\theta_2^{2n}$ and consequently $(w_1 - w_0\theta_2)^2 = f\sqrt[m]{\theta_1^{2n}}$ and $(w_1 - w_0\theta_1)^2 = g\sqrt[m]{\theta_2^{2n}}$ or $(w_1 - w_0\theta_1)^2 = -g\sqrt[m]{\theta_2^{2n}}$. We have proved

THEOREM 1. *Let $\boldsymbol{Q}(\theta_1) \neq \boldsymbol{Q}(i)$ or $\boldsymbol{Q}(w)$ and k be the maximal positive integer such that $(w_1 - w_0\theta_2)^2 = g\zeta_1\sqrt[k]{\theta_1^2}$ and $(w_1 - w_0\theta_1)^2 = g\zeta_2\sqrt[k]{\theta_2^2}$ have simultaneous solutions with w_1, w_0, $g \in \boldsymbol{Z}$ and $\zeta_1, \zeta_2 = \pm 1$ (which are identical if $\boldsymbol{Q}(\theta_1) \neq \boldsymbol{Q}$). Then the subgroup $H(f)$ of elements of finite order in $G(f)$ is isomorphic to $\boldsymbol{Z}_{2k}$ when k is odd or when k is even and $(\mathscr{X})^2 = (\mathscr{E})$ has a solution in $G(f)$ and is isomorphic to $\boldsymbol{Z}_k \times \boldsymbol{Z}_2$ when k is even and $(\mathscr{X})^2 = (\mathscr{E})$ has no solution in $G(f)$.*

The condition of the theorem implies that Q is a unit times a k-th power in $\boldsymbol{Z}$. By Theorem 4.5 of [1], $(X)^2 = (E)$ has a solution in $G(f)$ when and only when $-(P^2 - 4Q)$ or $-Q(P^2 - 4Q)$ is a square in $\boldsymbol{Z}$. Since $\boldsymbol{Q}(\theta_1) \neq Q(i)$ this can only happen when Q is the negative of a square and $P^2 - 4Q$ is a square, i.e., $f(x)$ is reducible over $\boldsymbol{Q}$.

EXAMPLE. $f(x) = (x-4)(x+9)$ so that $Q = -6^2$, k is even and $H(f) \cong \boldsymbol{Z}_4$. Direct calculation shows that $H(f) = \{[5, -19], [2, -5], [1, -35], [0, 1]\}$ and is obtained by solving $(w_1 + w_0 9)^2 = f(4)$ and $(w_1 - w_0 4)^2 = -f(-9)$ simultaneously.

Case 2. $\boldsymbol{Q}(\theta_1) = Q(i)$. The equation $(\mathrm{w}_1 - w_2\theta_2)^2 = ei$ has a solution $w_0, w_1, e \in \boldsymbol{Z}$; let us denote the resulting solution in $G(f)$ by $(\mathscr{Y})$. Then $(\mathscr{I})$, $(\mathscr{Y})$, $(\mathscr{Y})^2$ and $(\mathscr{Y})^3$ are all distinct and $(\mathscr{Y})^4 = (\mathscr{I})$. Furthermore $(\mathscr{Y})^2$ is derived from a solution of $(t_1 - t_0\theta_2)^2 = g \in \boldsymbol{Z}$ and so must be $(\mathscr{E})$. It follows that if we obtain all the solutions of the equation (1.5.t) with $\zeta = 1$ we get all elements of $H(f)$ combining these with the powers of $(\mathscr{Y})$.

Let $(\mathscr{W})$ and $(\mathscr{W}\mathscr{E})$ be solutions of (1.5.1) with $\zeta = 1$; then all solutions of (1.5.1) for all ζ are $(\mathscr{W})$, $(\mathscr{W}\mathscr{E})$, $(\mathscr{W}\mathscr{Y})$ and $(\mathscr{W}\mathscr{Y}\mathscr{E})$. Then two solutions of (1.5.2) are $(\mathscr{W})^2 = (\mathscr{W}\mathscr{E})^2$ and $(\mathscr{W}\mathscr{Y})^2 = (\mathscr{W}\mathscr{Y}\mathscr{E})^2 = (\mathscr{W})^2(\mathscr{E})$, all four solutions of (1.5.3) are $(\mathscr{W})^3$, $(\mathscr{W}\mathscr{E})^3$, $(\mathscr{W}\mathscr{Y})^3$ and $(\mathscr{W}\mathscr{Y}\mathscr{E})^3$ and similarly up to the solutions $(\mathscr{W})^k$, $(\mathscr{W}\mathscr{E})^k$ $(\mathscr{W}\mathscr{Y})^k$ and $(\mathscr{W}\mathscr{Y}\mathscr{E})^k$ obtained for (1.5.k) (the solutions of which are $(\mathscr{I})$, $(\mathscr{E})$, $(\mathscr{Y})$ and $(\mathscr{Y}\mathscr{E})$, for all possible values of ζ). If k is odd then the four solutions obtained are all distinct and so one is $(\mathscr{I})$—say $(\mathscr{W})^k = (\mathscr{I})$. Then $H(f) = \langle(\mathscr{W})\rangle \times \langle(\mathscr{Y})\rangle \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_4$ (direct product). If $k = 2 \pmod 4$, then the distinct solutions obtained are $(\mathscr{W})^k$ and $(\mathscr{W})^k(\mathscr{E})$ one of which must be $(\mathscr{I})$. So we have the same result. If $k = 0 \pmod 4$ all our solutions in $G(f)$ obtained for (1.5.k) are identical to $(\mathscr{W})^k$. Now we already have two solutions of the equations $(\mathscr{X})^2 = (\mathscr{E})$, namely $(\mathscr{X}) = (\mathscr{Y})$ and $(\mathscr{Y}\mathscr{E})$. Such an equation can have no, two or four solutions in $G(f)$ (see [1] 4.5). If then only two solutions exist, $(\mathscr{W})^k = (\mathscr{I})$ and again our group $H(f) = \langle(\mathscr{W})\rangle \times \langle(\mathscr{Y})\rangle \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_4$; if four solutions exist we have $(\mathscr{W})^k = (\mathscr{E})$ and $H(f) = \langle(\mathscr{W})\rangle \times \langle(\mathscr{W}\mathscr{Y})\rangle \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_k$. Thus

THEOREM 2. *Let $\boldsymbol{Q}(\theta_1) = \boldsymbol{Q}(i)$ and k be the maximal positive integer such that $(w_1 - w_0\theta_2)^2 = g\zeta\sqrt[k]{\theta_1^2}$ has a solution $w_1, w_0, g \in \boldsymbol{Z}$ and ζ a fourth root of unity. Then $H(f)$ is isomorphic to $\boldsymbol{Z}_{4k}$ if k is odd, to $\boldsymbol{Z}_k \times \boldsymbol{Z}_4$ if $k = 2 \pmod 4$ or if $k = 0 \pmod 4$ when $(\mathscr{X})^2 = (\mathscr{E})$ has only two solutions in $G(f)$ and to $\boldsymbol{Z}_k \times \boldsymbol{Z}_k$ when $k = 0 \pmod 4$ and $(\mathscr{X})^2 = (\mathscr{E})$ has four solutions in $G(f)$.*

Again the condition of the theorem implies that Q is a unit times a k-th power in $\boldsymbol{Z}$. Since $P^2 - 4Q$ is the negative of a square, the invariant $\varDelta(\mathscr{E}) = -(P^2 - 4Q)$ is a square and so $(\mathscr{X})^2 = (\mathscr{E})$ has four solutions in $G(f)$ only when Q is also a square in $\boldsymbol{Z}$ and then $G(f)$ has three elements of order two (see [1], 4.6).

Case 3. $\boldsymbol{Q}(\theta_1) = \boldsymbol{Q}(w)$. The equation $(w_1 - w_0\theta_2)^2 = ew$ has a solution $w_0, w_1, e \in \boldsymbol{Z}$; denote the resulting solution in $G(f)$ by $(\mathscr{Y})$. Then $(\mathscr{I})$, $(\mathscr{Y})$, $(\mathscr{Y})^2$, $(\mathscr{Y})^3$, $(\mathscr{Y})^4$, $(\mathscr{Y})^5$ are all distinct and $(\mathscr{Y})^6 = (\mathscr{I})$, Furthermore $(\mathscr{Y})^3 = (\mathscr{E})$; the equation $(\mathscr{X})^2 = (\mathscr{E})$ has no solution in $G(f)$ (since $-(P^2 - 4Q) = 3$ times a square in $\boldsymbol{Z}$ and so is not a square, and $-Q(P^2 - 4Q)$ cannot be a square in $\boldsymbol{Z}$ also). If $(\mathscr{W})$ is one solution in $G(f)$ derived from (1.5.1) with $\zeta = 1$, then all solutions derived from (1.5.1) for all ζ are $(\mathscr{W})$, $(\mathscr{W}\mathscr{Y})$, $(\mathscr{W}\mathscr{Y}^2)$, $(\mathscr{W}\mathscr{E})$, $(\mathscr{W}\mathscr{Y}\mathscr{E})$, $(\mathscr{W}\mathscr{Y}^2\mathscr{E})$. Among the solutions derived from (1.5.k) are $(\mathscr{W})^k$, $(\mathscr{W}\mathscr{Y})^k$, $(\mathscr{W}\mathscr{Y}^2)^k$, $(\mathscr{W}\mathscr{E})^k$, $(\mathscr{W}\mathscr{Y}\mathscr{E})^k$ and $(\mathscr{W}\mathscr{Y}^2\mathscr{E})^k$. If k is odd, then $(\mathscr{W})^k$ and $(\mathscr{W}\mathscr{E})^k$ provide the two distinct solutions obtained from (1.5.k) with $\zeta = 1$ and so at least one of them is $(\mathscr{I})$; say $(\mathscr{W})^k = (\mathscr{I})$, then $H(f) = \langle(\mathscr{W})\rangle \times \langle(\mathscr{Y})\rangle \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_6$. If k is even, neither $(\mathscr{W})^k$ nor $(\mathscr{W}\mathscr{E})^k$ can be $(\mathscr{E})$ (since otherwise $(\mathscr{X})^2 = (\mathscr{E})$ would have a solution in $G(f)$) and so both must be $(\mathscr{I})$, Hence again $H(f) = \langle(\mathscr{W})\rangle \times \langle(\mathscr{Y})\rangle \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_6$. Therefore

THEOREM 3. *Let $\boldsymbol{Q}(\theta_1) = \boldsymbol{Q}(w)$ and k be the maximal positive integer such that $(w_1 - w_0\theta_2)^2 = g\zeta \sqrt[k]{\theta_1^2}$ has a solution $w_1, w_0, g \in \boldsymbol{Z}$ and ζ a cube root of unity. Then $H(f) \cong \boldsymbol{Z}_k \times \boldsymbol{Z}_6$.*

2. **Prime divisors of elements of finite order.** At present the only known way to determine if a prime p, $(p, Q) = 1$, divides a general linear recurrence $\mathscr{W}$ is to examine any $p + 1$ consecutive terms of $\mathscr{W}$; p is a divisor of $\mathscr{W}$ if and only if it divides one of these terms. Such a characterization we shall call *local*. On the other hand every prime divides $(\mathscr{I})$ and a prime divides the element $(\mathscr{E})$ of order two in $G(f)$ if and only if its rank of apparition in $\mathscr{I}$ is even. M. Ward in [3] termed this a *global* characterization of the prime divisors of $(\mathscr{I})$ and $(\mathscr{E})$ and raised the question whether these are the only two recurrences (for a given companion polynomial $f(x)$, or in our terminology, in $G(f)$) for which the prime divisors can be so characterized. Here we show that there are other elements of finite order besides $(\mathscr{I})$ and $(\mathscr{E})$ where prime divisors can be globally characterized. Although we have not made an exhaustive study we suspect that there are other elements of finite order whose prime divisors are globally characterized. But it is not clear that every element of finite order has this property, for example, we know that an odd prime of odd rank of apparition in $\mathscr{I}$ is a divisor of one and only one of the two linear recurrences $(\mathscr{A})$, $(\mathscr{B})$ of order two in $G(f)$ (when they exist) but we cannot at present say which recurrence of the two such a prime divides. Nevertheless, we are tempted to conjecture that if an element of $G(f)$ has its prime

divisors globally characterized, then it is of finite order.

We consider the example of an element of order four considered previously; here $H(f) = \{(\mathscr{V}), (\mathscr{V})^2 = (\mathscr{E}), (\mathscr{V})^3 = (\mathscr{V}\mathscr{E}), (\mathscr{V})^4 = (\mathscr{I})\}$, where $\mathscr{V} = [5, -19]$ and $\mathscr{V}\mathscr{E} = [1, -35]$. Since both $(\mathscr{V})$ and $(\mathscr{V}\mathscr{E})$ generate $H(f)$ they have precisely the same prime divisors; by Theorem 4.3 of [1] there is a labelling of the terms of $\mathscr{V}$ and $\mathscr{W} = \mathscr{V}\mathscr{E}$, say v_n and w_n, such that $v_n w_n = de_{2n+k}$ for all $n \in \boldsymbol{Z}$, some $d \in \boldsymbol{Z}$ and $k = 0$ or 1. Comparing the first two products we see that $v_n w_n = -e_{2n+1}$, where $v_0 = 5$, $v_1 = -19$, $w_0 = 1$, $w_1 = -35$, $e_0 = 2$ and $e_1 = -5$. Since $i_{2n} = i_n e_n$ for nonnegative integers, where $\mathscr{I} = [0, 1]$, $i_0 = 0$, $i_1 = 1$ is the Lucas sequence of $G(f)$, it follows that the odd prime divisors of both $(\mathscr{V})$ and $(\mathscr{W})$ are precisely those of rank $4n + 2$, $n = 0, 1, \cdots$. Thus the prime divisors of $(\mathscr{V})$ and $(\mathscr{V}\mathscr{E})$ have been characterized globally.

Now consider the group $G(f)$, where $f(x) = x^2 - 5x + 7$. The group has two elements $(\mathscr{V}) = ([1, 3])$ and $(W) = (V^2) = ([1, 2])$ of order three. Since these two elements generate the same subgroup of $G(f)$ they have the same prime divisors. If we put $v_0 = 1$, $v_1 = 3$, $w_0 = 1$, $w_1 = 2$, $i_0 = 0$, $i_1 = 1$, then we deduce that $3v_n w_n i_n = i_{3n}$ for all n and consequently it follows that the prime divisors of $(\mathscr{V})$ are precisely those of rank $3n$, $n = 1, 2, \cdots$. Again the prime divisors of $(\mathscr{V})$ and $(\mathscr{V})^2$ have been characterized globally.

Both these examples admit of some generalization.

3. Elements which are locally finite everywhere. To say that an element $(\mathscr{W})$ of $G(f)$ is locally finite everywhere means that $(\mathscr{W})$ is of finite order modulo $G(f, p)$ for all primes p, i.e., $(\mathscr{W}) \in H(f, p)$ for all p (see after Corollary 3.4.1 of [1]). Now $H(f, p) \supseteqq K(f, p)$ with equality for all p, which are coprime to $Q(P^2 - 4Q)$. Here we discuss only the case when $Q = \pm 1$. Then $(\mathscr{W}) \in K(f) = \bigcap_p K(f, p)$ if and only if the *reduced* elements of (W) have invariant ± 1. It can be shown that $\Delta(\mathscr{W}) = \pm 1$ implies $(\mathscr{W}) = (I)$ except in the following two situations: when $f(x) = x^2 - 3x + 1$ with $\mathscr{W} = [1, 1]$ and $f(x) = x^2 + 3x + 1$ with $\mathscr{W} = [-1, 1]$. Referring to the remark after Theorem 4.4 of [1], we see that both these exceptional sequences are of order two; $K(x^2 - 3x + 1) = ((\mathscr{B}))$, $K(x^2 + 3x + 1) = ((\mathscr{A}))$ and $K(x^2 - Px \pm 1) = ((\mathscr{I}))$ in all other cases. Now if $(\mathscr{W}) \in \bigcap_p H(f, p)$, when $(\mathscr{W})^k \in K(f)$ for some $K \in \boldsymbol{Z}$ and so we may conclude by means of Theorem 3.7 of [1] that the elements which are locally finite everywhere are precisely the elements of finite order in $G(f)$.

REMARKS. (a) The sequence $\mathscr{B}$ given above is a sequence of alternate terms of the Fibonacci sequence, the sequence $\mathscr{A}$ is similarly related apart from signs, and the two exceptional groups $G(x^2 - 3x + 1)$ and $G(x^2 + 3x + 1)$ are isomorphic.

(b) If Q is not a unit the situation is quite different. To begin with things are complicated by the fact that one cannot use reduced elements alone in discussing the subgroup K. The above result that an element which is locally finite everywhere is of finite order is not true in general.

(c) We can generalize a result of A. Schinzel given in [2] to show that if $(\mathscr{W}) \in G(f, p)$ for all primes p with at most a finite number of exceptions, then $(\mathscr{W}) = (\mathscr{I})$.

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Received December 11, 1968, and in revised form July 22, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

THE ADJOINT GROUP OF LIE GROUPS

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Let G be a Lie group and let Aut(G) denote the group of automorphisms of G. If the subgroup Int(G) of inner-automorphisms of G is closed in Aut(G), then we call G a (CA) group (after Van Est.). In this note, we investigate (CA) property of certain classes of Lie groups. The main results are as follows:

THEOREM A. **Let G be an analytic group and suppose that there is no compact semisimple normal subgroup of G. If G contains a closed uniform (CA) subgroup H, then G is (CA).**

THEOREM B. **If G is an analytic group whose exponential map is surjective, then G is (CA).**

In [3], Garland and Goto proved that if an analytic group G contains a lattice, then G is (CA). Since a lattice in a solvable group is a uniform lattice, it is finitely generated and so the automorphism group of this uniform lattice is discrete, and thus this lattice is trivially a (CA) subgroup. Thus Theorem A generalizes the above theorem of Garland and Goto for solvable groups. Theorem B is an improvement of the well known theorem that every nilpotent analytic group is (CA) (see [2]). In §1, we introduce some notation and preliminary materials. §2 and §3 are devoted for the proofs of the main theorems together with their immediate corollaries.

1. Preliminaries and notations. The group Aut(G) of automorphisms of locally compact a topological group G may be regarded as a topological group, the topology being the (generalized) compact open topology defined as in [5]. Thus, if we denote by $N(C, V)$ the set of all $\theta \in \mathrm{Aut}(G)$ for which $\theta(x)x^{-1} \in V$ and $\theta^{-1}(x)x^{-1} \in V$ whenever $x \in C$, then the sets $N(C, V)$ form a fundamental system of neighborhoods of the identity element of Aut(G) as C ranges over the compact subsets of G and V over the set of neighborhoods of the identity element of G.

If G is an analytic group and $\mathscr{G}$ its Lie algebra, then Aut(G) may be identified with a closed subgroup of the linear group Aut($\mathscr{G}$) of automorphisms of $\mathscr{G}$. Under this identification, Int(G) coincides with the adjoint group Int($\mathscr{G}$), which is generated by e^{adX}, $X \in \mathscr{G}$ where ad denotes the adjoint representation of $\mathscr{G}$. Thus the (CA) property of analytic groups are entirely determined by their Lie algebras. In particular, if $\tilde{G}$ is a covering group of G and if G is (CA), then so is $\tilde{G}$. This fact is used in the proofs of the main theorems.

Throughout this paper the following notation is used: If A is a subgroup of G, then $\mathrm{Int}_G(A)$ denotes the subgroup of $\mathrm{Int}(G)$ which consists of inner automorphisms induced by elements of A. Thus $\mathrm{Int}_G(G)$ is merely equal to $\mathrm{Int}(G)$. The center of G is denoted with $Z(G)$. Also if $x \in G$, then I_x means the inner automorphism induced by x.

2. **Proof of Theorem A.** Let H be a closed uniform subgroup of an analytic group G, and H_0 its identity component. Then H/H_0 is finitely generated. In order to see this, let $\widetilde{G}$ be the simply connected covering group of G, $\widetilde{H}$ the complete inverse image of H under the covering projection and $\widetilde{H}_0$ the identity component of $\widetilde{H}$. Then since the covering projection induces an epimorphism $\widetilde{H}/\widetilde{H}_0 \to H/H_0$, it suffices to show that $\widetilde{H}/\widetilde{H}_0$ is finitely generated. Nothing that $\widetilde{G}/\widetilde{H}_0$ is simply connected (see, for example, Mostow [7], Corollary 1, p. 617), we can identify the discrete group $\widetilde{H}/\widetilde{H}_0$ with the fundamental group of the compact manifold $\widetilde{G}/\widetilde{H}$. As the fundamental group of a compact manifold is finitely presented, it follows that $\widetilde{H}/\widetilde{H}_0$ is, in particular, finitely generated.

Now we can apply a theorem of Hochschild ([5], Th. 2, p. 212) to see that if H is a closed subgroup of an analytic group, then $\mathrm{Aut}(H)$ is a Lie group.

The following lemma enables us to assume that G is simply connected.

LEMMA. *Let $\widetilde{H}$ be a compactly generated Lie group and A a closed discrete central subgroup of $\widetilde{H}$. Let $H = \widetilde{H}/A$. If H is a (CA) group, then so is $\widetilde{H}$. In fact,* $\mathrm{Int}(\widetilde{H})$ *is a topological extension of a discrete group by* $\mathrm{Int}(H)$.

Proof. Let $\pi\colon \widetilde{H} \to \widetilde{H}/A = H$ be the natural map and define χ: $\mathrm{Int}(\widetilde{H}) \to \mathrm{Int}(H)$ by $\chi(I_{\tilde{h}}) = I_{\pi(\tilde{h})}$, for $\tilde{h} \in \widetilde{H}$

(i) χ is continuous. To see this, note first that we can find a compact nighborhood $\widetilde{D}$ of 1 in $\widetilde{H}$ which generates $\widetilde{H}$. Now let C be a compact subset of H and U a neighborhood of 1 in H. Then we have to find a compact subset $\widetilde{C}$ of $\widetilde{H}$ and a neighborhood $\widetilde{U}$ of 1 in $\widetilde{H}$ so that $\chi(N(\widetilde{C}, \widetilde{U}) \cap \mathrm{Int}(\widetilde{H})) \subseteq N(C, U) \cap \mathrm{Int}(H)$. Since $\pi(\widetilde{D}) = D$ is also a compact neighborhood of 1 which generates H, we can find a positive integer k such that $C \subset D^k$ by using the compactness of C. Now letting $\widetilde{C} = \widetilde{D}^k$ and $\widetilde{U} = \pi^{-1}(U)$, it is easy to see that $(\widetilde{C}, \widetilde{U})$ is a desired pair. Hence χ is cotinuous.

(ii) χ is open. In fact, since H is (CA), the canonical map $H/Z(H) \to \mathrm{Int}(H)$ is an isomorpqism of topological groups. Hence (ii) follows from the following commutative diagram

$$\begin{array}{ccc} \tilde{H}/Z(\tilde{H}) & \longrightarrow & H/Z(H) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Int}(\tilde{H}) & \xrightarrow[\chi]{} & \operatorname{Int}(H) \end{array}$$

where the left vertical map is always continuous and the top one is open.

(iii) The kernel $\mathscr{K}$ of χ is a discrete subgroup of $\operatorname{Aut}(\tilde{H})$ and hence is closed in $\operatorname{Aut}(\tilde{H})$. To see this let $\overline{\mathscr{K}}$ be the closure of $\mathscr{K}$ in $\operatorname{Aut}(\tilde{H})$, and $\overline{\mathscr{K}}_0$ the identity component of $\overline{\mathscr{K}}$. Since $\operatorname{Aut}(\tilde{H})$ is a Lie group $\overline{\mathscr{K}}/\overline{\mathscr{K}}_0$ is discrete.

Since $\mathscr{K} = \operatorname{Ker}(\chi)$ and since A is central in $\tilde{H}$ every element of $\mathscr{K}$ induces the identity map on $H = \tilde{H}/A$ and on A. Hence $\theta \in \overline{\mathscr{K}}$ implies that $\theta = 1$ on A and $\theta = 1$ on $H = \tilde{H}/A$, which implies that $\theta(\tilde{h})\tilde{h}^{-1} \in A$ for $\tilde{h} \in \tilde{H}$.

Let $\tilde{h} \in \tilde{H}$ be arbitrary and define $\eta_{\tilde{h}}\colon \overline{\mathscr{K}} \to A$ by $\eta_{\tilde{h}}(\theta) = \theta(\tilde{h})\tilde{h}^{-1}$, $\theta \in \overline{\mathscr{K}}$. Then $\eta_{\tilde{h}}$ is continuous and thus $\eta_{\tilde{h}}(\overline{\mathscr{K}}_0)$ is connected in the discrete A. Since $\eta_{\tilde{h}}(\overline{\mathscr{K}}_0)$ contains 1, $\eta_{\tilde{h}}(\overline{\mathscr{K}}_0) = 1$ and this then implies that $\theta(\tilde{h}) = \tilde{h}$ for all $\theta \in \overline{\mathscr{K}}_0$. Since $\tilde{h}$ is arbitrary, $\overline{\mathscr{K}}_0 = 1$ and $\overline{\mathscr{K}}$ is discrete. We have thus shown that $\mathscr{K}$ is a discrete subgroup of $\operatorname{Aut}(H)$ and hence $\mathscr{K}$ is closed in $\operatorname{Aut}(\tilde{H})$.

(iv) Since $(\operatorname{Int}(\tilde{H})/\mathscr{K} \cong \operatorname{Int}(H)$, $\operatorname{Int}(\tilde{H})$ is closed in $\operatorname{Aut}(\tilde{H})$ as a locally compact subgroup of $\operatorname{Aut}(\tilde{H})$ and the lemma is proved.

Now we are ready to present the proof of Theorem A. Let $\tilde{G}$ denote the simply connected covering group of G and let π be the covering homomorphism.

Then, by the lemma $\pi^{-1}(H) = \tilde{H}$ is also uniform and (CA). Hence no generality will be lost in assuming that G is simply connected. By the assumption, $\operatorname{Int}(H)$ is closed in $\operatorname{Aut}(H)$. Thus the canonical map $H/Z(H) \to \operatorname{Int}(H)$ is an isomorphism of topological groups. Define φ: $\operatorname{Int}_G(H) \to \operatorname{Int}(H)$ to be the restricting homomorphism and let $\mathscr{K}$ be the closure of the kernel of φ, the closure being taken in $\operatorname{Aut}(G)$. Then $\operatorname{Int}_G(H)\mathscr{K}$ is a subgroup of $\operatorname{Aut}(G)$. We define $H/Z(H) \to \operatorname{Int}_G(H)\mathscr{K}/\mathscr{K}$ and $\operatorname{Int}_G(H)\mathscr{K}/\mathscr{K} \to \operatorname{Int}(H)$ to be the homomorphisms induced by the canonical maps $H \to \operatorname{Int}_G(H)$ and $\operatorname{Int}_G(H) \to \operatorname{Int}(H)$, respectively. Then the following diagram commutes:

$$\begin{tikzcd} & \operatorname{Int}_G(H)\mathscr{K}/\mathscr{K} \arrow[dr] & \\ H/Z(H) \arrow[ur] \arrow[rr] & & \operatorname{Int}(H) \end{tikzcd}$$

and all three maps are continuous and algebraically isomorphisms. Since the bottom one is an isomorphism of topological groups, $\operatorname{Int}(H)$

is topologically isomorphic with $\mathrm{Int}_G(H)\mathscr{K}/\mathscr{K}$ and thus the latter is a locally compact subspace of the quotient space $\mathrm{Aut}(G)/\mathscr{K}$. Hence it is closed in $\mathrm{Aut}(G)/\mathscr{K}$ and, accordingly, $\mathrm{Int}_G(H)\mathscr{K}$ is closed in $\mathrm{Aut}(G)$.

We next claim that $\mathscr{K} \subseteqq \mathrm{Int}(G)$. In fact, if $\theta \in \mathscr{K}$, define $P(g) = \theta(g)g^{-1}$, for $g \in G$. Then $P\colon G \to G$ is continuous, and $P(H) = \{1\}$ and G/H compact imply that $P(G)$ is compact. Thus we see that θ is an automorphism of bounded displacement in the sense of Tits [8] and θ is therefore an inner automorphism induced by a central element of the nilradical of G ([8], Lemma (6), p. 102). Thus $\mathscr{K} \subseteqq \mathrm{Int}(G)$.

By what we have shown, it is clear now that the closure $\overline{\mathrm{Int}_G(H)}$ of $\mathrm{Int}_G(H)$ is contained in $\mathrm{Int}(G)$. Since G/H is compact and since $G/H \to \mathrm{Int}(G)/\overline{\mathrm{Int}_G(H)}$ is continuous, $\mathrm{Int}(G)$ is compact, modulo $\overline{\mathrm{Int}_G(H)}$ and hence $\mathrm{Int}(G)$ is closed, proving that G is (CA).

COROLLARY. *If a solvable analytic group G contains a closed abelian uniform subgroup, then G is a (CA) group.*

COROLLARY. (*See, Garland and Goto* [3]). *If a solvable analytic group G contains a lattice, then G is a (CA) group.*

REMARK. In [6], we have shown that any extension of a simply connected (CA) group by a compact connected group is a (CA) group. Thus Theorem A generalizes this for the solvable case.

REMARK. We have failed to see whether or not the nonexistence of compact semi-simple normal subgroup in the theorem is necessary. This was needed in order to apply the result of Tits in the proof.

3. Proof of Theorem B. In order to prove Theorem B, we first note that an analytic group G is (CA) if and only if its radical is (CA)(See Van Est [2]). Thus we may assume that the group in the theorem is solvable.

Let $\mathscr{G}$ be a finite-dimensional real solvable Lie algebra and let G be an analytic group with its Lie algebra $\mathscr{G}$. If an exponential map $\exp\colon \mathscr{G} \to G$ is surjective, then the exponential map into its simply connected covering group is a bijection. Thus by the remark in §1, it suffices to prove:

THEOREM B′. *Let $\mathscr{G}$ be a finite-dimensional real solvable Lie algebra. If the exponential map is a bijection, then $\mathscr{G}$ is a (CA) Lie algebra* (*that is, the adjoint group* $\mathrm{Int}(\mathscr{G})$ *is closed in* $\mathrm{Aut}(\mathscr{G})$).

In order to prove this, we need the following lemma:

LEMMA. *Let $\mathscr{N}$ be the nilradical of $\mathscr{G}$. If $X \in \mathscr{N}$ then the one-parameter subgroup $\{e^{ad(tX)}: t \in R\}$ is closed in* $\mathrm{Aut}(\mathscr{G})$.

Proof of lemma. Let T denote the given one-parameter subgroup. We show if T is not closed, then T is trivial. In fact, if T is not closed in $\mathrm{Aut}(\mathscr{G})$, then the closure $\bar{T}$ of T is compact. Define φ: $\mathrm{Aut}(\mathscr{G}) \to \mathrm{Aut}(\mathscr{N})$ to be the restricting homomorphism. Since $\mathscr{N}$ is a characteristic ideal in $\mathscr{G}$, φ is well defined and is continuous.

Now let $ad_{\mathscr{N}}$ denote the adjoint representation of the nilpotent Lie algebra $\mathscr{N}$. Since $\mathscr{N}$ is nilpotent, $\mathrm{Int}(\mathscr{N})$ is closed in $\mathrm{Aut}(\mathscr{N})$ ([2], Proposition 1.2.2, p. 322), and thus $\overline{\varphi(T)} \subset \mathrm{Int}(\mathscr{N})$. By using the fact that the maximal compact subgroup of any nilpotent analytic group is contained in its center, it follows that $\mathrm{Int}(\mathscr{N})$ is always simply connected. Hence the compact subgroup $\overline{\varphi(T)}$ must be trivial, which means that $ad_{\mathscr{N}} X = 0$ and so X is central in $\mathscr{N}$.

Next we show that X is central in $\mathscr{G}$. In order to see this, note first that $[X, \mathscr{G}] \subseteqq \mathscr{G}' \subseteqq \mathscr{N}$, $\mathscr{G}'$ being the commutator subalgebra of $\mathscr{G}$. Thus X being a central element of $\mathscr{N}$ implies that $ad(X)^2 = 0$. Therefore $e^{ad(tX)} = 1 + ad(tX)$ for $t \in R$. Let $Y \in \mathscr{G}$ be arbitrary. Thus we have

$$\begin{aligned}\exp(R[X, Y]) &= \exp(ad(RX)(Y)) = \exp(e^{ad(RX)} - 1)(Y) \\ &= \exp((T-1)(Y)) .\end{aligned}$$

Since $\bar{T}$ is compact, the closure of $T - 1$ is compact in the matrix topology of $\mathrm{End}(\mathscr{G})$, the ring of endomorphisms of the vector space $\mathscr{G}$. Therefore, the continuity of exp implies that $\exp((T-1)(Y))$ is bounded in G. Consequently, the one-parameter subgroup $\exp(R[X, Y])$ is relatively compact. But G is simply connected and thus this one-parameter subgroup must be trivial, which implies that $ad(X) = 0$ and we have proved that X is central in $\mathscr{G}$. Therefore $T = 1$ as desired.

Proof of Theorem B′. By a theorem of Goto ([4], Theorem III, p. 165), it suffices to show that every one-parameter subgroup of $\mathrm{Int}(\mathscr{G})$ is closed in $\mathrm{Aut}(\mathscr{G})$. Noting that every one-parameter subgroup of $\mathrm{Int}(\mathscr{G})$ is of the form $e^{ad(RX)}$ for some $X \in \mathscr{G}$, assume that there is a nonzero X such that $T = e^{ad(RX)}$ is not closed in $\mathrm{Aut}(\mathscr{G})$. We see from the lemma that X is not in $\mathscr{N}$.

Next we select a decreasing sequence of ideals of $\mathscr{G}$:

$$\mathscr{G}_0 = \mathscr{G} > \mathscr{G}_1 > \mathscr{G}_2 > \cdots > \mathscr{G}_{n+1} = (0)$$

such that $\dim_R((\mathscr{G}_i/\mathscr{G}_{i+1}) \leqq 2$. Let A_i denote the endomorphism on $\mathscr{G}_i/\mathscr{G}_{i+1}$ which is induced by $ad(X)$, $i = 0, 1, \cdots, n$. Then there exists

p such that $A_p \neq 0$. For, if $A_i = 0$ for all i, then $ad(X)$ would be a nilpotent transformation and hence $X \in \mathscr{N}$, which is impossible. Since T is relatively compact in $\mathrm{Aut}(\mathscr{G})$, so is $S = e^{RA_p}$ in $\mathrm{Aut}(\mathscr{G}_p/\mathscr{G}_{p+1})$. Since A_p is nonzero, S is nontrivial and thus $\dim_R(\mathscr{G}_p/\mathscr{G}_{p+1}) = 2$. Since a maximal compact subgroup of $\mathrm{Aut}(\mathscr{G}_p/\mathscr{G}_{p+1})$ is a circle group, it follows that S is a circle group in $\mathrm{Aut}(\mathscr{G}_p/\mathscr{G}_{p+1})$. Now let $\pi: \mathscr{G} \rightarrow \mathscr{G}/\mathscr{G}_{p+1}$ be the natural homomorphism and let $\mathscr{H}$ be the sub-algebra of $\mathscr{G}/\mathscr{G}_{p+1}$ which is generated by $\pi(X)$ and $\mathscr{G}_p/\mathscr{G}_{p+1}$. Then from what we have seen above, it is easy to see that $\mathscr{H}$ is the Lie algebra of the group of the rigid motions on the plane. Thus exp is not a bijection by the well known theorem of Dixmier ([1], Th. 3, p. 120). Hence every one-parameter subgroup of $\mathrm{Int}(\mathscr{G})$ is closed in $\mathrm{Aut}(\mathscr{G})$, which proves the Theorem B′.

In the proof of Theorem B′, we have actually shown that $\mathrm{Int}(\mathscr{G})$ contains no compact subgroups. Hence we have:

COROLLARY. *Let G be a solvable analytic group such that the exponential map is surjective. Then* $\mathrm{Int}(G)$ *is simply connected.*

COROLLARY. *Let G be as above. Then $Z(G)$ is connected.*

Proof. By Theorem B, $G/Z(G) = \mathrm{Int}(G)$ is an isomorphism of topological groups. Since $\mathrm{Int}(G)$ is simply connected, it follows that $Z(G)$ is connected.

REMARK. The coverse of the Theorem B is false. The group of rigid motions on the plane is perhaps the simplest example.

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Received July 28, 1969. This research was supported in part by the National Science Foundation Grant GP 12261.

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1. 1970

COMMUTATIVITY IN LOCALLY COMPACT RINGS

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A structure theorem is given for all locally compact rings such that x belongs to the closure of $\{x^n: n \geqq 2\}$, in particular, all such rings are commutative, a result which extends a well-known theorem of Jacobson. Similarly we show the commutativity of semisimple locally compact rings satisfying topological analogues of properties studied by Herstein.

Jacobson has shown that a ring is commutative if for every x there is some $n(x) \geqq 2$ such that $x^{n(x)} = x$ [5, Th. 1, p. 212]. Herstein has generalized this result, and certain of his and other generalizations are of interest here. A ring is commutative if (and only if) for all x and y there is some $n(x, y) \geqq 2$ such that $(x^{n(x,y)} - x)y = y(x^{n(x,y)} - x)$ [4, Th. 2]; a ring is commutative if (and only if) for all x and y there is some $n(x, y) \geqq 2$ such that $xy - yx = (xy - yx)^{n(x,y)}$ [3, Th. 6]; a semisimple ring is commutative if (and only if) for all x and y there is some $n(x, y) \geqq 1$ such that $x^{n(x,y)}y = yx^{n(x,y)}$ [4, Th. 1] or if for all x and y there are $n, m \geqq 1$ such that $x^n y^m = y^m x^n$ [1, Lemma 1]. The investigation of analogous conditions for topological rings is the major concern of this paper.

1. A topological analogue of Jacobson's condition.

If $x^n = x$ for some $n \geqq 2$, then an inductive argument shows that $x^{k(n-1)+1} = x$ for all $k \geqq 1$. A possible topological analogue of Jacobson's condition would thus be that for every x there is some $n(x) \geqq 2$ such that $\lim_k x^{k(n(x)-1)+1} = x$. But this implies that $x^{n(x)} = x$, since

$$x^{n(x)} = x^{n(x)-1}x = x^{n(x)-1} \lim_k x^{k(n(x)-1)+1} = \lim_k x^{(k+1)(n(x)-1)+1} = x .$$

Thus all topological rings having this property have Jacobson's property and hence are commutative.

A less trivial analogue of Jacobson's condition is that for every x in the topological ring A, x belongs to the closure of $\{x^n: n \geqq 2\}$. In our investigation of these rings, rings with no nonzero topological nilpotents play an important role. Recall that an element x of a topological ring is a *topological nilpotent* if $\lim_n x^n = 0$. We shall prove that a locally compact ring has no nonzero topological nilpotents if and only if it is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring B that is the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields. From this it is easy to derive a structure theorem for locally compact rings having the topological analogue of Jacobson's property mentioned above.

LEMMA 1. *If A is a locally compact ring with no nonzero topological nilpotents, then A is totally disconnected.*

Proof. The connected component C of zero in A is a closed ideal of A and so is itself a connected locally compact ring with no nonzero topological nilpotents. By hypothesis, C is not annihilated by any of its nonzero elements, for if $xC = (0)$, then $x^2 = 0$, so $x = 0$. Thus C is a finite-dimensional algebra over the real numbers (cf. [6, Th. III]). As the radical of a finite-dimensional algebra is nilpotent, C is a semisimple algebra. If $C \neq (0)$, then by Wedderburn's Theorem, C has an identity e, and clearly $(1/2)e$ would then be a nonzero topological nilpotent contrary to our hypothesis. Thus $C = (0)$, and so A is totally disconnected.

LEMMA 2. *A compact ring A has no nonzero topological nilpotents if and only if A is the Cartesian product of finite fields.*

Proof. Necessity: By Lemma 1, A is totally disconnected. Thus the radical $J(A)$ of A is topologically nilpotent [11, Th. 14], and hence is the zero ideal. Thus A is a compact semisimple ring, and so A is topologically isomorphic to the Cartesian product of a family of finite simple rings [11, Th. 16]. A finite simple ring is a matrix ring over a finite field, and unless the matrix ring is just the finite field itself, it will have nonzero nilpotent elements. Thus as A has no nonzero nilpotents, A is topologically isomorphic to the Cartesian product of a family of finite fields. Sufficiency: Clearly zero is the only topological nilpotent in the Cartesian product of a family of finite fields.

LEMMA 3. *If A is a ring with no nonzero nilpotents, then every idempotent is in the center of A.*

Proof. If e is an idempotent and if $a \in A$, an easy calculation shows that $(ae - eae)^2 = 0$, hence $ae - eae = 0$. Similarly, $ea = eae$ and thus $ae = ea$.

We recall that the local direct sum of a family $(A_\gamma)_{\gamma \in \Gamma}$ of topological rings with respect to open subrings $(B_\gamma)_{\gamma \in \Gamma}$ is the subring of the Cartesian product $\prod_\gamma A_\gamma$ consisting of all (a_γ) such that $a_\gamma \in B_\gamma$ for all but finitely many γ, topologized by declaring all neighborhoods of zero in the topological ring $\prod_\gamma B_\gamma$ to be a fundamental system of neighborhoods of zero in the local direct sum. It is easy to see that the local direct sum equipped with this topology is indeed a topological ring.

THEOREM 1. *A locally compact ring A has no nonzero topological nilpotents if and only if A is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring B (possibly the zero ring) that is topologically isomorphic to the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields.*

Proof. Necessity: As A is totally disconnected by Lemma 1, A contains a compact open subring F [7, Lemma 4]. By Lemma 2, F is topologically isomorphic to the product of finite fields. Consequently there exists in F a summable orthogonal family $(e_\gamma)_{\gamma \in \Gamma}$ of idempotents such that Fe_γ is a finite field and $\sum_{\gamma \in \Gamma} e_\gamma = e$, the identity of F.

By Lemma 3, e is in the center of A, so Ae and $A(1-e) = \{a - ae: a \in A\}$ are ideals. The continuous mappings $a \to ae$ and $a \to (a - ae)$ are the projections from A onto Ae and $A(1-e)$. Thus A is the topological direct sum of Ae and $A(1-e)$. As e is the identity of F, $F \cap A(1-e) = (0)$. Thus as F is open, $A(1-e)$ is discrete and hence has no nonzero nilpotents.

As F is open and as $Ae_\gamma \cap F = Fe_\gamma$, a finite field, Ae_γ is discrete and is an ideal as e_γ is in the center of A. Consequently Ae_γ has no nonzero nilpotents. It will therefore suffice to show that $B = Ae$ is topologically isomorphic to the local direct sum of the descrete rings Ae_γ, with respect to the finite subfields Fe_γ.

Let B' be the local direct sum of the Ae_γ's with respect to the Fe_γ's. Let $K: b \to (be_\gamma) \in \prod_\gamma Ae_\gamma$. Clearly $b \to be_\gamma$ is a continuous homomorphism for each γ, hence K is a continuous homomorphism from B into $\prod_\gamma Ae_\gamma$. If $b \in B$, then (be_γ) is summable and $\sum_\gamma be_\gamma = b(\sum_\gamma e_\gamma) = be = b$. Therefore as F is open in B, $be_\gamma \in F \cap Ae_\gamma = Fe_\gamma$ for all but finitely many $\gamma \in \Gamma$. Thus $K(B) \subseteqq B'$.

The mapping K is an isomorphism onto $K(B)$, since if $x \in B$ and if $xe_\gamma = 0$ for all $\gamma \in \Gamma$, then $x = xe = x(\sum_\gamma e_\gamma) = \sum_\gamma xe_\gamma = 0$. Let $y_\beta \in Fe_\beta$, and let $x_\gamma = 0$ for all $\gamma \neq \beta$, $x_\beta = y_\beta$; then $(x_\gamma) = K(y_\beta) \in K(F)$ since $(e_\gamma)\gamma$ is an orthogonal family. Thus $K(F)$ contains a dense subring of $\prod_\gamma Fe_\gamma$, and hence $K(F) = \prod_\gamma Fe_\gamma$ as $K(F)$ is compact. As the restriction of K to F is thus a continuous isomorphism from conpact F onto $\prod_\gamma Fe_\gamma$, F is topologically isomorphic to $\prod_\gamma Fe_\gamma$ under K.

Thus it sufficices to show that $K(B) \supseteqq B'$, for K is then, by the definition of the local direct sum, a topological isomorphism from B onto B'. If $(b_\gamma e_\gamma) \in B'$, then $b_\gamma e_\gamma \in Fe_\gamma$ for all but finitely many γ, say $\gamma_1, \cdots, \gamma_n$. Call this set Γ_1 and let $\Gamma - \Gamma_1 = \Gamma_2$. Thus $\sum_{\gamma \in \Gamma_1} b_\gamma e_\gamma \in B$ and $b_\gamma e_\gamma \in F$ for all $\gamma \in \Gamma_2$. Hence as F is topologically isomorphic to $\prod_\gamma Fe_\gamma$, $b' = \sum_{\gamma \in \Gamma_2} b_\gamma e_\gamma \in B$. Thus $b = b' + \sum_{\gamma \in \Gamma_1} b_\gamma e_\gamma \in B$, and $be_\gamma = b_\gamma e_\gamma$, so $K(b) = (b_\gamma e_\gamma)$. The sufficiency is clear.

We will call a ring A a *Jacobson ring* if given any $x \in A$ there is an $n(x) \geqq 2$ such that $x^{n(x)} = x$. All Jacobson rings are commutative [5, Th. 1, p. 212], and in extending this result to topological rings we give the following definition, noting that it reduces to Jacobson's condition in the discrete case.

DEFINITION. A topological ring A is a *J-ring* if for each $x \in A$, x belongs to the closure of $\{x^n : n \geqq 2\}$.

LEMMA 4. *If A is a J-ring, then A has no nonzero topological nilpotents.*

Proof. If $\lim_n x^n = 0$, then since x belongs to the closure of $\{x^n : n \geqq 2\}$, we conclude that $x = 0$.

THEOREM 2. *A locally compact ring A is a J-ring if and only if A is the topological direct sum of a discrete Jacobson ring and a ring B which is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields.*

Proof. Necessity: By Theorem 1 and Lemma 4, A is the topological direct sum of a discrete ring C and a ring B which is topologically isomorphic to the local direct sum of a family of discrete rings with respect to finite subfields. As each of these rings is an ideal of A, each is a discrete J-ring and so is a Jacobson ring.

Sufficiency: Let B be the local direct sum of a family of discrete Jacobson rings $B_\gamma, \gamma \in \Gamma$ with respect to finite subfields $F_\gamma, \gamma \in \Gamma$. Let $(x_\gamma) \in B$ and let U be a neighborhood of zero in B. Then we may assume that there is a finite subset Δ of Γ such that $x_\gamma \in F_\gamma$ for all $\gamma \notin \Delta$ and $U = \prod_\gamma G_\gamma$, where $G_\gamma = F_\gamma$ for all $\gamma \notin \Delta$. For each $\gamma \in \Delta$, let $n(\gamma) > 1$ be such that $x_\gamma^{n(\gamma)} = x_\gamma$. Let $n = 1 + \prod_{\gamma \in \Delta} (n(\gamma) - 1)$. An inductive argument shows that $x_\gamma^n = x_\gamma$ for all $\gamma \in \Delta$. Hence $(x_\gamma)^n - (x_\gamma) \in U$. Thus B is a J-ring, and consequently A is also a J-ring.

As all Jacobson rings are commutative we have the following analogue of Jacobson's Theorem:

COROLLARY. *A locally compact J-ring is commutative.*

THEOREM 3. *A locally compact ring A is a Jacobson ring if and only if there exists $N \geqq 2$ such that A is the topological direct sum of a discrete Jacobson ring and a ring B that is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields of order $\leqq N$.*

Proof. Necessity: Let $|B_\gamma| =$ the order of B_γ. By Theorem 2 it suffices to show that $\sup |B_\gamma| < +\infty$. If $\sup |B_\gamma| = +\infty$, then there exists $(x_\gamma) \in \prod_\gamma B_\gamma$ such that the orders of the x_γ's are unbounded. Consequently for no n does $x_\gamma^n = x_\gamma$ for all γ, i.e., for no n does $(x_\gamma)^n = (x_\gamma)$.

Sufficiency: Let $(A_\gamma)_{\gamma \in \Gamma}$ be a family of discrete Jacobson rings with finite subfields B_γ such that $|B_\gamma| \leqq N$ for all γ. Let (x_γ) be in the local direct sum of the A_γ's with respect to the B_γ's. There exists a finite subset Δ of Γ such that if $\gamma \notin \Delta$, $x_\gamma \in B_\gamma$. Since each A_γ is a Jacobson ring, for $\gamma \in \Delta$ there is $n(\gamma)$ such that $x_\gamma^{n(\gamma)} = x_\gamma$.

If $x_\gamma^{n(\gamma)} = x_\gamma$, an inductive argument shows that $x_\gamma^{k(n(\gamma)-1)+1} = x_\gamma$ for all k. If $x_\gamma \in B_\gamma$, then $|B_\gamma| \leqq N$, so since $|B_\gamma| - 1 < N$, $x_\gamma^{1+k(N!)} = x_\gamma$ for all k. Let $n = 1 + [(N!) \prod_{\gamma \in \Delta} (n(\gamma) - 1)]$. Then $x_\gamma^n = x_\gamma$ for all γ, i.e., $(x_\gamma)^n = (x_\gamma)$.

2. Analogues of four of Herstein's results. An analogue for topological rings of the first of Herstein's conditions that are mentioned above is that for all x and y, $xy - yx$ is in the closure of $\{x^n y - yx^n : \geqq 2\}$, and we say such a topological ring is an *H_1-ring*. An analogue of the second of Herstein's conditions is that for all x and y, $xy - yx$ is in the closure of $\{(xy - yx)^n : n \geqq 2\}$, and we say such a topological ring is an *H_2-ring*. (If $(xy - yx)^{n(x,y)} = xy - yx$, then

$$(xy - yx)^{k[n(x,y)-1]+1} = xy - yx$$

for all $k \geqq 1$; hence another topological analogue is the assumption that for each $x, y \in A$, there exists $n(x, y) \geqq 2$ that $\lim_k (xy - yx)^{k[n(x,y)-1]+1} = xy - yx$; however by an argument similar to that of the first paragraph of § 1, this condition implies that $(xy - yx)^{n(x,y)} = xy - yx$.) Similarly an analogue of the third of Herstein's conditions is that for all x, y in A, $\lim_n x^n y - yx^n = 0$, and we say such topological rings are *H_3-rings*, just as we will call *H_4-rings* those topological rings in which for all x, y there is an $m(x, y) \geqq 1$ such that $\lim_n x^n y^{m(x,y)} - y^{m(x,y)} x^n = 0$. We shall prove that those *H_i-rings* which are semisimple and locally compact are commutative, $i = 1, 2, 3, 4$.

LEMMA 5. *All idempotents in an H_i-ring, $i = 1, 2, 3, 4$, commute.*

Proof. Let e and f be idempotents in such a ring A. Then $(efe - ef)^2 = 0$, so $\{(efe - ef)^n e - e(efe - ef)^n : n \geqq 2\} = \{0\}$. Therefore, if A is an H_1-ring, then $(efe - ef)e - e(efe - ef) = 0$, so

$$0 = (efe - ef)e = e(efe - ef) = efe - ef .$$

If A is an H_2-ring, then $(ef)e - e(ef) = efe - ef = 0$ since $efe - ef$ is in the closure of $\{[(ef)e - e(ef)]^n : n \geqq 2\} = \{0\}$. Similarly in either case

$efe = fe$, so $ef = fe$. As $0 = \lim_n e^n f - fe^n = \lim_n e^n f^m - f^m e^n = ef - fe$, the assention also holds for H_3 and H_4-rings.

Since it is clear that all subrings and quotient rings determined by closed ideals of H_i-rings are H_i-rings, $i = 1, 2, 3, 4$, and since all idempotents in such rings commute, we see that the following is applicable.

LEMMA 6. *Let P be a property of Hausdorff topological rings such that:*

(1) *if A is a Hausdorff topological ring with property P, then every subring of A has property P and A/B has property P where B is any closed ideal of A,*

(2) *if A has property P, then all idempotents in A commute. If A is a locally compact primitive ring with property P, then A is a division ring.*

Proof. Since A is a semisimple ring, A is the topological direct sum of a connected ring B and a totally disconnected ring C, where B is a semisimple algebra over R of finite dimension [7, Th. 2]. As A is primitive, either $A = B$ or $A = C$. In the former case A is a matrix ring since it is primitive, and so has idempotents which do not commute unless it is a division ring.

It suffices, therefore, to consider the case in which A is totally disconnected. We shall first prove the assertion under the additional assumption that A is a Q-ring (i.e., the set of quasi-invertible elements is a neighborhood of zero). We may consider A to be a dense ring of linear operators on a vector space E over a division ring D. If E is not one-dimensional, then E has a two-dimensional subspace M with basis $\{z_1, z_2\}$. Let $B = \{a \in A: a(M) \subseteqq M\}$, and let

$$N = \{a \in A: a(M) = (0)\} = K_1 \cap K_2$$

where $K_i = \{a \in A: a(z_i) = 0\}$, $i = 1, 2$.

There exists $u \in A$ such that $u(z_1) = z_1$, and hence $x - xu \in K_1$, for all $x \in A$. If $v \notin K_1$, then there exists $w \in A$ such that $wv(z_1) = z_1$, so as $u = wv + (u - wv)$ and $u - wv \in K_1$, $A = Au + K_1 = Av + K_1$. Therefore K_1, and similarly K_2, is a regular maximal left ideal, an observation of the referee that simplifies the proof. Hence K_1 and K_2 are closed (cf. [11, Th. 2]), so N is a closed ideal of B. By hypothesis B/N is therefore a Hausdorff topological ring having property P. Thus all idempotents in B/N commute; but B/N is isomorphic to the ring of all linear operators on M, a ring containing idempotents which do not commute. Hence E is one-dimensional and A is a division ring.

Next we shall show that A is necessarily a Q-ring, from which the result follows by preceding. As A is totally disconnected A has a compact open subring D [7, Lemma 4]. If $D = J(D)$, the radical of D, then D and hence A are Q-rings. Assume therefore that $J(D) \subset D$. We shall show that $D/J(D)$ is a finite ring and hence is discrete.

The radical, $J(D)$, of D is closed [8, Th. 1], $D/J(D)$ is compact semisimple ring and thus $D/J(D)$ is topologically isomorphic to the Cartesian product of a family $(F_\gamma)_{\gamma \in \Gamma}$ of finite simple rings with identities $(f_\gamma)_{\gamma \in \Gamma}$ [11, Th. 16]. As $J(D)$ is topologically nilpotent [11, Th. 14], D is suitable for building idempotents [12, Lemma 4] (cf. [11, Lemma 12]). Suppose that Γ has more than one element, say $\{\alpha, \beta\} \subseteqq \Gamma$. Then there are nonzero orthogonal idempotents e_α, e_β in D such that $e_\alpha + J(D)$, $e_\beta + J(D)$ correspond, respectively, under the isomorphism to (f_γ^α), (f_γ^β) where $f_\gamma^\lambda = 0 \in F_\gamma$ if $\gamma \neq \lambda$ and $f_\lambda^\lambda = f_\lambda$. Let ϕ be the canonical mapping $x \rightarrow x + J(D)$ from D onto $D/J(D)$. As $(f_\gamma^\alpha) + (f_\gamma^\beta)$ annihilates the open neighborhood $\prod_{\gamma \in \Gamma} G_\gamma$ of zero where $G_\alpha = \{0\}$, $G_\beta = \{0\}$, and $G_\gamma = F_\gamma$ for $\gamma \neq \alpha, \beta$, we conclude that $\phi(e_\alpha + e_\beta)$ annihilates a neighborhood V of zero in $D/J(D)$. Consequently $U = \phi^{-1}(V)$ is a neighborhood of zero in D, and $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) \subseteqq J(D)$ (cf. [7, proof of Th. 11]). Therefore as $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) = U \cap (e_\alpha + e_\beta)A(e_\alpha + e_\beta)$, $(e_\alpha + e_\beta)U(e_\alpha + e_\beta)$ is a neighborhood of zero in $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ consisting of quasi-invertable elements, so $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a Q-ring. As $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is primitive [6, Proposition 1, p. 48] and is clearly closed, $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a locally compact, primitive Q-ring with property P, so $(e_\alpha + e_\beta)$ $A(e_\alpha + e_\beta)$ is a division ring. But it contains nonzero e_α, e_β satisfying $e_\alpha e_\beta = 0$, a contradiction. Thus Γ can contain only one element, so $D/J(D)$ is isomorphic to a finite ring. Hence $J(D)$, being closed in D, is open in D and thus in A, so A is a Q-ring.

LEMMA 7. *If A is an H_i-ring, $i = 1, 2, 3, 4$ and if A is a locally compact division ring, then A is a field.*

Proof. If A is discrete and is an H_i-ring ($i = 1, 2, 3, 4$) then A is commutative [3, Th. 2; 4, Th. 1; 3, Th. 1; 1, Lemma 1].

If A is not discrete, then A has a nontrivial absolute value giving its topology, and A is a finite-dimensional algebra over its center, on which the absolute value is nontrivial [10, Th. 8].

If A is an H_1-ring and x is nonzero in A, then there exists some nonzero z in the center of A such that $|z| < 1/|x|$. Thus $|xz| < 1$, so $\lim_n (xz)^n = 0$. Hence for any $y \in A$, $\lim_n (xz)^n y - y(xz)^n = 0$, so as $(xz)y - y(xz)$ is in the closure of $\{(xz)^n y - y(xz)^n : n \geqq 2\}$, $0 = (xz)y - y(xz) = z(xy - yx)$. Hence $xy = yx$, as $z \neq 0$. Thus A is commutative.

If A is an H_2-ring and if $x, y \in A$ satisfy $xy - yx \neq 0$, then there exists some nonzero z in the center such that $|z| < 1/|xy - yx|$. Thus

$|(xz)y - y(xz)| < 1$, so $\lim_n [(xz)y - y(xz)]^n = 0$. Hence $0 = (xz)y - y(xz) = (xy - yx)z$, so $xy - yx = 0$ as $z \neq 0$, a contradiction. Thus A is commutative.

Assume that A is an H_3-ring. As A is a division ring, A is either totally disconnected or connected [7, Th. 2].

Case 1. A is totally disconnected. Then the topology of A is given by a nonarchimedean absolute value. Suppose A is not commutative. Then as A is a finite-dimensional and hence an algebraic extension of its center C, there exists some $x \notin C$ having minimal degree $m > 1$ over C. Let y be arbitrary in A, and assume that for no $1 \leqq i \leqq m - 1$, does $x^i y = yx^i$. Hence $x^i y - yx^i \neq 0$, $1 \leqq i \leqq m - 1$, and we claim $\{x^i y - yx^i : 1 \leqq i \leqq m - 1\}$ is a linearly independent set over C. Suppose $\sum_{i=i}^{m-1} \beta_i(x^i y - yx^i) = 0$, where $\beta_i \in C$, and let $z = \sum_{i-i}^{m-1} \beta_i x^i$. Then $zy = yz$. By the definition of m, either $z \in C$ on z has degree $\geqq m$ over C. Suppose $z \notin C$. Then $C[x]$ has dimension m over C, so m is the degree of z as $z \in C[x]$. Therefore $C[x] = C[z]$, so as $zy = yz$, every element of $C[x]$ commutes with y, contrary to our assumption. Thus $z \in C$; let $-\beta_0 = z$. Then $\sum_{i=0}^{m-1} \beta_i x^i = 0$, so $\beta_i = 0$, $0 \leqq i \leqq m-1$ since $\{1, x, \cdots, x^{m-1}\}$ is linearly independent over C.

Since x is algebraic of degree m over the center C of A, there exist $\alpha_i \in C$, $0 \leqq i \leqq m - 1$, such that $x^m = \sum_{i=0}^{m-1} \alpha_i x^i$; thus for all $n \geqq m$, there exist $\alpha_{i,n} \in C$, $0 \leqq i \leqq m - 1$, such that $x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i$. We may also assume that $|x| > 1$, since all our assumption on x are true for any λx, $\lambda \in C^*$. We note that there is therefore some r such that $|x|^r \geqq |\alpha_i|$, $0 \leqq i \leqq m - 1$.

Since $x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i$,

$$x^n y - yx^n = \sum_{i=i}^{m-1} \alpha_{i,n}(x^i y - yx^i) ;$$

so $\lim_n x^n y - yx^n = 0$ if and only if $\lim_n \alpha_{i,n} = 0$, $1 \leqq i \leqq m - 1$.

Since $|x^n| \leqq \max\{|\alpha_{i,n}|\,|x|^i : 0 \leqq i \leqq m - 1\}$, if $|\alpha_{i,n}| < 1, 1 \leqq i \leqq m - 1$, then $|x|^n \leqq |\alpha_{0,n}|$. Let r_0 be such that $|x|^{r_0} > |x| + 1$. Since $\lim_n \alpha_{i,n} = 0, 1 \leqq i \leqq m - 1$, there exists $n_0 > r + r_0$ such that $|\alpha_{i,n}| < 1$, for all $n \geqq n_0$ and all i such that $1 \leqq i \leqq m - 1$. But for any $n > n_0$,

$$\begin{aligned} x^{n+1} &= \sum_{i=0}^{m-2} \alpha_i, x^{i+1} + \alpha_{m-1,n}\left(\sum_{i=0}^{m-1} \alpha_i x^i\right) \\ &= \alpha_{m-1,n}\alpha_0 + \sum_{i=1}^{m-1} [\alpha_{i-1,n} + (\alpha_{m-1,n})\alpha_i]x^i , \end{aligned}$$

so

$$\begin{aligned} |\alpha_{1,n+1}| &= |\alpha_{0,n} + \alpha_{m-1,n}\alpha_1| \geqq |\alpha_{0,n}| - |\alpha_{m-1,n}|\,|\alpha_1| \\ &\geqq |x|^n - |\alpha_1| \geqq |x|^{r+r_0} - |x|^r = |x^r|\,(|x|^{r_0} - 1) > 1 . \end{aligned}$$

a contradiction. Hence A is commutative.

Case 2. A is connected. Then the center C of A contains the real number field R, A is finite-dimensional over R, so the degree of each element of A over R is less than or equal to 2, and the topology is given by an absolute value. Suppose $x \notin C$. Then $\deg x = 2$; let $x^2 = \alpha_1 + \alpha_2 x$, and for each $n \geqq 2$, let $x^n = \alpha_{1,n} + \alpha_{2,n}x$, where $\alpha_{1,n}$, $\alpha_{2,n} \in R$. As before we may assume that $|x| > 1$. Let r be such that $|x|^r > \max\{|\alpha_1|, |\alpha_2|\}$. Let $y \in A$ be such that $xy \neq yx$. Then $0 = \lim_n (x^n y - yx^n) = \lim_n \alpha_{2,n}(xy - yx)$, so $\lim_n \alpha_{2,n} = 0$. Let $n_0 > r$ be such that $|\alpha_{2,n}| < 1$ for all $n \geqq n_0$. But if $n \geqq n_0$ is such that $|x|^n > 3 |x|^r$, then

$$|x|^n = |\alpha_{1,n} + \alpha_{2,n}x| \leqq |\alpha_{1,n}| + |\alpha_{2,n}||x| < |\alpha_{1,n}| + |x| ,$$

so $|x^n| - |x| < |\alpha_1, n|$. As

$$x^{n+1} = \alpha_{1,n}x + \alpha_{2,n}(\alpha_1 + \alpha_2 x) = \alpha_{2,n}\alpha_1 + (\alpha_{1,n} + \alpha_{2,n}\alpha_2)x ,$$

$$|\alpha_{2,n+1}| = |\alpha_{1,n} + (\alpha_{1,n})\alpha_2| \geqq |\alpha_{1,n}| - |\alpha_{2,n}||\alpha_2| .$$

Hence $|\alpha_{2,n+1}| \geqq (|x|^n - |x|) - |x|^r \geqq 3|x|^r - |x|^r - |x|^r = |x|^r > 1$, a contradiction. Hence A is commutative.

Finally let A be an H_4-ring. If for all x and y, $\lim_n x^n y - yx^n = 0$, then A is an H_3-ring and so a field; so assume there are x and y in A such that $\lim_n x^n y - yx^n \neq 0$. Let $W = \{w \in A: \lim_n x^n w - wx^n = 0\}$. Clearly W is a division subring of A, and since $y \notin W$, W is a proper division subring. By hypothesis, for all $a \in A$ there is an $r \geqq 1$ such that $a^r \in W$; thus A is a field [2, Th. B].

THEOREM 4. *All H_i-rings that are locally compact and semisimple are commutative*, $i = 1, 2, 3, 4$.

Proof. P is a primitive ideal of such a ring A if and only if $P = (B: A)$ (by definition $(B: A) = \{x \in A: Ax \subseteqq B\}$) where B is a regular maximal to left ideal [5, Corollary to Proposition 2, p. 7]. Let $e \in A$ be such that $x - ex \in B$ for all $x \in A$. If $x \in (B: A)$, then $ex \in B$, so $x \in B$. Hence $(B: A) \subseteqq B$.

If B is closed, then $(B: A)$ is closed for if (x_α) is a directed set of elements of $(B: A)$ converging to x, then for all $a \in A$, $ax_\alpha \in B$, whence $ax = \lim ax_\alpha \in B$.

As A is semisimple, $(0) = \bigcap \{B: B$ is a closed regular maximal left ideal$\} \supseteqq \bigcap \{P: P$ is a closed primitive ideal$\}$ [8, Th. 1]. By Lemma 6 and 7, A/P is a field if P is a closed primitive ideal. Thus for all $x, y \in A$, $xy - yx \in P$, so $xy - yx \in \bigcap \{P: P$ is a closed primitive ideal$\} = (0)$.

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Received August 16, 1968. The results in this paper are taken from the author's doctoral dissertation, written at Duke University under Professor Seth Warner.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

RINGS OF FUNCTIONS WITH CERTAIN LIPSCHITZ PROPERTIES

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Let (X, d) denote a metric space, $L_c(X)$ the ring of real valued functions on X which are Lipschitz on each compact subset of X, $L_1(X)$ the ring of real valued functions on X which are locally Lipschitz relative to the completion of X, and $L_c{}^*(X)$, $L_1{}^*(X)$ the bounded elements of $L_c(X)$, $L_1(X)$. The relations between equality of these rings and the topological properties of X are studied. It is shown that a subspace (S, d) of (X, d) is L_c-embedded (or $L_c{}^*$-embedded) in (X, d) if and only if S is closed. Further, every subspace of (X, d) is L_1- and $L_1{}^*$-embedded in (X, d).

Su [3] investigated algebraic properties of the rings $L_c(X)$ and $L_c^*(X)$ similar to those of $C(X)$ and $C^*(X)$ by Gillman and Jerrison [2].

2. **Equality of rings.** Let f denote a real valued function defined on X. f is Lipschitz on $S \subset X$ if and only if there is a real number m, called a Lipschitz constant for f on S, such that if x, $y \in S$, then $|f(x) - f(y)| \leqq md(x, y)$. f is locally Lipschitz on X if and only if for each $x \in X$, there is a neighborhood N of x such that f is Lipschitz on N. If comp X denotes the completion of X, then f is locally Lipschitz with respect to comp X if and only if for each $x \in \operatorname{comp} X$ there is a neighborhood N of x such that f is Lipschitz on $N \cap X$.

THEOREM 2.1. *$f \in L_c(X)$ if and only if f is locally Lipschitz on X.*

Sufficiency. Let f be locally Lipschitz on X and S a compact subset of X. Then there exists a finite collection $N_1, N_2, \cdots, N_m$ of open sets covering S, on each of which f is Lipschitz and thus bounded. Assuming f is not Lipschitz on S implies that there exists a sequence $\{x_n\}$ from S converging to $x \in S$ and a sequence $\{y_n\}$ from S such that $|f(x_n) - f(y_n)|/d(x_n, y_n) > n$ for each positive integer n. Since f is bounded on S, it follows that $\{y_n\}$ converges to x. Since $x \in N_j$ for some $j = 1, 2, \cdots, m$, f is not Lipschitz on N_j which contradicts the definition of N_j.

Necessity. Let $f \in L_c(X)$ and $x \in X$. Assuming f is not locally Lipschitz at x implies there exists sequences $\{x_n\}$ and $\{y_n\}$ such that

$d(x, x_n) < 1/n$, $d(x, y_n) < 1/n$, and $|f(x_n) - f(y_n)|/d(x_n, y_n) > n$. Then $\{p: p \in \{x_n\}, p \in \{y_n\}$, or $p = x\}$ is a compact subset of X on which f is not Lipschitz.

COROLLARY 2.2. *$f \in L_c^*(x)$ if and only if f is locally Lipschitz on X and bounded.*

Proof. Follows immediately from the definition of $L_c^*(X)$.

COROLLARY 2.3. *$L_1(X) \subset L_c(X)$ and $L_1^*(X) \subset L_c^*(X)$.*

Proof. If f is locally Lipschitz relative to com X, then f is locally Lipschitz.

LEMMA 2.4. *If K is a uniformly bounded set of Lipschitz functions defined on $S \subset X$ and there is a real number m which is a Lipschitz constant for each element of K, then $f(x) = \sup\{g(x): g \in K\}$ for each $x \in S$ is Lipschitz on S and m is a Lipschitz constant for f on S.*

Proof. f exists since K is a uniformly bounded set. Assume $x \in S$, $y \in S$, and

$$f(y) - f(x) - md(x, y) = e > 0 . \tag{1}$$

Let $g \in K$ such that

$$f(y) - g(y) < e , \tag{2}$$

then

$$g(y) - g(x) \leqq md(x, y) . \tag{3}$$

Combining (2) and (3) yields $f(y) - g(x) - md(x, y) < e$, which when combined with (1) gives $f(x) < g(x)$. This contradicts the definition of f.

LEMMA 2.5. *Suppose each of c and $r > 0$, $p \in X$, and for*

$$\text{each } x \in X,\ f(x) = \begin{cases} (c/r)\{r - d(x, p)\} & \text{for } d(x, p) \leqq r, \\ 0 & \text{otherwise} \end{cases}$$

then f is Lipschitz on X and (c/r) is a Lipschitz constant for f on X.

Proof. Let $g(x) = (c/r)\{r - d(x, p)\}$ for each $x \in X$. Then for x, $y \in X$,

$$g(x) - g(y) = g(x) - g(p) + g(p) - g(y) ,$$
$$g(x) - g(y) = -(c/r)d(x, p) + (c/r)d(y, p) ,$$

and $g(x) - g(y) \leqq (c/r)d(x, y)$ by the triangle property. Since $\sup\{g, 0\}$ is Lipschitz with a Lipschitz constant $\sup\{(c/r), 0\}$ by Lemma 2.4, the conclusion follows.

THEOREM 2.6. *Each of the following is equivalent to each of the others*:

(1) $L_1(X) = L_c(x)$,

(2) $L_1^*(X) = L_c^*(X)$, *and*

(3) X *is complete.*

Proof. (1) $\Rightarrow$ (2) obviously. The remaining order is (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Assume (2) and that X is not complete. Then there exists an $x \in (\text{comp}\, X) - X$ and a sequence $\{x_n\}$ of distinct points in X such that $\{x_n\}$ converges to x. For each odd integer n, let

$$r_n = \frac{1}{3}\inf\{y\colon y = d(x_n, x_m) \quad \text{for} \quad m \neq n \quad \text{or} \quad y = (1/n)\} ,$$

$$C(x_n, r_n) = \{t \in X\colon d(t, x_n) \leqq r_n\} ,$$

and

$$f_n(t) = \begin{cases} (1/r_n)\{r_n - d(x_n, t)\} & \text{for} \quad t \in C(x_n, r_n) \\ 0 \quad \text{otherwise} \end{cases}$$

for each $t \in X$. Let $f(t) = \sup\{f_n(t)\}$ for each $t \in X$. If S is a compact subset of X, then S can intersect at most a finite number of the elements of $\{C(x_n, r_n)\}$ and since only a finite number of elements of $\{f_n\}$ are nonzero on S, by Lemma 2.4 f is Lipschitz on S and $f \in L_c^*(X)$. For each neighborhood N in comp X of x, there is a point $t \in N$ and a point $y \in N$ such that $f(t) = 1$ and $f(y) = 0$. Thus $f \notin L_1(X)$ and by contradiction, (2) $\Rightarrow$ (3).

If (3) is true, $f \in L_1(X)$ if and only if f is locally Lipschitz. Thus by Theorem 2.1, $L_1(X) = L_c(X)$ and (3) $\Rightarrow$ (1).

THEOREM 2.7. $L_c(X) = L_c^*(X)$ *if and only if* X *is compact.*

Proof. If X is compact, then each element of $L_c(X)$ is bounded.

Assume $L_c(X) = L_c^*(X)$ and X is not compact. Then there exists a sequence $\{x_n\}$ of distinct points in X which has no convergent subsequence. Let

$$r_n = \frac{1}{3}\inf\left\{y\colon y = d(x_n, x_m) \quad \text{for} \quad n \neq m \quad \text{or} \quad y = \frac{1}{n}\right\} ,$$

and

$$f(x) = \begin{cases} (n/r_n)\{r_n - d(x_n, x)\} & \text{for} \quad d(x_n, x) \leqq r_n \\ 0 \quad \text{otherwise} \end{cases}$$

for each $x \in X$. By an argument similar to the one for Theorem 2.6, $f \in L_c(X)$. Since $f(x_n) = n$ for each n, $f \in L_c(X) - L_c^*(X)$ which contradicts the assumption.

THEOREM 2.8. $L_1(X) = L_1^*(X)$ *if and only if* comp X *is compact.*

Proof. Each element of $L_1(X)$, $L_1^*(X)$ can be uniquely extended to an element of $L_1(\text{comp } X) = L_c(\text{comp } X)$, $L_1^*(\text{comp } X) = L_c^*(\text{comp } X)$. Since $L_c(\text{comp } X) = L_c^*(\text{comp } X)$ if and only if comp X is compact by Theorem 2.7, the conclusion follows.

3. If A denotes one of L_1, L_1^*, L_c, L_c^* and $S \subset X$, then the statement that S is A-embedded in X means that if $f \in A(S)$, there is a $g \in A(X)$ such that $g \mid S = f$ where $g \mid S = \{(x, y) \in g : x \in S\}$.

THEOREM 3.1. *If S is a subset of X, then each of the following is equivalent to each of the others*:

(1) *S is L_c-embedded in X,*
(2) *S is L_c^*-embedded in X, and*
(3) *S is closed.*

Proof. Czipszer and Geher [1] proved that if S is a closed subset of X and f is a real valued locally Lipschitz function with domain S, then there is a real valued locally Lipschitz function g with domain X such that $g \mid S = f$. Furthermore, they proved that if f is bounded, then there exists a bounded such g. Consequently, by Theorem 2.1, $(3) \Rightarrow (1)$ and $(3) \Rightarrow (2)$.

Assume (2) and S is not closed. Then there exists a sequence $\{x_n\}$ of distinct points in S and a point $x \in X - S$ such that $\{x_n\}$ converges to x. Construct f as in Theorem 2.6. Then $f \in L_c^*(S)$ which has no extension to X in $L_c(X)$. Thus $(2) \Rightarrow (3)$. Note that this also shows $(1) \Rightarrow (3)$.

COROLLARY 3.2. *Every subset of X is L_1-embedded and L_1^*-embedded in X.*

Proof. If $S \subset X$, then every element of $L_1(S)$ has a unique extension to the closure of S in comp X and by Theorems 2.6 and 3.1

an extension in $L_1(\text{comp}\, X)$ which when restricted to X is an element of $L_1(X)$.

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Received April 3, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol, 32, No. 1, 1970

TOTALLY POSITIVE DIFFERENTIAL SYSTEMS

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Totally positive (TP), and strictly totally positive (STP) differential systems are defined. These real, first order, linear systems are characterized by the form of their coefficient matrices, and by the decrease of the number of sign changes of their solution vectors as functions of the independent variable. A bound is given for the combined number of zeros of the first and last components of any particular solution vector of STP system and a similar result is obtained for TP systems. Examples show that no such bounds exist for the number of zeros of any other component.

In this paper we consider real differential systems of the form

$$y'(t) = A(t)y(t) \ . \tag{1.1}$$

Here the solutions $y(t)$ are real column vectors $y(t) = (y_1(t), \cdots, y_n(t))$ and $A(t)$ is a given $n \times n$ matrix $(a_{ij}(t))_1^n$ whose elements $a_{ij}(t)$ are real functions which are continuous in the open interval (a, b), $-\infty \leqq a < b \leqq \infty$. Together with the vector differential equation (1.1) we consider also the corresponding matrix differential equation

$$Y'(t) = A(t)Y(t) \ , \tag{1.2}$$

where $Y(t) = (y_{ij}(t))_1^n$. Let $Y(t)$ be any solution of (1.2); for each integer p, $1 \leqq p \leqq n$, we denote the p th compound of $Y(t)$ by $C_p(Y(t))$. In § 2 we construct for each p, $1 \leqq p \leqq n$, a $\binom{n}{p} \times \binom{n}{p}$ matrix $B^{(p)}(t)$, such that

$$[C_p(Y(t))]' = B^{(p)}(t)C_p(Y(t)) \ . \tag{1.3}$$

($B^{(1)}(t) = A(t)$.) The elements of $B^{(p)}(t)$ are easily expressed by the given n^2 elements $a_{ij}(t)$ of $A(t)$ (Theorem 1). Special cases of these compound systems were previously considered: Mikusiński [6] considered the differential system satisfied by the 2×2 Wronskians of the solutions of the equation $u^{(n)}(t) + p(t)u(t) = 0$ and Nehari [7] considered all compound systems (1.3) in the case where (1.1) is equivalent to an n th order linear differential equation. We remark that for $p = n - 1$ (1.3) is closely related to the system adjoint to (1.2); and for $p = n$ (1.3) reduces to Liouville's equation

$$\varDelta(t)' = \left(\sum_{i=1}^{n} a_{ii}(t)\right)\varDelta(t) \ , \tag{1.4}$$

where $\varDelta(t) = C_n(Y(t))$ is the determinant of $Y(t)$. We state an immediate consequence of Theorem 1, showing a characteristic difference between the elements $a_{ij}(t)$ with $|i-j|=1$ and the other off-diagonal elements of $A(t)$, as a corollary.

A real $n \times n$ matrix is totally positive (TP) is all its minors are nonnegative, and the matrix is strictly totally positive (STP) if all its minors are positive. For each r, $a < r < b$, we denote the fundamental solution $Y(t)$ of (1.2) satisfying

$$Y(r) = I\,, \tag{1.5}$$

$(I = (\delta_{ij})_1^n)$, by $Y(t) = Y(t, r)$. *We call the system* (1.2), *and the corresponding system* (1.1), *totally positive* (TP) *in* (a, b) *if for each pair* (r, t), $a < r \leqq t < b$, $Y(t, r)$ *is* TP. *If for each pair* (r, t), $a < r < t < b$, $Y(t, r)$ *is* STP *then the systems* (1.2) *and* (1.1) *are called strictly totally positive* (STP) *in* (a, b). In § 3 we characterize these systems by the form of the matrix $A(t) = (a_{ij}(t))_1^n$. The system (1.2) is TP in (a, b) if and only if $A(t)$ is a (variable) Jacobi matrix (i.e., $a_{ij}(t) \equiv 0$ for $|i-j| \geqq 2$) with nonnegative off-diagonal elements (i.e., $a_{i,i+1}(t) \geqq 0$, $a_{i+1,i}(t) \geqq 0$, $i = 1, \cdots, n-1$). This result (Theorem 2) was first proved by Loewner [5]. Our proof (based on Corollary 1) is quite elementary and leads also to the following modification of Loewner's result: The system (1.2) is STP in (a, b) if and only if $A(t)$ satisfies the above conditions and none of the functions $a_{i,i+1}(t)$ and $a_{i+1,i}(t)$ vanishes identically in any interval contained in (a, b) (Theorem 3).

In § 4 we consider vector solutions $y(t)$ of a STP system. The system (1.1) is shown to be STP in (a, b) if and only if $S^+(y(s)) \leqq S^-(y(r))$ holds for all nontrivial solutions $y(t)$ and all pairs (r, s), $a < r < s < b$, (Theorem 4). This result on the number of sign changes, following from the variation-diminishing properties of STP matrices, leads now to results on the number of zeros of the components $y_1(t)$ and $y_n(t)$ of any given vector solution $y(t)$ of (1.1). The combined number of zeros of these two extreme components cannot exceed $n-1$ (Theorem 5). No such restriction exists for the interior components $y_2(t), \cdots, y_{n-1}(t)$. We illustrate this dissimilarity between the extreme and the interior components by examples in the last section (§ 6). In § 5 we consider vector solutions of TP systems and the results are now weakened versions of the corresponding results for STP systems. We rely strongly on the recent book by Karlin [4], but we give all necessary definitions in order to keep this paper reasonably selfcontained.

2. The compound differential systems. For given integers n and p, $1 \leqq p \leqq n$, we consider the p-tuples of increasing integers

$$1 \leqq \begin{matrix} i_1 < i_2 < \cdots < i_p \\ j_1 < j_2 < \cdots < j_p \end{matrix} \leqq n \,,$$

and we arrange these $N = \binom{n}{p}$ p-tuples in lexicographic order. We denote these p-tuples of indices also by

$$\alpha = (i_1, i_2, \cdots, i_p) \,, \qquad \beta = (j_1, \cdots, j_p) \,.$$

For a $n \times n$ matrix $Y = (y_{ij})_1^n$, we denote the minor, determined by these rows and columns, by

$$Y\begin{pmatrix} i_1, i_2, \cdots, i_p \\ j_1, j_2, \cdots, j_p \end{pmatrix} .$$

The p th compound $C_p(Y)$ of Y is the $N \times N$ matrix having these minors (in lexicographic order) as elements. The elements of the $N \times N$ matrix $B^{(p)}$ are denoted by $b_{\alpha\beta} = b(i_1, \cdots, i_p \mid j_1, \cdots, j_p)$. In the following the matrices A and $B^{(p)}$ will be continuous functions of t, Y and $C_p(Y)$ will therefore be continuously differentiable functions of t. Using this notation we obtain the following relation between the given system (1.2) and its compound systems (1.3).

THEOREM 1. *Let $Y(t) = (y_{ij}(t))_1^n$, $a < t < b$, $-\infty \leqq a < b \leqq \infty$, be any solution of the differential system*

$$Y'(t) = A(t)Y(t) \,, \tag{1.2}$$

where $A(t) = (a_{ij}(t))_1^n$ and the n^2 real functions $a_{ij}(t)$ are continuous in (a, b). The p th compound $C_p(Y(t))$ of $Y(t)$, $1 \leqq p \leqq n$, satisfies in (a, b) the equation

$$[C_p(Y(t))]' = B^{(p)}(t)C_p(Y(t)) \,. \tag{1.3}$$

The matrix $B^{(p)}(t) = (b_{\alpha\beta}(t))_1^N$, $N = \binom{n}{p}$, is given by

$$\begin{aligned} &b_{\alpha\beta}(t) \\ &\quad = b(i_1, \cdots, i_p \mid j_1, \cdots, j_p) \end{aligned} = \begin{cases} 0 \text{ if at most } p-2 \text{ of the indices } i \text{ of } \alpha \text{ coincide with indices } j \text{ of } \beta; \\ (-1)^{\ell+m} a_{i_\ell j_m} \text{ if exactly } p-1 \text{ of the indices of } \alpha \text{ coincide with indices of } \beta, \text{ but } i_\ell \neq j_m,\ 1 \leqq \ell, m \leqq p; \\ \sum_{\ell=1}^{p} a_{i_\ell j_\ell} \text{ if } i_\ell = j_\ell,\ \ell = 1, \cdots, p. \end{cases} \tag{2.1}$$

Proof. We choose two p-tuples of increasing indices $\alpha = (i_1, \cdots, i_p)$,

$\gamma = (k_1, \cdots, k_p)$ with

$$1 \leqq \begin{matrix} i_1 < i_2 < \cdots < i_p \\ k_1 < k_2 < \cdots < k_p \end{matrix} \leqq n ,$$

and consider the minor $Y\begin{pmatrix} i_1, \cdots, i_p \\ k_1, \cdots, k_p \end{pmatrix}$. Differentiating this minor by rows and using (1.2) we obtain

$$Y\begin{pmatrix} i_1, \cdots, i_p \\ k_1, \cdots, k_p \end{pmatrix}' = \begin{vmatrix} \sum_{\nu=1}^{n} a_{i_1\nu} y_{\nu k_1} & \cdots & \sum_{\nu=1}^{n} a_{i_1\nu} y_{\nu k_p} \\ \cdots & \cdots & \cdots \\ y_{i_p k_1} & \cdots & y_{i_p k_p} \end{vmatrix} + \cdots + \begin{vmatrix} y_{i_1 k_1} & \cdots & y_{i_1 k_p} \\ \cdots & \cdots & \cdots \\ \sum_{\nu=1}^{n} a_{i_p\nu} y_{\nu k_1} & \cdots & \sum_{\nu=1}^{n} a_{i_p\nu} y_{\nu k_p} \end{vmatrix} .$$

We rewrite this as

$$\begin{aligned} Y\begin{pmatrix} i_1, \cdots, i_p \\ k_1, \cdots, k_p \end{pmatrix}' &= \sum_{\nu=1}^{n} a_{i_1\nu} Y\begin{pmatrix} \nu, i_2, \cdots, i_p \\ k_1, k_2, \cdots, k_p \end{pmatrix} \\ &+ \sum_{\nu=1}^{n} a_{i_2\nu} Y\begin{pmatrix} i_1, \nu, i_3, \cdots, i_p \\ k_1, k_2, k_3, \cdots, k_p \end{pmatrix} + \cdots \\ &+ \sum_{\nu=1}^{n} a_{i_p\nu} Y\begin{pmatrix} i_1, \cdots, i_{p-1}, \nu \\ k_1, \cdots, k_{p-1}, k_p \end{pmatrix} . \end{aligned} \tag{2.2}$$

The row indices on the r.h.s are, in general, not in increasing order and the pn determinants appearing there are hence, in general, not minors of Y. But each of these determinants either vanishes or is equal to a minor of Y or is equal to (-1) times a minor. We thus can write (2.2) in the form

$$\begin{aligned} Y\begin{pmatrix} i_1, \cdots, i_p \\ k_1, \cdots, k_p \end{pmatrix}' &= \sum_{\beta} b_{\alpha\beta} Y\begin{pmatrix} j_1, \cdots, j_p \\ k_1, \cdots, k_p \end{pmatrix} \\ &= \sum_{1 \leqq j_1 < \cdots < j_p \leqq n} b(i_1, \cdots, i_p \mid j_1, \cdots, j_p) Y\begin{pmatrix} j_1, \cdots, j_p \\ k_1, \cdots, k_p \end{pmatrix} . \end{aligned} \tag{2.3}$$

To obtain (2.1) we compare (2.2) and (2.3). We first note that on the r.h.s. of (2.2) appear only p-tuples of row indices for which at least $p-1$ of the indices belong to the p-tuple $\alpha = (i_1, \cdots, i_p)$. This gives the first part of (2.1). Secondly, if ν does not belong to α, then the p-tuples

$$(\nu, i_2, \cdots, i_p), (i_1, \nu, i_3, \cdots, i_p), \cdots, (i_1, \cdots, i_{p-1}, \nu) ,$$

appearing as row indices on the r.h.s. of (2.2), have to be rearranged by putting the index $\nu = j_m$ in its proper place in order to obtain an increasing p-tuple which in (2.3) is denoted by $\beta = (j_1, \cdots, j_p)$. For the p-tuples corresponding to the first sum on the r.h.s. of (2.2) this may be achieved by $m+1$ transpositions, for those of the second sum by $m+2$ transpositions and, in general, for those of the ℓth sum by $m+\ell$ transpositions. This implies the second part of (2.1). Finally, if we choose $\nu = i_1$ in the first sum on the r.h.s. of (2.2), $\nu = i_2$ in the second sum and so on, we obtain the last part of (2.1) and we have thus proved Theorem 1.

We illustrate this result by expressing the elements $b_{\alpha\beta}$ of $B^{(p)}$ in terms of the elements a_{ij} in the simplest cases: $n = 3, p = 2, n = 4, p = 2$ and $n = 4, p = 3$.

i_1	i_2	j_1 j_2 1 2	j_1 j_2 1 3	j_1 j_2 2 3
1	2	$a_{11}+a_{22}$	a_{23}	$-a_{13}$
1	3	a_{32}	$a_{11}+a_{33}$	a_{12}
2	3	$-a_{31}$	a_{21}	$a_{22}+a_{33}$

$n = 3,\ p = 2$

i_1	i_2	j_1 j_2 1 2	j_1 j_2 1 3	j_1 j_2 1 4	j_1 j_2 2 3	j_1 j_2 2 4	j_1 j_2 3 4
1	2	$a_{11}+a_{22}$	a_{23}	a_{24}	$-a_{13}$	$-a_{14}$	0
1	3	a_{32}	$a_{11}+a_{33}$	a_{34}	a_{12}	0	$-a_{14}$
1	4	a_{42}	a_{43}	$a_{11}+a_{44}$	0	a_{12}	a_{13}
2	3	$-a_{31}$	a_{21}	0	$a_{22}+a_{33}$	a_{34}	$-a_{24}$
2	4	$-a_{41}$	0	a_{21}	a_{43}	$a_{22}+a_{44}$	a_{23}
3	4	0	$-a_{41}$	a_{31}	$-a_{42}$	a_{32}	$a_{33}+a_{44}$

$n = 4,\ p = 2$

i_1	i_2	i_3	j_1 j_2 j_3 1 2 3	j_1 j_2 j_3 1 2 4	j_1 j_2 j_3 1 3 4	j_1 j_2 j_3 2 3 4
1	2	3	$a_{11}+a_{22}+a_{33}$	a_{34}	$-a_{24}$	a_{14}
1	2	4	a_{43}	$a_{11}+a_{22}+a_{44}$	a_{23}	$-a_{13}$
1	3	4	$-a_{42}$	a_{32}	$a_{11}+a_{33}+a_{44}$	a_{12}
2	3	4	a_{41}	$-a_{31}$	a_{21}	$a_{22}+a_{33}+a_{44}$

$n = 4,\ p = 3$

We remark that each diagonal element a_{ii} appears as a summand in $\binom{n-1}{p-1}$ diagonal elements of $B^{(p)}$. Each a_{ij}, $i \neq j$, appears, possibly with the sign -1, $\binom{n-2}{p-1}$ times as an off-diagonal element of $B^{(p)}$. In each row and each column of $B^{(p)}$ $p(n-p)$ off-diagonal elements are of the form $\pm a_{ij} (i \neq j)$ and, for $2 \leqq p \leqq n-2$, the remaining off-diagonal elements are zeros. (2.1) implies also the following symmetry of the dependence of $B^{(p)}$ on A: if, for $\alpha \neq \beta$, $b_{\alpha\beta} = a_{ij}$, $b_{\alpha\beta} = -a_{ij}$ or $b_{\alpha\beta} = 0$ then $b_{\beta\alpha} = a_{ji}$, $b_{\beta\alpha} = -a_{ji}$ or $b_{\beta\alpha} = 0$ respectively.

For $p = n$, (2.1) gives $B^{(n)}(t) = b(1, \cdots, n \mid 1, \cdots, n) = \sum_{i=1}^{n} a_{ii}(t)$, and the differential system of the n th compound $\Delta(t) = C_n(Y(t))$ is Liouville's equation

$$\Delta(t)' = \left(\sum_{i=1}^{n} a_{ii}(t)\right)\Delta(t) . \tag{1.4}$$

We now consider the case $p = n-1$. Let $Y(t)$ be a fundamental solution of (1.2), then

$$\begin{aligned} \Delta(t)(Y(t)^{-1})^T &= \left((-1)^{i+j} Y\begin{pmatrix} 1 \cdots i-1, i+1 \cdots n \\ 1 \cdots j-1, j+1 \cdots n \end{pmatrix}\right)_1^n \\ &= \widetilde{C}_{n-1}(Y(t)) . \end{aligned} \tag{2.4}$$

Here the superscript T denotes the transposition operation, and if $M = (m_{ij})_1^n$ we define $\widetilde{M} = (\widetilde{m}_{ij})_1^n$ by

$$\widetilde{m}_{ij} = (-1)^{i+j} m_{n+1-i,n+1-j}, \quad i, j = 1, \cdots, n .$$

With this notation $[C_{n-1}(Y(t))]' = B^{(n-1)}(t) C_{n-1}(Y(t))$ gives

$$[\widetilde{C}_{n-1}(Y(t))]' = \widetilde{B}^{(n-1)}(t) \widetilde{C}_{n-1}(Y(t)) . \tag{2.5}$$

$(Y(t)^{-1})^T$ is a solution of the system adjoint to (1.2):

$$(Y(t)^{-1})^{T\prime} = -A(t)^T (Y(t)^{-1})^T . \tag{2.6}$$

Differentiating (2.4) and using also (1.4), (2.5) and (2.6) we obtain

$$-A(t)^T = \widetilde{B}^{(n-1)}(t) - \left(\sum_{i=1}^{n} a_{ii}(t)\right) I . \tag{2.7}$$

(2.7) gives the connection between the adjoint equation and the equation for the $(n-1)$ st compound.

In the next section we use the following consequence of Theorem 1.

COROLLARY 1. *Let* $A(t) = (a_{ij}(t))_1^n$ *and* $B^{(p)}(t) = (b_{\alpha\beta}(t))_1^N$ *be the coefficient matrices of the system* (1.2) *and its compound systems,* $1 \leqq p \leqq n$. *Then,*

(i) *None of the matrices* $B^{(p)}(t)$ *contains elements of the form*

$-a_{i,i+1}(t)$, $-a_{i+1,i}(t)$, $i = 1, \cdots, n-1$.

(ii) *For each pair* (i, j), $|i-j| \geqq 2$, $i, j = 1, \cdots, n$, $-a_{ij}(t)$ *is an off-diagonal element of* $B^{(2)}(t)$.

Proof. (i) Formula (2.1) implies that $\pm a_{k,k+1}(t)$ and $\pm a_{k+1,k}(t)$, $k = 1, \cdots, n-1$, can appear as elements of $B^{(p)}(t)$ only if they are an element $b(i_1, \cdots, i_p \mid j_1, \cdots, j_p)$, where $p-1$ of the indices of the two p-tuples $\alpha = (i_1, \cdots, i_p)$ and $\beta = (j_1, \cdots, j_p)$ coincide, but $i_\ell \neq j_m$, and where the set $\{i_\ell, j_m\}$ is the set $\{k, k+1\}$. If a given $(p-1)$-tuple of increasing indices, which contains neither k nor $k+1$, is completed to a p-tuple of increasing indices by inserting k or $k+1$, then it is necessary to insert either one of them at the same place, i.e., between the same two elements of the $(p-1)$-tuple. Hence $\ell = m$. and (2.1) implies (i).

(ii) If $1 \leqq i < i+2 \leqq j \leqq n$ then (2.1) gives $b(i, i+1 \mid i+1, j) = -a_{ij}$; and if $1 \leqq j < j+2 \leqq i \leqq n$ then $b(j+1, i \mid j, j+1) = -a_{ij}$.

3. Positive, strictly positive, totally positive and strictly totally positive systems. Totally positive (TP) and strictly totally positive (STP) systems were defined in the introduction. To define positive and strictly positive systems we agree to call a real $n \times n$ matrix positive if all its elements are nonnegative; and the matrix is strictly positive if all its elements are positive. *The differential system*

$$Y'(t) = A(t)Y(t) \ , \tag{1.2}$$

is called positive in (a, b), *if for each pair* (r, t), $a < r \leqq t < b$, $Y(t, r)$ *is positive.* (Here $Y(t) = Y(t, r)$ is the fundamental solution of (1.2) satisfying (1.5).) (1.2) *is strictly positive in* (a, b) *if for each pair* (r, t), $a < r < t < b$, $Y(t, r)$ *is strictly positive.* We start with a criterion for the positivity of the system.

LEMMA 1. *Let the* n^2 *real functions* $a_{ij}(t)$, $i, j = 1, \cdots, n$, *be continuous in* (a, b), $-\infty \leqq a < b \leqq \infty$, *and set* $A(t) = (a_{ij}(t))_1^n$. *The differential system* (1.2) *is positive in* (a, b) *if and only if all off-diagonal elements* $a_{ij}(t)$, $i \neq j$, $i, j = 1, \cdots, n$, *are nonnegative in* (a, b).

This lemma is known [1, p. 173, exercise 2]. For completeness, and also in view of the proof of the next lemma, we prove Lemma 1.

Proof. To show the necessity of the condition, suppose to the contrary that there exist indices i^* and j^*, $i^* \neq j^*$, and a point r in (a, b) such that $a_{i^*j^*}(r) < 0$. Let $Y(t, r) = (y_{ij}(t))_1^n$ be the solution of (1.2) satisfying (1.5). Then $y'_{i^*j^*}(r) = 0$ and $y_{i^*j^*}(r) = a_{i^*j^*}(r) < 0$. Hence,

$y_{i^*j^*}(t) < 0$ for all t in some interval $(r, r+\varepsilon)$, $\varepsilon > 0$, and the system (1.2) is not positive.

We first prove sufficiency in the special case where all diagonal elements $a_{ii}(t)$ of $A(t)$ vanish identically in (a, b). Each element of $A(t)$ is thus nonnegative, and the Peano-Baker expansion

$$Y(t, r) = I + \int_r^t A(\tau)d\tau + \int_r^t A(\tau)\int_r^\tau A(\tau_1)d\tau_1 d\tau + \cdots, \tag{3.1}$$

shows that the same holds for each element of $Y(t, r)$, $a < r \leqq t < b$.

To prove sufficiency in the general case (of arbitrary diagonal elements $a_{ii}(t)$ of $A(t)$) we choose a point r, $r \in (a, b)$, and define

$$p_i(t, r) = \exp \int_r^t a_{ii}(\tau)d\tau,\ a < t < b,\ i = 1, \cdots, n\ . \tag{3.2}$$

Using these n positive functions we now build the diagonal matrix

$$P_r(t) = \operatorname{diag}(p_1(t, r), \cdots, p_n(t, r))\ , \qquad a < t < b\ . \tag{3.3}$$

If $Y(t)$ is an arbitrary solution of (1.2) we define $\tilde{Y}_r(t)$ by

$$Y(t) = P_r(t)\tilde{Y}_r(t)\ , \qquad a < t < b\ . \tag{3.4}$$

(1.2) and (3.2) to (3.4) imply that each $\tilde{Y}_r(t)$ satisfies the equation

$$\tilde{Y}_r'(t) = \tilde{A}_r(t)\tilde{Y}_r(t)\ , \qquad a < t < b\ , \tag{3.5}$$

where $\tilde{A}_r(t) = (\tilde{a}_{ij}(t, r))_1^n$ is defined by

$$\tilde{a}_{ij}(t, r) = a_{ij}(t)\frac{p_j(t, r)}{p_i(t, r)},\ i \neq j,\ i, j = 1, \cdots, n\ , \qquad a < t < b\ , \tag{3.6}$$

and

$$\tilde{a}_{ii}(t, r) = 0,\ i = 1, \cdots, n\ , \qquad a < t < b\ . \tag{3.7}$$

The matrix $\tilde{A}_r(t)$ has thus, together with the given matrix $A(t)$, nonnegative off-diagonal elements but its diagonal elements vanish identically. By the special case considered above, it follows that the system (3.5) is positive in (a, b). Let now $\tilde{Y}_r(t, r)$ be the fundamental solution $\tilde{Y}_r(t)$ of (3.5) which satisfies $\tilde{Y}_r(r) = I$. Then $\tilde{Y}_r(t, r)$ is positive for all t in $[r, b)$. As $P_r = I$, it follows from (3.4) that

$$Y(t, r) = P_r(t)\tilde{Y}_r(t, r)\ , \tag{3.8}$$

where $Y(t, r)$ is the solution of (1.2) satisfying (1.5). (3.8) implies that this matrix $Y(t, r)$ is positive for all t in $[r, b)$. Since r was arbitrary in (a, b), this completes the proof of Lemma 1.

For the next lemma it is convenient to use the following terminology. We denote the set of the n^2 elements $a_{ij}(t)$ of $A(t)$ by S.

With each subset F of S we associate a matrix $C = (c_{ij})_1^n$ in the following way: $c_{ij} = 1$ if $a_{ij}(t) \in F$, $c_{ij} = 0$ if $a_{ij}(t) \notin F$. Then we call F *irreducible* or *reducible* if the associated matrix C is, respectively, irreducible or reducible. If we associate with F a directed graph Γ of n vertices $P_1, \cdots, P_n$, having a (directed) arc from P_i to P_j if and only if $a_{ij} \in F$, then F is irreducible if and only if Γ is strongly connected. (A matrix $C = (c_{ij})_1^n$ is reducible if the index set $\{1, \cdots, n\}$ can be split into two nonvoid sets $\{i_1, \cdots, i_\ell\}$ and $\{j_1, \cdots, j_m\}$, $\ell + m = n$ such that $c_{i_\lambda j_\mu} = 0$ for $\lambda = 1, \cdots, \ell$, $\mu = 1, \cdots, m$. If no such partition of the index set exists, then C is irreducible. A directed graph Γ is strongly connected if and only if for every ordered pair (P_i, P_j) of its vertices there exists a (directed) path leading from P_i to P_j. The matrix C is irreducible if and only if the corresponding graph Γ is strongly connected. [9, pp. 18–20].)

LEMMA 2. *Let the n^2 real functions $a_{ij}(t), i, j = 1, \cdots, n$, be continuous in (a, b), $-\infty \leqq a < b \leqq \infty$, and set $A(t) = (a_{ij}(t))_1^n$. Let S be the set of the n^2 functions $a_{ij}(t)$. For each r, $r \in (a, b)$, the subset $F(r)$ of S is defined in the following way: $a_{ij}(t) \in F(r)$ if and only if $a_{ij}(t)$ does not vanish identically in any interval $[r, r+\varepsilon]$, $0 < \varepsilon < b - r$. The differential system* (1.2) *is strictly positive in (a, b) if and only if the following two conditions hold:*

(a) *Each off-diagonal element $a_{ij}(t), i \neq j, i, j = 1, \cdots, n$, is nonnegative in (a, b).*

(b) *For each r, $a < r < b$, the set $F(r)$ is irreducible.*

Proof. The necessity of condition (a) follows from Lemma 1. We prove the necessity of (b) by negation and thus assume that there exists r, $r \in (a, b)$, such that $F(r)$ is reducible. As the graph $\Gamma(r)$ is thus not strongly connected it follows that there exists ε, $0 < \varepsilon < b - r$ and two indices $i^*, j^*, i^* \neq j^*$, such that for every given ordered set $(i_0, i_1, \cdots, i_\ell)$ of indices (with repetition), for which $i_0 = i^*$, $i_\ell = j^*$, at least one function $a_{i_\nu i_{\nu+1}}(t)$, $\nu = 0, \cdots, \ell - 1$, vanishes identically in $[r, r+\varepsilon]$. For $\ell = 1$ this implies

$$\int_r^{r+\varepsilon} a_{i^*j^*}(\tau)d\tau = 0 . \tag{3.9$_1$}$$

For $\ell = 2$ we obtain

$$\int_r^{r+\varepsilon} \sum_{\nu=1}^{n} a_{i^*\nu}(\tau) \int_r^{\tau} a_{\nu j^*}(\tau_1)d\tau_1 d\tau = 0 , \tag{3.9$_2$}$$

and similar equalities hold for $\ell \geqq 3$. Using these equalities it follows from (3.1) that the off-diagonal element $y_{i^*j^*}(r+\varepsilon, r)$ of the matrix $Y(r+\varepsilon, r)$ vanishes and $Y(r+\varepsilon, r)$ is thus not strictly positive.

We prove sufficiency of conditions (a) and (b) again first in the special case where all diagonal elements $a_{ii}(t)$ of $A(t)$ vanish identically in (a, b). By (b), the set $F(r)$ is, for each $r \in (a, b)$, irreducible and in this special case $F(r)$ does not contain diagonal elements $a_{ii}(t)$. This and (a) imply that for any given r, $r \in (a, b)$, and any ordered pair (i^*, j^*) of (not necessarily distinct) indices there exists an ordered set $(i_0, i_1, \cdots, i_\ell)$ of indices, $i_0 = i^*$, $i_\ell = j^*$ and $i_\nu \neq i_{\nu+1}$ for $\nu = 0, \cdots, \ell - 1$, such that

$$\int_r^t a_{i_\nu i_{\nu+1}}(\tau)d\tau > 0 ,$$

for all t in (r, b) and all ν, $\nu = 0, \cdots, \ell - 1$. But this implies that for all such t

$$\int_r^t a_{i^* i_1}(\tau)\int_r^\tau a_{i_1 i_2}(\tau_1) \cdots \int_r^{\tau_{\ell-2}} a_{i_{\ell-1} j^*}(\tau_{\ell-1})d\tau_{\ell-1} \cdots d\tau_1 d\tau > 0 ,$$

and it follows that the element in the place (i^*, j^*) of the $(\ell + 1)$ th summand of the r.h.s. of (3.1) is, for $t \in (r, b)$, positive. As r and the pair of indices were arbitrary it follows that the system (1.2) is, in this special case, strictly positive in (a, b).

The sufficiency of conditions (a) and (b) in the general case (of arbitrary diagonal elements $a_{ii}(t)$ of $A(t)$) follows again by reduction to the special case (formulas (3.2) to (3.8)). We now use also the fact that if the set $F(r)$ is irreducible, so is the set $\tilde{F}(r)$ which is obtained from $F(r)$ by deletion of its diagonal elements and by multiplication of its off-diagonal elements with positive functions. This completes the proof of Lemma 2.

These criteria for positivity and strict positivity and the corollary of § 2 lead to the main results of this section.

THEOREM 2. *Let the n^2 real functions $a_{ij}(t)$, $i, j = 1, \cdots, n$, be continuous in (a, b), $-\infty \leqq a < b \leqq \infty$, and set $A(t) = (a_{ij}(t))_1^n$. The differential system*

$$Y'(t) = A(t)Y(t) , \tag{1.2}$$

is TP *in (a, b) if and only if the following two conditions hold:*

(a) $a_{ij}(t) = 0$, $|i - j| \geqq 2$, $i, j = 1, \cdots, n$, $a < t < b$.

(b) $a_{i,i+1}(t) \geqq 0$, $a_{i+1,i}(t) \geqq 0$, $i = 1, \cdots, n - 1$, $a < t < b$.

Proof. As total positivity of the system (1.2) implies its positivity, it follows from Lemma 1 that all off-diagonal elements $a_{ij}(t)$, $i \neq j$, of $A(t)$ have to be nonnegative in (a, b). If an element $a_{ij}(t)$, $|i - j| \geqq 2$, were to be positive for some t part (ii) of Corollary 1 would imply that the matrix $B^{(2)}(t)$ of the second compound system has an off-diagonal element which is somewhere negative, and Lemma 1, applied to this second compound system, then shows that (1.2) is not TP. Conditions (a) and (b) are thus necessary. Their sufficiency follows from part (i) of Corollary 1 and the sufficiency part of Lemma 1, applied to all compound systems (1.3). (We remark that we also use that the p th compound of the unit matrix $I = (\delta_{ij})_1^n$ is again $I = (\delta_{\alpha\beta})_1^N$. Hence if $Y(t) = Y(t, r)$ is the solution of (1.2) which satisfies (1.5), then its compound also satisfies $C_p(Y(r)) = I$.)

THEOREM 3. *Let the n^2 real functions $a_{ij}(t)$, $i, j = 1, \cdots, n$, be continuous in (a, b), $-\infty \leqq a < b \leqq \infty$, and set $A(t) = (a_{ij}(t))_1^n$. The differential system*

$$Y'(t) = A(t)Y(t) \ , \tag{1.2}$$

is STP *in (a, b) if and only if the following three conditions hold*:

(a) $a_{ij}(t) = 0, |i - j| \geqq 2, i, j = 1, \cdots, n, \quad a < t < b$.

(b) $a_{i,i+1}(t) \geqq 0, a_{i+1,i}(t) \geqq 0, i = 1, \cdots, n - 1, \ a < t < b$.

(c) *None of the $2n - 2$ functions mentioned in* (b) *vanishes identically in any interval $[r, s]$, $a < r < s < b$.*

Proof. The necessity of conditions (a) and (b) follows from Theorem 2. To prove that condition (c) is necessary, we consider the (0, 1) matrix $C^* = (c_{ij}^*)_1^n$ where $c_{ij}^* = 0$ if $|i - j| \neq 1$, and $c_{ij}^* = 1$ if $|i - j| = 1$. Then the following statement holds. (i) *C^* is irreducible, and* (ii) *if any element equal to* 1 *of C^* is replaced by* 0 *then the new matrix is reducible.* This is easily seen by considering the corresponding directed graph Γ^*. Assume now that condition (c) is not satisfied and that one of the $2n - 2$ functions $a_{i,i+1}(t)$ and $a_{i+1,i}(t)$ vanishes identically in a certain interval $[r, s]$. Part (ii) of the italicized statement implies that the set $F(r)$, defined in Lemma 2, is reducible and Lemma 2 implies that the system (1.2) is not strictly positive in (a, b). This contradicts the assumption of the present theorem and condition (c) is thus necessary.

To prove the sufficiency of conditions (a) to (c), we consider also the (0, 1) matrices $C^{*(p)}$, $1 \leqq p \leqq n$, which are built from the elements c_{ij}^* of $C^* = C^{*(1)}$ by the rule (2.1). Namely,

$$C^{*(p)} = (c_{\alpha\beta}^*)_1^N, N = \binom{n}{p} \quad \text{and} \quad c_{\alpha\beta}^* = c^*(i_1, \cdots, i_p \mid j_1, \cdots, j_p) = 0$$

except if exactly $p - 1$ of the indices of α coincide with $p - 1$ indices of β and the two remaining indices satisfy $|i_{\swarrow} - j_{\swarrow}| = 1$; in this case $c^*_{\alpha\beta} = 1$. *For each* $p, 1 \leqq p \leqq n, C^{*(p)}$ *is irreducible.* (For $p = 1$ this is part (i) of the former italicized statement.) This is again easily seen by considering the corresponding graph $\Gamma^{*(p)}$. ($\Gamma^{*(p)}$ has N vertices $P_\alpha = P(i_1, \cdots, i_p)$, $P_\beta = P(j_1, \cdots, j_p)$, etc. There are arcs (in both directions) between P_α and P_β if $p-1$ of the indices of α and β coincide and $|i_{\swarrow} - j_{\swarrow}| = 1$. Clearly there exists a path of length $\sum_{\nu=1}^{p}(i_\nu - \nu)$ leading from P_α to the first vertex P_{α^*} ($\alpha^* = 1 = (1, \cdots, p)$) and similarly there exists a path leading from P_{α^*} to P_β. $\Gamma^{*(p)}$ is thus strongly connected). Using part (i) of Corollary 1 and the irreducibility of $C^{*(p)}, 1 \leqq p \leqq n$, it follows that the present conditions (a) to (c) imply the validity of conditions (a) and (b) of Lemma 2 for each compound system (1.3). Each of these systems is therefore strictly positive in (a, b) and (1.2) is thus STP. This completes the proof of Theorem 3.

4. Vector solutions of strictly totally positive systems. Our next result refers to the number of sign changes of a given nontrivial vector solution $y(t)$ of a STP system (1.1). We use the standard notation [2, 4]. If $x = (x_1, \cdots, x_n)$ is a real vector, $x \neq 0$, then $S^-(x)$ denotes the number of sign changes in the sequence obtained from $x_1, x_2, \cdots, x_n$ by deleting all zero terms; $S^+(x)$ denotes the maximum number of sign changes possible by allowing each zero to be replaced ± 1 (or equivalently, $S^+(x) = \overline{\lim}_{y \to x} S^-(y)$).

THEOREM 4. (i) *Let the differential system*

$$y'(t) = A(t)y(t) \, , \tag{1.1}$$

be STP *in* (a, b), $-\infty \leqq a < b \leqq \infty$ *and let* $y(t)$ *be a nontrivial solution. Then*

$$S^+(y(s)) \leqq S^-(y(r)) \quad \textit{for all } (r, s) \textit{ satisfying} \quad a < r < s < b \, . \tag{4.1}$$

(ii) *Conversely, if* (4.1) *is valid for every nontrivial solution* $y(t)$ *of the system* (1.1), *then this system is* STP *in* (a, b).

Proof. (i) Let $Y(t) = Y(t, r)$ be the fundamental solution of

$$Y'(t) = A(t)Y(t) \, , \tag{1.2}$$

satisfying

$$Y(r) = I \, . \tag{1.5}$$

For all s and r in (a, b)

$$y(s) = Y(s, r)y(r) \, . \tag{4.2}$$

By assumption the matrix $Y(s, r)$ is, for $r < s$, STP. (4.1) follows from the variation-diminishing property of such matrices [4, p. 219, Th. 1.2, (a)].

(ii) Let the index $k, 1 \leqq k \leqq n$, and the point $r, r \in (a, b)$, be given and consider nontrivial solutions $y(t)$ of (1.1) which satisfy

$$y_k(r) = 0 . \tag{4.3}$$

(4.2) and (4.3) give

$$y(s) = Y_k(s, r)c . \tag{4.4}$$

Here c is the $(n-1)$ vector $(y_1(r), \cdots, y_{k-1}(r), y_{k+1}(r), \cdots, y_n(r))$ and $Y_k(s, r)$ is the $n \times (n-1)$ matrix obtained from $Y(s, r)$ by deletion of the k th column. By assumption (4.1), we have for $r < s$,

$$S^+(y(s)) \leqq S^-(y(r)) = S^-(c) .$$

As this holds for every nonnull vector c, it follows that $Y_k(s, r)$ is, for $r < s$, strictly sign-regular of order $n-1$ [4, p. 219, Th. 1.2, (b)]; i.e., all minors of $Y_k(s, r)$ are nonzero and, for each $p, 1 \leqq p \leqq n-1$, all minors of order p have the same sign, possibly dependent on p. But as $Y(r, r) = I$, it follows that $Y_k(r, r)$ has for each $p, 1 \leqq p \leqq n-1$, a minor equal to 1. It follows, by continuity, that all minors of $Y_k(s, r)$, $r < s$, are positive. As k was an arbitrary index, this implies that all minors, up to the order $n-1$, of $Y(s, r)$ are positive for $r < s$. But the determinant of $Y(s, r)$ is always positive and we have thus proved that the system (1.1) is STP.

We remark that by the last two theorems property (4.1), for all nontrivial solutions $y(t)$, is equivalent to the properties (a) to (c) of $A(t)$ stated in Theorem 3. A direct proof of this equivalence, without use of the variation-diminishing properties of the STP matrix $Y(s, r)$, seems to be rather tedious.

The next theorem, and the examples in the final section, will give some information about the number of points at which each component of a fixed solution of an STP system (1.1) may vanish. It might be of interest to consider here briefly the case of such systems with constant coefficients $A(t) = A$. A is thus a Jacobi matrix with positive off-diagonal elements. But the class of Jacobi matrices B with negative off-diagonal elements was studied in detail by Gantmacher and Krein [2, Ch. 2, § 1.]. For $A(=-B)$ it follows that A has n distinct real characteristic values $\lambda_j, \lambda_1 < \lambda_2 < \cdots < \lambda_n$, (and that for the characteristic vector $u^{(j)} = (u_{ij}, \cdots, u_{nj})$, corresponding to λ_j, $S^+(u^{(j)}) = S^-(u^{(j)}) = n - j, j = 1, \cdots, n$). Every solution $y(t)$ of the corresponding system (1.1) is therefore of the form

$$y_i(t) = \sum_{j=1}^{n} c_j u_{ij} e^{\lambda_j t} , \qquad i = 1, \cdots, n ,$$

and it follows that in this case each component $y_i(t)$ of a nontrivial solution $y(t)$ vanishes at most $n-1$ times. (Note that for any system (1.1) there always exist nontrivial solutions $y(t)$ satisfying $(n-1)$ homogeneous conditions.) As already mentioned in the introduction a more precise statement holds for the total number of zeros of $y_1(t)$ and $y_n(t)$ for any STP system (Theorem 5, (ii)); and the examples will show that, for any n, $n \geqq 3$, there exist STP systems with variable $A(t)$ having a solution $y(t)$ for which each interior component $y_i(t)$, $i = 2, \cdots, n-1$, vanishes infinitely many times in $(-\infty, \infty)$.

To facilitate the proof of Theorem 5 we now state some evident properties of the functions S^+ and S^- as a lemma.

LEMMA 3. *Let $x = (x_1, \cdots, x_n)$, be a real nonnull vector. Then*

$$0 \leqq S^-(x) \leqq S^+(x) \leqq n-1 . \tag{4.5}$$

If m components of x vanish, $1 \leqq m \leqq n-1$, then

$$S^+(x) \geqq m, S^-(x) \leqq n-m-1 . \tag{4.6}$$

If $x_1 = 0$, or if $x_n = 0$, then

$$S^+(x) - S^-(x) \geqq 1 . \tag{4.7}$$

If $x_1 = 0$ and $x_n = 0$, then

$$S^+(x) - S^-(x) \geqq 2 . \tag{4.8}$$

Part (i) of Theorem 4, and Lemma 3, now imply the following theorem.

THEOREM 5. *Let the differential system*

$$y'(t) = A(t)y(t) \tag{1.1}$$

be STP *in (a, b), $-\infty \leqq a < b \leqq \infty$, and let $y(t) = (y_1(t), \cdots, y_n(t))$ be a nontrivial solution.*

(i) *If $S^-(y(r)) = 0$, $r \in (a, b)$, then no component of $y(t)$ vanishes in (r, b). If $S^+(y(s)) = n-1$, $s \in (a, b)$, then no component of $y(t)$ vanishes in (a, s).*

(ii) *Let k and ℓ be nonnegative integers and assume that*

$$y_1(\alpha_i) = 0, i = 1, \cdots, k, a < \alpha_1 < \cdots < \alpha_k < b ,$$

and that

$$y_n(\beta_j) = 0, j = 1, \cdots, \ell, a < \beta_1 < \cdots < \beta_\ell < b .$$

Then $k + \ell \leqq n - 1$. *Moreover, if* $k + \ell = n - 1$, *then no component of* $y(t)$ *vanishes in* $(a, \min(\alpha_1, \beta_1)) \cup (\max(\alpha_k, \beta_\ell), b)$.

(iii) *Assume that* m *components of* $y(r)$, $r \in (a, b)$, *vanish, and that*

$$y_1(\alpha_i) = 0, i = 1, \cdots, k, r < \alpha_1 < \cdots < \alpha_k < b ,$$

and that

$$y_n(\beta_j) = 0, j = 1, \cdots, \ell, r < \beta_1 < \cdots < \beta_\ell < b .$$

Then $k + \ell \leqq n - m - 1$. *Moreover, if* $k + \ell = n - m - 1$, *then no component of* $y(t)$ *vanishes in* $(\max(\alpha_k, \beta_\ell), b)$. *A similar statement holds for the number of zeros of* $y_1(t)$ *and* $y_n(t)$ *in* (a, r).

Proof. (i) $S^-(y(r)) = 0$ and (4.1) imply $S^+(y(t)) = 0, r < t < b$, and the first inequality of (4.6) implies that no component of $y(t)$ vanishes. $S^+(y(s)) = n - 1$ and (4.1) imply $S^-(y(t)) = n - 1, a < t < s$, and the other inequality of (4.6) gives the desired conclusion.

(ii) Denote the union of the sets $\{\alpha_i\}_1^k$ and $\{\beta_j\}_1^\ell$ by

$$\{t_\nu\}_1^p, t_1 < \cdots < t_p, (\max(k, \ell) \leqq p \leqq k + \ell) .$$

Then

$$\begin{aligned} k + \ell &\leqq \sum_{\nu=1}^{p} [S^+(y(t_\nu)) - S^-(y(t_\nu))] \\ &= \sum_{\nu=2}^{p} [S^+(y(t_\nu)) - S^-(y(t_{\nu-1}))] + S^+(y(t_1)) - S^-(y(t_p)) \\ &\leqq S^+(y(t_1)) - S^-(y(t_p)) \leqq n - 1 . \end{aligned} \tag{4.9}$$

Here the first inequality sign follows from (4.7) and (4.8), the second inequality sign follows from (4.1) and the last one from (4.5). This proves the main assertion of (ii). If $k + \ell = n - 1$, then (4.9) implies $S^-(y(t_p) = 0$ and $S^+(y(t_1)) = n - 1$ and the remaining assertion of (ii) now follows from (i).

(iii) Let $t_1, \cdots, t_p$ have the same meaning as above. (4.9), the assumption $r < t_1$ and (4.1), and (4.6) give

$$\begin{aligned} k + \ell &\leqq S^+(y(t_1)) - S^-(y(t_p)) \\ &\leqq S^-(y(r)) - S^-(y(t_p)) \leqq n - m - 1 . \end{aligned} \tag{4.10}$$

If $k + \ell = n - m - 1$, then (4.10) and $S^-(y(r) \leqq n - m - 1$ imply $S^-(y(t_p)) = 0$ and no component of $y(t)$ vanishes in (t_p, b). For zeros to the left of r, $a < t_1 < \cdots < t_p < r$, we obtain

$$\begin{aligned} k + \ell &\leqq S^+(y(t_1)) - S^-(y(t_p)) \\ &\leqq S^+(y(t_1)) - S^+(y(r)) \leqq n - m - 1 . \end{aligned}$$

If $k + \ell = n - m - 1$ this gives $S^+(y(t_1)) = n - 1$ and no component vanishes in (a, t_1). This completes the proof of Theorem 5.

We remark that the constants $n-1$ of part (ii) and $n-m-1$ of part (iii) of this theorem, are the best possible as there always exist nontrivial solutions of (1.1) satisfying $n-1$ conditions $y_{i_\nu}(t_\nu) = 0, 1 \leqq i_\nu \leqq n, a < t_\nu < b, \nu = 1, \cdots, n-1$. We conclude this section with another direct consequence of (4.1). *Let r and s be given points, $a < r < s < b$, and assume that $y(t)$ is a nontrivial solution of* (1.1) *such that k components of $y(r)$ and l components of $y(s)$ vanish. Then $k + l \leqq n-1$. Moreover, if $k + l = n-1$, then there exists-except for a multiplicative constant-precisely one nontrivial solution $y(t)$ of* (1.1) *satisfying the given set of conditions $y_{i_\nu}(r) = 0, y_{j_\mu}(s) = 0, \nu = 1, \cdots, k, \mu = 1, \cdots, l$.* To prove the first part, we remark that, by (4.6), $S^-(y(r)) \leqq n-k-1$ and $S^+(y(s)) \geqq l$. (4.1) gives therefore $k + l \leqq n-1$. Assume now $k + l = n-1$ and let $y(t)$ and $u(t)$ be two solutions satisfying the given set of $(n-1)$ conditions. We can then form a linear combination $v(t) = c_1 y(t) + c_2 u(t)$ such that $k+1$ components of $v(r)$ and the former $l = n-k-1$ components of $v(s)$ vanish. $v(t)$ violates the first part of the above statement unless it reduces to the trivial solution. Hence $u(t) = cy(t)$ (cf. [7, p. 507]). This statement can also be obtained directly from the strict total positivity of the matrix $Y(s, r)$.

5. **Vector solutions of totally positive systems.** According to Theorem 4, the inequality (4.1) is characteristic for STP systems. It follows from (4.1) that $S^-(y(t))$ and $S^+(y(t))$ are decreasing functions of t. These consequences of (4.1) characterize the larger class of TP systems.

THEOREM 6. (i) *Let the differential system*

$$y'(t) = A(t)y(t) \,, \tag{1.1}$$

be TP *in (a, b), $-\infty \leqq a < b \leqq \infty$, and let $y(t)$ be a nontrivial solution. Then*

$$S^-(y(s)) \leqq S^-(y(r)) \quad \text{for all } (r, s) \text{ satisfying } \quad a < r \leqq s < b \,, \tag{5.1}$$

and

$$S^+(y(s)) \leqq S^+(y(r)) \quad \text{for all } (r, s) \text{ satisfying } \quad a < r \leqq s < b \,. \tag{5.2}$$

(ii) *Conversely, if* (5.1) *is valid for every nontrivial solution $y(t)$ of the system* (1.1), *or if* (5.2) *is valid for every $y(t)$, then the system* (1.1) *is* TP *in (a, b).*

Proof. (i) We obtain the necessity of (5.1) and (5.2) by an approximation procedure. Let the constant matrix $C^* = (c^*_{ij})^n_1$ be defined as in the proof of Theorem 3 ($c^*_{ij} = 1$ if $|i-j| = 1$, otherwise

$c_{ij}^* = 0$). If the system (1.1) is TP in (a, b), then it follows from Theorems 2 and 3 that the system

$$y_\varepsilon'(t) = A_\varepsilon(t) y_\varepsilon(t) , \qquad A_\varepsilon(t) = A(t) + \varepsilon C^* , \tag{5.3}$$

is, for $\varepsilon > 0$, STP in (a, b). To prove (5.1) let the solution $y(t)$ of (1.1) and the point r be given. For any $\varepsilon > 0$, let $y_\varepsilon(t)$ be the solution of (5.3) satisfying

$$y_\varepsilon(r) = y(r) . \tag{5.4}$$

(4.1) and (5.4) imply that for any $\varepsilon > 0$, and for any s, $s \in (r, b)$,

$$S^+(y_\varepsilon(s)) \leqq S^-(y(r)) . \tag{5.5}$$

By a standard theorem on differential equations (cf. [3, p. 55, Corollary 4.1])

$$\lim_{\varepsilon \to 0} y_\varepsilon(s) = y(s) .$$

This and the relation

$$S^-(\lim_{\varepsilon \to 0} y_\varepsilon(s)) \leqq \varliminf_{\varepsilon \to 0} S^+(y_\varepsilon(s)) ,$$

[4, p. 217, Lemma 1.1] imply

$$S^-(y(s)) \leqq \varliminf_{\varepsilon \to 0} S^+(y_\varepsilon(s)) . \tag{5.6}$$

(5.5) and (5.6) imply (5.1).

To obtain (5.2) let the solution $y(t)$ of (1.1) and the point s be given. For any $\varepsilon > 0$, let $\tilde{y}_\varepsilon(t)$ be the solution of (5.3) satisfying

$$\tilde{y}_\varepsilon(s) = y(s) . \tag{5.4$'$}$$

(4.1) and (5.4′) imply that for any $\varepsilon > 0$, and for any r, $r \in (a, s)$

$$S^+(y(s)) \leqq S^-(\tilde{y}_\varepsilon(r)) . \tag{5.5$'$}$$

For $\varepsilon \to 0$,

$$\varlimsup_{\varepsilon \to 0} S^-(\tilde{y}_\varepsilon(r)) \leqq S^+(y(r)) . \tag{5.6$'$}$$

(5.5′) and (5.6′) imply (5.2). This completes the proof of part (i). (We remark that (5.1) follows also directly from a theorem of Schoenberg [8, Satz 1] (cf. [2, p. 290] and [4, p. 21]) applied to the vector equation (4.2). Moreover (5.1) and (5.2) are equivalent as we shall show in Lemma 4.)

(ii) To prove the first half of this converse assertion, we assume the validity of (5.1) for all nontrivial solutions $y(t)$ of (1.1). We now proceed as in the proof of part (ii) of Theorem 4. The index k and the point r are fixed and we consider only nontrivial solutions of (1.1) which satisfy

$$y_k(r) = 0 . \tag{4.3}$$

Defining c as before and now using $S^-(y(s)) \leqq S^-(c), r < s$, we find that the $n \times (n-1)$ matrix $Y_k(s, r)$ (which is of rank $n-1$) is, for $r < s$, sign-regular of order $n-1$ [4, p. 222, Th. 1.4]; i.e., for each $p, 1 \leqq p \leqq n-1$, all nonvanishing minors of order p of $Y_k(s, r)$ have the same sign. But, for each p, $Y_k(r, r)$ has a positive minor of this order and not all minors of order p of $Y_k(s, r)$ can vanish. It follows, by continuity, that all minors of $Y_k(s, r)$ are nonnegative for $r < s$ and we thus proved the first half of (ii). (This follows again directly from the converse theorem of Schoenberg [8, Satz 2]). The second half of (ii) follows from the first half and the following lemma.

LEMMA 4. *Let the n^2 real functions $a_{ij}(t), i, j = 1, \cdots, n$, be continuous in (a, b), $-\infty \leqq a < b \leqq \infty$ and set $A(t) = (a_{ij}(t))_1^n$ and let (1.1) be the corresponding differential system. If, for each nontrivial solution $y(t)$, $S^-(y(t))$ is a decreasing function of t in (a, b), then the same holds for $S^+(y(t))$. Conversely, if $S^+(y(t))$ is, for each nontrivial solution $y(t)$, a decreasing function of t, then the same holds for $S^-(y(t))$.*

Proof. We shall use Theorem 2 and the (already proved) parts of Theorem 6 relating to (5.1), i.e., the first half of part (i) and the first half of part (ii). Let $y(t)$ be a nontrivial solution of (1.1) and define $y^*(t) = (y_1^*(t), \cdots, y_n^*(t))$ by

$$y_i^*(t) = (-1)^i y_i(t), i = 1, \cdots, n , \qquad a < t < b . \tag{5.7}$$

This and (1.1) imply that

$$\frac{dy^*}{dt} = B(t)y^*(t) , \qquad a < t < b , \tag{5.8}$$

where $B(t) = (b_{ij}(t))_1^n$ is given by

$$b_{ij}(t) = (-1)^{i+j} a_{ij}(t), i, j = 1, \cdots, n , \qquad a < t < b . \tag{5.9}$$

We now define

$$u(\tau) = y^*(-\tau) , \qquad -b < \tau < -a , \tag{5.10}$$

and

$$C(\tau) = -B(-\tau) , \qquad -b < \tau < -a . \tag{5.11}$$

(5.8), (5.10) and (5.11) give

$$\frac{du}{d\tau} = C(\tau)u(\tau)\ , \qquad -b < \tau < -a\ . \tag{5.12}$$

(5.7) and (5.10) imply, that for each t, $a < t < b$,

$$S^+(y(t)) = (n-1) - S^-(y^*(t)) = (n-1) - S^-(u(-t))\ . \tag{5.13}$$

We now assume that $S^-(y(t))$ is, for each $y(t)$, a decreasing function of t. By the first half of Theorem 6, (ii), and by Theorem 2, it follows that $A(t)$ is a Jacobi matrix with nonnegative off-diagonal elements. (5.9) and (5.11) show that the same holds for $C(\tau)$, hence using once more Theorem 2 and the first half of Theorem 6, (i), it follows that $S^-(u(\tau))$ is a decreasing function of τ; $S^-(y^*(t))$ is thus an increasing function of t, and (5.13) implies that $S^+(y(t))$ is a decreasing function of t. Conversely, assume that $S^+(y(t))$ is, for each $y(t)$, a decreasing function of t. $S^-(u(\tau))$ is then also a decreasing function of τ, $C(\tau)$ is a Jacobi matrix with nonnegative off-diagonal elements, and the same holds for $A(t)$. $S^-(y(t))$ decreases therefore for each $y(t)$. This proves Lemma 4 and we have thus completed the proof of Theorem 6.

(We shall use formulas (5.7) to (5.13) in the proof of the following lemma. We remark here that Lemma 4 is only a special case of the following statement: If the real $n \times n$ matrix M is nonsingular, and if for every pair of nonnull vectors (x, z), $z = Mx$, $S^-(z) \leqq S^-(x)$, then $S^+(z) \leqq S^+(x)$ holds also for all these pairs. This follows easily from the above mentioned theorems of Schoenberg, by obvious analogues of (5.7) and (5.9) and a well-known formula for the minors of the inverse matrix [4, p. 5].)

For the proof of our final theorem we need the following lemma.

LEMMA 5. *Let the differential system*

$$y'(t) = A(t)y(t) \tag{1.1}$$

be TP *in* (a, b), $-\infty \leqq a < b \leqq \infty$, *let* $y(t) = (y_1(t), \cdots, y_n(t))$ *be a nontrivial solution, and let the points* r *and* s *satisfy* $a < r < s < b$.

(i) *If*

$$y_1(r) = 0,\ y_1(s) \neq 0\ , \tag{5.14}$$

or if

$$y_n(r) = 0,\ y_n(s) \neq 0\ , \tag{5.15}$$

then

$$(5.16) \qquad S^+(y(r)) - S^+(y(s)) \geqq 1 .$$

Moreover, if both (5.14) *and* (5.15) *hold, then*

$$(5.17) \qquad S^+(y(r)) - S^+(y(s)) \geqq 2 .$$

(ii) *If*

$$(5.14') \qquad y_1(r) \neq 0, y_1(s) = 0 ,$$

or if

$$(5.15') \qquad y_n(r) \neq 0, y_n(s) = 0 ,$$

then

$$(5.16') \qquad S^-(y(r)) - S^-(y(s)) \geqq 1 .$$

Moreover, if both (5.14′) *and* (5.15′) *hold, then*

$$(5.17') \qquad S^-(y(r)) - S^-(y(s)) \geqq 2 .$$

Proof. (i) We assume that (5.14) holds for a given pair (r, s), $a < r < s < b$. By the continuity of $y(t)$, and by considering, if necessary, $-y(t)$ instead of $y(t)$, it follows that there exist points (r_1, s_1), $r \leqq r_1 < s_1 \leqq s$, such that

$$(5.18) \qquad y(r_1) = 0 , \qquad \text{and } y_1(t) > 0 \text{ for all } t \text{ in } (r_1, s_1] ,$$

and such that no component $y_i(t)$ for which $y_i(r_1) \neq 0$ vanishes in $[r_1, s_1]$. We now consider the possible values of $y_2(r_1)$. (a) If $y_2(r_1) > 0$, then our choice of $[r_1, s_1]$ implies that also $y_2(s_1) > 0$. The pair $(y_1(t), y_2(t))$ contributes in this case to $S^+(y(r_1))$ and gives no contribution to $S^+(y(s_1))$, and the remaining pairs $(y_i(t), y_{i+1}(t))$, $i = 2, \cdots, n-1$, cannot contribute more to $S^+(y(s_1))$ than to $S^+(y(r_1))$. Hence, in this case,

$$(5.19) \qquad S^+(y(r_1)) - S^+(y(s_1)) \geqq 1 .$$

(b) The assumption $y_2(r_1) < 0$ implies $y_2(s_1) < 0$. These inequalities and (5.18), and once more, the fact that components which are $\neq 0$ at r_1 remain so in $[r_1, s_1]$, give $S^-(y_1(r_1)) < S^-(y_1(s_1))$. This contradicts (5.1) and this case is thus excluded. There remains the case (c) $y_2(r_1) = 0$. $y_2(t)$ cannot vanish identically in $[r_1, s_1]$ as then the first component of the equation (1.1), i.e., $y_1' = a_{11}y_1 + a_{12}y_2$ contradicts (5.18). Furthermore, $y_2(t)$ cannot become negative in $(r_1, s_1]$, as $y_2(r_1) = 0$, $y_2(t) < 0$ and (5.18) would again give $S^-(y(r_1)) < S^-(y(t))$, $r < t$. Hence there exists a point s_2, $s_2 \in (r_1, s_1]$ such that $y_2(s_2) > 0$, and we obtain

$$(5.19') \qquad S^+(r_1) - S^+(s_2) \geqq 1 .$$

As $r \leqq r_1 < s_2 \leqq s_1 \leqq s$, (5.19), (5.19′) and (5.2) imply (5.16). The assumption (5.15) gives the same conclution. As t increases from r to s, the decrease of $S^+(y(t))$ is, under the assumption (5.14) due to the pair $(y_1(t), y_2(t))$. Under the assumption (5.15), it is due to the pair $(y_{n-1}(t), y_n(t))$, and it therefore follows that the simultaneous validity of (5.14) and (5.15) implies (5.17).

(ii) This part now follows from part (i) by the previously used transformation (formulas (5.7) to (5.13)). Together with the system (1.1) also the system (5.12) is TP. (5.14′) becomes $u_1(-s) = 0$, $u_1(-r) \neq 0$ and part (i) gives $S^+(u(-s)) - S^+(u(-r)) \geqq 1$. This and

$$S^+(u(-t)) = S^+(y^*(t)) = n - 1 - S^-(y(t))$$

gives (5.16′) and we have thus completed the proof of the lemma.

In Theorem 5 we obtained results on the behavior of solutions $y(t)$ of a STP system (1.1). *If the system* (1.1) *is* TP, *but not* STP, *then none of the assertions of Theorem* 5 *remains valid.* To show this, let $A(t) = (a_{ij}(t))_1^n$ be a Jacobi matrix with nonnegative off-diagonal elements in (a, b) and assume that for a given index q, $1 \leqq q \leqq n - 1$, and a given interval (α, β), $a \leqq \alpha < \beta \leqq b$, the element $a_{q+1,q}(t)$ vanishes identically in (α, β). We now consider (1.1) only in this subinterval (α, β). Here (1.1) may be satisfied by solution vectors $y(t)$ for which $y_{q+1}(t) \equiv \cdots \equiv y_n(t) \equiv 0$. If we consider only such solutions $y(t)$, then the vector consisting of their first q components $\tilde{y}(t) = (y_1(t), \cdots, y_q(t))$ satisfies an equation of the form

$$\tilde{y}'(t) = \tilde{A}(t)\tilde{y}(t) , \qquad \alpha < t < \beta , \tag{5.20}$$

where

$$\tilde{A}(t) = (a_{ij}(t))_1^q , \qquad \alpha < t < \beta . \tag{5.21}$$

This q th order system (5.21) is again TP (possibly even STP) in (α, β), and we obtain a q-dimensional subspace of the solutions of (1.1) by adding the $n - q$ zero components $y_{q+1}(t) \equiv \cdots \equiv y_n(t) \equiv 0$ to an arbitrary solution of (5.20). These solutions of (1.1) do not satisfy the assertions of Theorem 5. Indeed, let $r \in (\alpha, \beta)$ and choose $y_1(r) = \cdots = y_q(r) = 1$. Then $S^-(y(r)) = 0$, but the $n - q$ last components of $y(t)$ vanish identically in (r, β). If we choose $y_i(r) = (-1)^i$, $i = 1, \cdots, q$, then $S^+(y(r)) = n - 1$, but the last components vanish identically (α, r). This shows that part (i) of Theorem 5 is not valid for the present system. Parts (ii) and (iii) are not valid as $y_n(t) \equiv 0$ in (α, β). If we assume that an element of the first superdiagonal $a_{q,q+1}(t)$ vanishes identically in (α, β), then we have to consider solutions of (1.1) for which $y_1(t) \equiv y_2(t) \equiv \cdots \equiv y_q(t) \equiv 0$ in (α, β) and the remaining components satisfy a system of order $n - q$. Theorem 5 does therefore not hold for TP systems; the following weakened version is however valid for such systems.

THEOREM 7. *Let the differential system*

$$y'(t) = A(t)y(t) \tag{1.1}$$

be TP *in* (a, b), $-\infty \leqq a < b \leqq \infty$, *and let* $y(t) = (y_1(t), \cdots, y_n(t))$ *be a nontrivial solution.*

(i) *If* $S^+(y(r)) = 0$, $r \in (a, b)$, *then no component of* $y(t)$ *vanishes in* $[r, b)$. *If* $S^-(y(s)) = n - 1$, $s \in (a, b)$, *then no component of* $y(t)$ *vanishes in* $(a, s]$.

(ii) *Let*

$$(a<)\gamma_0 < \alpha_1 < \gamma_1 < \alpha_2 < \cdots < \gamma_{k-1} < \alpha_k < \gamma_k(<b)$$

be $2k + 1$ *points, such that for each* i, $i = 1, \cdots, k$, *at least one of the following two conditions holds.*

$$y_1(\gamma_{i-1}) \neq 0,\ y_1(\alpha_i) = 0,\ y_1(\gamma_i) \neq 0\ , \tag{5.22}$$

or

$$y_n(\gamma_{i-1}) \neq 0,\ y_n(\alpha_i) = 0,\ y_n(\gamma_i) \neq 0\ . \tag{5.23}$$

Set $m_i = 1$ *if only one of these two conditions holds for the index* i, *and* $m_i = 2$ *if both conditions hold,* $i = 1, \cdots, k$, *and let* $\tilde{k} = \sum_{i=1}^{k} m_i$. *Then* $\tilde{k} \leqq n - 1$. *Moreover, if* $\tilde{k} = n - 1$, *then no component of* $y(t)$ *vanishes in* $(a, \gamma_0] \cup [\gamma_k, b)$.

(iii) *Let all the assumptions of* (ii) *hold and, in addition, assume that* m *components of* $y(r)$, $r \in (a, b)$ *vanish and that either* $r \leqq \gamma_0$ *or* $\gamma_k \leqq r$. *Then* $\tilde{k} \leqq n - m - 1$.

Proof. (i) (5.1), (5.2) and (4.6) yield these two assertions.

(ii) We have

$$\begin{aligned}\tilde{k} &\leqq \sum_{i=1}^{k} [S^+(y(\alpha_i)) - S^+(y(\gamma_i))] = \sum_{i=2}^{k} [S^+(y(\alpha_i)) - S^+(y(\gamma_{i-1}))] \\ &\quad + S^+(y(\alpha_1)) - S^+(y(\gamma_k)) \leqq S^+(y(\alpha_1)) - S^+(y(\gamma_k)) \leqq n - 1\ .\end{aligned} \tag{5.24}$$

The first inequality sign follows from (5.22) and (5.23) by Lemma 5. The second inequality sign follows by (5.2). This proves $\tilde{k} \leqq n - 1$. If $\tilde{k} = n - 1$, then (5.24) implies $S^+(y(\gamma_k)) = 0$, hence part (i) implies that no component of $y(t)$ vanishes in $[\gamma_k, b)$. To show that, if $\tilde{k} = n - 1$, no component vanishes in $(a, \gamma_0]$ either, we use

$$\begin{aligned}\tilde{k} &\leqq \sum_{i=1}^{k} [S^-(y(\gamma_{i-1})) - S^-(y(\alpha_i))] = \sum_{i=2}^{k} [S^-(y(\gamma_{i-1})) - S^-(y(\alpha_{i-1}))] \\ &\quad + S^-(y(\gamma_0)) - S^-(y(\alpha_k)) \leqq S^-(y(\gamma_0)) - S^-(y(\alpha_k)) \leqq n - 1\ .\end{aligned} \tag{5.25}$$

(5.25) and $\tilde{k} = n - 1$ imply $S^-(y(\gamma_0)) = n - 1$, which gives the desired nonvanishing in $(\alpha, \gamma_0]$.

(iii) If $r \leqq \gamma_0$, then (5.25), (5.1) and (4.6) imply

$$\tilde{k} \leqq S^-(y(\gamma_0)) - S^-(y(\alpha_k)) \leqq S^-(y(r)) - S^-(y(\alpha_k) \leqq n - m - 1 .$$

If $\gamma_k \leqq r$, then (5.24), (5.2) and (4.6) give

$$\tilde{k} \leqq S^+(y(\alpha_1)) - S^+(y(\gamma_k)) \leqq S^+(y(\alpha_1)) - S^+(y(r)) \leqq n - m - 1 .$$

This completes the proof of Theorem 7.

6. Examples. We conclude this paper with a few examples. All our examples are STP systems (1.1) and for each example we consider only one particular vector solution $y(t)$. We thus replace (1.1) in each case by a vector equality where the matrix $A(t)$, the particular solution $y(t)$ and its derivative $y'(t)$ are shown explicitly. As the case $n = 2$ is trivial, we start with an example for $n = 3$.

$$\begin{pmatrix} -\sin t \\ -\cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3}{2} - \cos t & 0 & \frac{3}{2} + \cos t \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 + \cos t \\ -\sin t \\ -2 + \cos t \end{pmatrix} . \tag{6.1}$$

This shows that, for $n = 3$, there exists a system (1.1) which is STP in $(-\infty, \infty)$ and for which the interior component $y_2(t)$ of a particular solution $y(t)$ vanishes infinitely many times. However, in this example the extreme components $y_1(t)$ and $y_3(t)$ do not vanish at all.

The next examples show that the assertion of Theorem 5, is, for $n = 3$, essentially all that can be said about the number of zeros of the components of any particular solution $y(t)$ of a STP system. Let α and β, $\alpha < \beta$, be zeros of the extreme components $y_1(t)$ and $y_3(t)$. Theorem 5 (ii) implies that these extreme components have no other zeros and that $y_2(t)$ does not vanish outside the interval (α, β); however, no restrictions on the number of zeros of $y_2(t)$ in (α, β) are given by Theorem 5. We combine system (6.1) with two other systems to show that *we may obtain an* (*except for its parity*) *arbitrary number of zeros of* $y_2(t)$ *in the interval bounded by the zeros of the extreme components.* The matrix in (6.1) and the vector given there, will be referred to as $A(t)$ and $y(t)$. We now consider the equality

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3}{4}(2 - \tau) & 0 & \frac{3}{4}(2 + \tau) \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 + \tau \\ 2 \\ -2 + \tau \end{pmatrix} . \tag{6.2}$$

Note that the corresponding system

$$(6.2') \qquad u'(\tau) = B(\tau)u(\tau) ,$$

is STP in $[-2, 2]$. Furthermore we note that if now $u(\tau)$ denotes the particular solution shown in (6.2), then, for each integer k, the equalities

$$(6.3) \qquad B(0) = A\left(-\frac{\pi}{2} + 2k\pi\right),\ u(0) = y\left(-\frac{\pi}{2} + 2k\pi\right) ,$$

(and hence also $u'(0) = y'(-\pi/2 + 2k\pi)$ hold. (6.3) allows us to combine the examples (6.1) and (6.2) at their respective points $t = -\pi/2 + 2k\pi$ and $\tau = 0$.

We also consider the equality

$$(6.4) \qquad \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3}{4}(2+\tau) & 0 & \frac{3}{4}(2-\tau) \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2-\tau \\ -1 \\ -2-\tau \end{pmatrix} .$$

The corresponding system

$$(6.4') \qquad v'(\tau) = C(\tau)v(\tau)$$

is again STP in $[-2, 2]$ and, for each integer k,

$$(6.5) \qquad C(0) = A\left(\frac{\pi}{2} + 2k\pi\right),\ v(0) = y\left(\frac{\pi}{2} + 2k\pi\right) ,$$

and we thus may combine (6.1) and (6.4).

For any nonnegative integer k, we now define the system

$$(6.6)_1 \qquad [y^{(1)}(t)]' = A^{(1)}(t)y^{(1)}(t) ,$$

in $[\alpha_1, \beta_1]$, $\alpha_1 = -2 - \pi/2$, $\beta_1 = \pi/2 + 2k\pi + 2$, by setting

$$A^{(1)}(t) = \begin{cases} B\left(t + \frac{\pi}{2}\right) , & -2 - \frac{\pi}{2} \leqq t \leqq -\frac{\pi}{2} , \\ A(t) , & -\frac{\pi}{2} \leqq t \leqq \frac{\pi}{2} + 2k\pi , \\ C\left(t - \frac{\pi}{2} - 2k\pi\right) , & \frac{\pi}{2} + 2k\pi \leqq t \leqq \frac{\pi}{2} + 2k\pi + 2 . \end{cases}$$

This systems is STP in $[\alpha_1, \beta_1]$ and has the particular solution

$$y^{(1)}(t) = \begin{cases} u\left(t + \frac{\pi}{2}\right) , & -2 - \frac{\pi}{2} \leqq t \leqq \frac{\pi}{2} , \\ y(t) , & -\frac{\pi}{2} \leqq t \leqq \frac{\pi}{2} + 2k\pi , \\ v\left(t - \frac{\pi}{2} - 2k\pi\right) , & \frac{\pi}{2} + 2k\pi \leqq t \leqq \frac{\pi}{2} + 2k\pi + 2 . \end{cases}$$

$y_1^{(1)}(\alpha_1) = y_1^{(1)}(\beta_1) = 0$, $y_3^{(1)}(t) < 0$ in $[\alpha_1, \beta_1]$, and $y_2^{(1)}(t)$ vanishes at the $2k+1$ points $t = \ell\pi$, $\ell = 0, 1, \cdots, 2k$.

If we define the system

$$(6.6)_2 \qquad [y^{(2)}(t)]' = A^{(2)}(t)y^{(2)}(t) ,$$

in $[\alpha_2, \beta_2]$, $\alpha_2 = -2 + \pi/2$, $\beta_2 = 3\pi/2 + 2k\pi + 2$, $(k = 0, 1, \cdots)$, by setting

$$A^{(2)}(t) = \begin{cases} C\left(t - \dfrac{\pi}{2}\right) , & -2 + \dfrac{\pi}{2} \leqq t \leqq \dfrac{\pi}{2} , \\ A(t) , & \dfrac{\pi}{2} \leqq t \leqq \dfrac{3\pi}{2} + 2k\pi , \\ B\left(t - \dfrac{3\pi}{2} - 2k\pi\right) , & \dfrac{3\pi}{2} + 2k\pi \leqq t \leqq \dfrac{3\pi}{2} + 2k\pi + 2 , \end{cases}$$

then this system has a solution $y^{(2)}(t)$, for which $y_3^{(2)}(\alpha_2) = y_3^{(2)}(\beta_2) = 0$, $y_1^{(2)}(t) > 0$ in $[\alpha_2, \beta_2]$, and $y_2^{(2)}(t)$ has again an odd number of zeros in $[\alpha_2, \beta_2]$.

Defining $A^{(3)}(t)$ in $[\alpha_3, \beta_3]$, $\alpha_3 = -2 - \pi/2$, $\beta_3 = 3\pi/2 + 2k\pi + 2$, by

$$A^{(3)}(t) = \begin{cases} B\left(t + \dfrac{\pi}{2}\right) , & -2 - \dfrac{\pi}{2} \leqq t \leqq \dfrac{\pi}{2} , \\ A(t) , & -\dfrac{\pi}{2} \leqq t \leqq \dfrac{3\pi}{2} + 2k\pi , \\ B\left(t - \dfrac{3\pi}{2} - 2k\pi\right) , & \dfrac{3\pi}{2} + 2k\pi \leqq t \leqq \dfrac{3\pi}{2} + 2k\pi + 2 , \end{cases}$$

we obtain a system which is STP in $[\alpha_3, \beta_3]$ and has a solution $y^{(3)}(t)$, for which $y_1^{(3)}(\alpha_3) = y_3^{(3)}(\beta_3) = 0$, and $y_2^{(3)}(t)$ has now an even number of zeros in $[\alpha_3, \beta_3]$. By using first C, then A and then again C, we obtain similarly a fourth example for which $y_3^{(4)}(\alpha_4) = y_1^{(4)}(\beta_4) = 0$ and $y_2^{(4)}(t)$ has again an even number of zeros in $[\alpha_4, \beta_4]$. These four examples establish the italicized statement preceding (6.2); the parity restriction on the number of the zeros of $y_2(t)$ follows easily by the proof of part (ii) of Theorem 5. We remark that there is no need to consider different systems for all four examples and all nonnegative integers k. All these cases can be illustrated by considering distinct solutions $y^{(j,k)}$, $j = 1, \cdots, 4$, $k = 0, 1, \cdots$, of a single system (1.1) which is STP in $(-\infty, \infty)$. The corresponding matrix $A(t)$ is given by $a_{11}(t) = a_{13}(t) = a_{31}(t) = a_{33}(t) = 0$ and $a_{12}(t) = a_{32}(t) = 1$ for all t, while the elements $a_{21}(t)$ and $a_{23}(t)$ are determined in the disjoint intervals $[\alpha_{jk}, \beta_{jk}]$ by the above formulas and are (otherwise arbitrary) continuous nonnegative functions of t, which only vanish at some of the end points α_{jk} and β_{jk}.

For $n = 4$ our example is

$$
\begin{pmatrix} \sin t \\ \cos t \\ \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \dfrac{-1}{3(2-\cos t)} & 1 & 0 & 0 \\ 1+\cos t & -\dfrac{5}{3} & \dfrac{4}{3}-\sin t & 0 \\ 0 & \dfrac{4}{3}+\sin t & -\dfrac{5}{3} & 1-\cos t \\ 0 & 0 & 1 & \dfrac{-1}{3(2+\cos t)} \end{pmatrix} \times \begin{pmatrix} 2-\cos t \\ \dfrac{1}{3}+\sin t \\ -\dfrac{1}{3}+\sin t \\ -2-\cos t \end{pmatrix}. \tag{6.7}
$$

The corresponding system is STP in $(-\infty, \infty)$ and the interior components $y_2(t)$ and $y_3(t)$ of the particular solution $y(t)$ have infinitely many zeros. (6.7) is the special case $n = 4$ of a general example, valid for any $n, n \geqq 4$. The nonzero elements of the Jacobi matrix $A(t) = (a_{ij}(t))_1^n$ are in this general case given by

$$
\begin{aligned}
a_{11} &= \frac{3-n}{(n-1)(2-\cos t)},\ a_{12} = 1,\ a_{21} = 1+\cos t\ , \\
a_{22} &= 5-n-\frac{8}{n-1},\ a_{23} = n-4+\frac{4}{n-1}-\sin t\ , \\
a_{ii-1} &= \frac{n-1}{2}\left(1+\frac{1}{2}\cos t\right),\ a_{ii} = 1-n\ , \\
a_{ii+1} &= \frac{n-1}{2}\left(1-\frac{1}{2}\cos t\right),\ i = 3, \cdots, n-2\ , \\
a_{n-1,n-2} &= n-4+\frac{4}{n-1}+\sin t,\ a_{n-1,n-1} = 5-n-\frac{8}{n-1}\ , \\
a_{n-1,n} &= 1-\cos t,\ a_{nn-1} = 1,\ a_{nn} = \frac{3-n}{(n-1)(2+\cos t)}\ .
\end{aligned} \tag{6.8}
$$

The particular solution is the vector $y(t)$, whose components are

$$
\begin{aligned}
y_1(t) &= 2-\cos t\ , \\
y_i(t) &= \frac{n+1-2i}{n-1}+\sin t\ , \qquad i = 2, \cdots, n-1\ , \\
y_n(t) &= -2-\cos t\ ,
\end{aligned}
$$

and all interior components vanish infinitely many times.

I am grateful to Professor Z. Nehari and Dr. M. Lavie for their valuable advice offered during many discussions.

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Received June 5, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

THE BENDING OF SPACE CURVES INTO PIECEWISE HELICAL CURVES

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It is the purpose of this paper to show that a regular C^3 space curve Γ in a Euclidean 3-space, whose curvature $\kappa \neq 0$, can be bent into a piecewise helix (i.e., a curve that is a helix but for a finite number of corners) in such a way that the piecewise helix remains within a tubular region about C of arbitrarily small preassigned radius. Moreover, we shall show that the bending can be carried out in such a way that either (a) the piecewise helix is circular or (b) the piecewise helix has the same curvature as Γ at corresponding points except possibly at corners, of (c) if the torsion of Γ is nowhere zero, then the piecewise helix has the same torsion as Γ at corresponding points except possibly at corners.

Also we shall show that if, in addition, Γ has a bounded fourth derivative, then an explicit formula can be given for a sufficient number n of helices that make up the piecewise helix, where n depends on Γ and the radius of the tubular region about Γ. In this case, we shall also show how the determination of the piecewise helix can be reduced to a problem in simple integration.

1. Bendability.

DEFINITION 1. A curve is called a *piecewise helix* if it consists of a finite number of segments, each of which is a helix (i.e., a curve whose tangent makes a constant angle with a fixed direction). A point at which two consecutive helices meet will be called a *corner* of the piecewise helix.

REMARK. If, in particular, between corners the helix is a circular helix, then the piecewise helix will be called a *piecewise circular helix*.

THEOREM 1. *Let Γ: $r(s)$, $s =$ arc length, $0 \leqq s \leqq l$, be a regular $C^3[0, l]$[1] curve whose curvature $\kappa(s)$ is nowhere zero. Then for any given $\varepsilon > 0$*

(a) *there exists a piecewise circular helix Γ_1^*: $h_1^*(s)$, $s =$ arc length, $0 \leqq s \leqq l$, such that:*

$$|r(s) - h_1^*(s)| < \varepsilon\,, \qquad 0 \leqq s \leqq l\,;$$

[1] (I.e., $r(s)$ can be extended to lie in C^3 on some open set containing $0 \leqq s \leqq l$.)

(b) *there exists a piecewise helix* Γ_2^*: $h_2^*(s)$, $s =$ *arc length*, $0 \leqq s \leqq l$, *such that*:

$$|r(s) - h_2^*(s)| < \varepsilon\,, \qquad 0 \leqq s \leqq l$$

and Γ_2^* *has the same curvature as* Γ *at corresponding points, except possibly at the corners of* $h_2^*(s)$;

(c) *provided the torsion* $\tau(s)$ *is nowhere zero, there exists a piecewise helix* Γ_3^*: $h_3^*(s)$, $s =$ *arc length*, $0 \leqq s \leqq l$, *such that*:

$$|r(s) - h_3^*(s)| < \varepsilon\,, \qquad 0 \leqq s \leqq l$$

and Γ_3^* *has the same torsion as* Γ *at corresponding points, except possibly at corners of* $h_3^*(s)$.

REMARK. In each case the curve Γ is "bent" into a piecewise helix.

Proof. We shall prove (b) and indicate what minor modifications are necessary to prove (a) and (c). Let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion respectively of Γ. Then $\kappa(s) \in C^1[0, l]$ and $\tau(s) \in C^0[0, l]$ since $r(s) \in C^3[0, l]$. By hypothesis, $\kappa(s) \neq 0$; therefore,

$$f(s) = \frac{\tau(s)}{\kappa(s)}$$

is continuous and thus uniformly continuous on $[0, l]$. Let

$$|\kappa(s)| \leqq \kappa_{\max} \qquad \text{on } 0 \leqq s \leqq l \tag{1.1}$$

and

$$|f(s)| \leqq f_{\max} \qquad \text{on } 0 \leqq s \leqq l \tag{1.2}$$

and choose $\delta(\varepsilon) > 0$ such that

$$|f(s_2) - f(s_1)| < \alpha\varepsilon \tag{1.3}$$

provided $|s_2 - s_1| \leqq \delta$, where

$$\alpha = \langle \kappa_{\max} l^2 \sqrt{6} \exp\{l\kappa_{\max}\sqrt{2(1 + f_{\max}^2)}\}\rangle^{-1}\,. \tag{1.4}$$

Let

$$n = n(\varepsilon) = \text{smallest integer} \geqq \frac{l}{\delta}$$

and

$$\begin{aligned} I_0 &= \{s: 0 \leqq s \leqq \delta\} \\ I_j &= \{s: j\delta < s \leqq (j + 1)\delta\}\,, \qquad j = 1, 2, \cdots, n - 2 \\ I_{n-1} &= \{s: (n - 1)\delta < s \leqq l\}\,. \end{aligned} \tag{1.4.1}$$

Then $I_j (0 \leqq j \leqq n-1)$ form a disjoint covering of $[0, l]$, each of length $\leqq \delta$.

Let

$$\tau_j(s) = f_j \kappa(s),\ s \in I_j\ , \qquad j = 0, 1, \cdots, n-1 \tag{1.5}$$

where

$$f_j = \begin{cases} f[(j+1)\delta] & \text{for } j = 0, 1, \cdots, n-2\ . \\ f[l] & \text{for } j = n-1 \end{cases}$$

By the fundamental theorem for space curves there exists a unique curve $h_j(s)$, $s \in I_j$, for which:

(i) its curvature and torsion are respectively $\kappa(s)$ and $\tau_j(s)$ as defined by (1.5), and

(ii) its position $h_j(s)$, tangent $t_j(s)$, principal normal $n_j(s)$ and binomial $b_j(s)$ satisfy the initial conditions:

$$h_j(j\delta) = r(j\delta),\ t_j(j\delta) = e_1(j\delta),\ n_j(j\delta) = e_2(j\delta),\ b_j(j\delta) = e_3(j\delta) \tag{1.6}$$

where $e_1(s)$, $e_2(s)$ and $e_3(s)$ are the tangent, principal normal and binormal of $r(s)$ respectively, and s is the arc length parameter of h_j.

Moreover, if

$$\Phi_j(s) = \begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix}, \qquad A_j = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & f_j \\ 0 & -f_j & 0 \end{bmatrix}, \tag{1.7}$$

then $\Phi_j(s)$ satisfies the differential equation:

$$\Phi_j'(s) = \kappa(s) A_j \Phi_j(s)\ . \tag{1.8}$$

Also, because $\tau_j(s)/\kappa(s) = f_j =$ constant on I_j, $h_j(s)$ is a helix on I_j.

By the Frenet formulae for Γ, we have:

$$\Psi'(s) = \kappa(s) A(s) \Psi(s)\ , \qquad 0 \leqq s \leqq l\ , \tag{1.9}$$

where

$$\Psi(s) = \begin{bmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{bmatrix},\ A(s) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & f(s) \\ 0 & -f(s) & 0 \end{bmatrix}. \tag{1.10}$$

Considering both (1.8) and (1.9) as differential equations on I_j, we obtain:

$$\Phi_j(s) = \Phi_j(j\delta) + \int_{j\delta}^{s} \kappa(t) A_j \Phi_j(t) dt,\ s \in I_j\ , \qquad j = 0, 1, \cdots, n-1 \tag{1.11}$$

and

$$\Psi(s) = \Psi(j\delta) + \int_{j\delta}^{s} \kappa(t)A(t)\Psi(t)dt,\ s \in I_j\ , \qquad j = 0, 1, \cdots, n-1 \tag{1.12}$$

Since by (1.6) $\Phi_j(j\delta) = \Psi(j\delta)$, we see that if

$$||(c_{ij})|| = \sqrt{\sum_{i,j=1}^{3} c_{ij}^2}\ ,$$

then

$$\begin{aligned} ||\Psi(s) - \Phi_j(s)|| \leqq & \int_{j\delta}^{s} |\kappa(t)|\, ||A(t) - A_j||\, ||\Psi(t)||\, dt \\ & + \int_{j\delta}^{s} |\kappa(t)|\, ||A_j||\, ||\Psi(t) - \Phi_j(t)||\, dt\ . \end{aligned} \tag{1.13}$$

But by (1.7), (1.10), and (1.3)

$$||A(t) - A_j|| = \sqrt{2[f(t) - f_j]^2} < \sqrt{2}\alpha\varepsilon \qquad \text{for } t \in I_j\ .$$

Also we have

$$||\Psi(t)|| = \sqrt{3}$$

and by (1.7)

$$||A_j|| = \sqrt{2(1 + f_j^2)} \leqq \sqrt{2(1 + f_{\max}^2)}\ .$$

Thus

$$||\Psi(s) - \Phi_j(s)|| < M\delta\varepsilon\alpha + N\int_{j\delta}^{s} ||\Psi(t) - \Phi_j(t)||\, dt\ , \qquad s \in I_j\ , \tag{1.14}$$

where

$$\begin{aligned} M &= \kappa_{\max}\sqrt{6} \\ N &= \kappa_{\max}\sqrt{2(1 + f_{\max}^2)}\ . \end{aligned}$$

Let

$$C = \sup_{t \in I_j} ||\Psi(t) - \Phi_j(t)||\ ,$$

then by (1.14)

$$||\Psi(s) - \Phi_j(s)|| < M\delta\varepsilon\alpha + NC(s - j\delta) \tag{1.15}$$

from which we see upon combining (1.14) and (1.15) that

$$
\begin{aligned}
&\|\Psi(s) - \Phi_j(s)\| \\
&\quad < M\delta\varepsilon\alpha\left[1 + N(s - j\delta) + N^2\frac{(s - j\delta)^2}{2!} + \cdots + N^k\frac{(s - j\delta)^k}{k!}\right] \\
&\qquad\qquad + \frac{(s - j\delta)^{k+1}}{(k + 1)!} CN^{k+1} < M\delta\varepsilon\alpha e^{N\delta} \qquad (1.16) \\
&\qquad\qquad\qquad < Ml\varepsilon\alpha e^{Nl} \\
&\qquad\qquad\qquad < \varepsilon/l\,, \qquad\qquad s \in I_j\,,
\end{aligned}
$$

by the definition (1.4) of α.

If we let

$$
\Phi^*(s) \equiv \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} = \Phi_j(s) \equiv \begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix}, \ s \in I_j\,, \qquad j = 0, 1, \cdots, n = 1\,,
$$

then $\Phi^*(s)$ is piecewise continuous on $[0, l]$ with discontinuities possibly at

$$
s = j\delta\,, \qquad j = 0, 1, 2\cdots, n - 1\,,
$$

and by (1.16) since I_j is a cover of $[0, l]$,

$$
\|\Psi(s) - \Phi^*(s)\| < \varepsilon/l \qquad \text{for } 0 \leqq s \leqq l\,. \tag{1.17}
$$

Let

$$
h^*(s) = r(0) + \int_0^s t(\sigma)d\sigma\,, \qquad \text{for } 0 \leqq s \leqq l\,. \tag{1.18}
$$

Then $h^*(s)$ is a piecewise helix Γ_2^* for which

$$
h^{*\prime}(s) = h_j'(s)\,, \qquad \text{for } s \in I_j, j = 0, 1, \cdots, n - 1\,.
$$

Thus for $0 \leqq s \leqq l$:

$$
\begin{aligned}
|r(s) - h^*(s)| &\leqq \int_0^s |e_1(s) - t(s)|\, ds \\
&\leqq \int_0^l \|\Psi(s) - \Phi^*(s)\|\, ds \\
&< \varepsilon
\end{aligned}
$$

by (1.17).

Next we note that s is the arc length of $h^*(s)$ since

$$
|h^{*\prime}(s)| = |h_j'(s)| = |t_j(s)| = 1\,, \qquad \text{for } s \in I_j\,.
$$

Moreover on the interior of I_j:

$$
|h^{*\prime\prime}(s)| = |h_j''(s)| = \text{curvature} = \kappa(s)
$$

by construction of $h_j(s)$.

This completes the proof of part (b). For the proof of part (a) and part (c), only obvious slight modifications are necessary. In part (a), we need only the additional fact that a helix is circular if the curvature and torsion are both constant.

2. Explicit results.

If we allow $r(s)$ to have one more bounded derivative we have:

THEOREM 2. *If in addition to the assumptions of Theorem* 1, *we also assume that* $r(s)$ *has a bounded fourth derivative on* $[0, l]$, *then we can choose* $n(\varepsilon)$ *in part* 2 *to be*

$$n(\varepsilon) = \textit{smallest integer} > \frac{g^* l}{\alpha\varepsilon} \tag{2.1}$$

where

$$\alpha = \langle \kappa_{\max} l^2 \sqrt{6} \exp \{ l \kappa_{\max} \sqrt{2(1 + f_{\max}^2)} \} \rangle^{-1} \tag{2.2}$$

$$\begin{aligned} &\left| \frac{r' \cdot (r'' \times r'''')}{[r'' \cdot r'']^{3/2}} - \frac{3[r''' \cdot r''][r' \cdot (r'' \times r''')]}{[r'' \cdot r'']^{5/2}} \right| < g^*, \quad 0 \leqq s \leqq l \, , \\ &\left| \frac{r' \cdot (r'' \times r''')}{[r'' \cdot r'']^{3/2}} \right| < f_{\max}, \ [r'' \cdot r'']^{1/2} < \kappa_{\max} \, , \qquad 0 \leqq s \leqq l \, , \end{aligned} \tag{2.3}$$

REMARK. A similar result holds for parts (a) and (c).

Proof. Since

$$\kappa(s) = [r''(s) \cdot r''(s)]^{1/2}$$

and

$$\tau(s) = \frac{r' \cdot (r'' \times r''')}{r'' \cdot r''}$$

the expression in the first inequality in (2.3) is simply the derivative of

$$f(s) = \frac{\tau(s)}{\kappa(s)} \, .$$

Thus

$$| f(s_2) - f(s_1) | = \left| \int_{s_1}^{s_2} f'(s) ds \right| < g^*[s_2 - s_1] \, .$$

If we choose

$$\delta = \frac{\alpha\varepsilon}{g^*} \, , \tag{2.4}$$

where α is given by (2.2) = (1.4), then

$$|f(s_2) - f(s_1)| < \alpha\varepsilon$$

whenever $|s_2 - s_1| < \delta$. This, by the proof of part (b) of Theorem 1, gives the result since

$$n(\varepsilon) = \text{smallest integer} > \frac{g^* l}{\alpha\varepsilon} = \frac{l}{\delta}\,.$$

THEOREM 3. *Let Γ: $r(s)$, s = arc length, $0 \leqq s \leqq l$, be a regular space curve with bounded fourth derivative and nowhere-zero curvature. Denote the curvature, torsion, tangent, principal normal and binormal of Γ by $\kappa(s)$, $\tau(s)$, $e_1(s)$, $e_2(s)$ and $e_3(s)$. For any given $\varepsilon > 0$, let $n(\varepsilon)$, δ, and I_j ($j = 0, 1, \cdots, n$) be given by* (2.1), (2.4) *and* (1.4.1), *respectively. Put*

$$t_j(s) = \frac{1}{m^2}\{[f_j^2 + \cos(g_j(s)m)]e_1(j\delta) + [m \sin(g_j(s)m)]e_2(j\delta) + f_j[1 - \cos(g_j(s)m)]e_3(j\delta)\}$$

where

$$f_j = \tau[(j+1)\delta]\}/\{\kappa[(j+1)\delta]\},\ m = +\sqrt{1+f_j^2},\ g_j(s) = \int_{j\delta}^{s} \kappa(\sigma)d\sigma$$

and let

$$t(s) = t_j(s),\ s \in I_j,\ j = 0, 1, \cdots, n\,.$$

Then the curve

$$\Gamma^*\colon h^*(s) = r(0) + \int_0^s t(\sigma)d\sigma,\ s = \text{arc length},\ 0 \leqq s \leqq l\,,$$

is a piecewise helix such that

$$|r(s) - h^*(s)| < \varepsilon,\ 0 \leqq s \leqq l\,,$$

and Γ^* has the same curvature as Γ at corresponding points except possibly at the corners.

Proof. From (1.7)

$$\Phi_j(s) = \begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix}$$

satisfies the system of differential equations

$$\Phi_j'(s) = \kappa(s)A_j\Phi_j(s) \qquad \text{on } I_j , \tag{1.8}$$

where A_j is given by (1.7). The solution of (1.8) for which $\Phi_j(j\delta) = \Psi(j\delta)$ is given by

$$\Phi_j(s) = e^{g_j(s)A_j}\Psi(j\delta) .$$

The eigenvalues of A_j are 0, im and $-im$ and the corresponding eigenvectors are:

$$T_1 = \begin{bmatrix} f_j \\ 0 \\ 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 \\ im \\ -f_j \end{bmatrix}, T_3 = \begin{bmatrix} 1 \\ -im \\ -f_j \end{bmatrix} .$$

Also the matrix $T = (T_1, T_2, T_3)$ has the inverse

$$T^{-1} = \frac{1}{2m^2}\begin{bmatrix} 2f_j & 0 & 2 \\ 1 & -im & -f_j \\ 1 & im & -f_j \end{bmatrix} .$$

Thus

$$T^{-1}e^{g_j(s)A_j}T = e^{g_j(s)D_j} ,$$

where

$$D_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & im & 0 \\ 0 & 0 & -im \end{bmatrix}$$

and

$$\begin{aligned} e^{g_j(s)A_j} &= Te^{g_j(s)D_j}T^{-1} \\ &= \frac{1}{m^2}\begin{bmatrix} f_j^2 + \cos g_j(s)m, & m \sin g_j(s)m, & f_j(1 - \cos g_j(s)m) \\ * & * & * \\ * & * & * \end{bmatrix} . \end{aligned}$$

From this it follows that

$$\begin{bmatrix} t_j(s) \\ n_j(s) \\ b_j(s) \end{bmatrix} \equiv \Phi_j(s) = e^{g_j(s)A_j}\begin{bmatrix} e_1(j\delta) \\ e_2(j\delta) \\ e_3(j\delta) \end{bmatrix} = \frac{1}{m^2} \times$$

$$\times \begin{bmatrix} [f_j^2 + \cos g_j(s)m]e_1(j\delta) + [m \sin g_j(s)m]e_2(j\delta) + f_j[1 - \cos g_j(s)m]e_3(j\delta) \\ * \qquad\qquad * \qquad\qquad * \\ * \qquad\qquad * \qquad\qquad * \end{bmatrix}$$

which gives (2.5) and the theorem is proved.

REMARKS. By using the definition of torsion as given by Hartman and Wintner [1], p. 771, [3] p. 202, the continuity requirement of Theorem 1 can be relaxed from C^3 to C^2. A question of further interest would be to consider the bending of normal curves, see for example, Nomizu [2] and Wong and Lai [4].

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Received March 17, 1969, and revised form June 13, 1969. The author wishes to acknowledge support for this research by the National Aeronautics Space Administration, NASA Grant NGR 05-010-008.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

ANALYTIC INTERPOLATION OF CERTAIN MULTIPLIER SPACES

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Let W_p denote the space of all functions on the circle which are the uniform limit of a sequence of trigonometric polynomials which is bounded as a sequence of multipliers for $l_p, 1 \leqq p \leqq 2$. Let U_s be the interpolation space $[W_2, W_1]_s$ (see 1.1). Our main result, Theorem 2.4, states that for a compact subset E of the circle, $U_s \mid E = C(E)$ if and only if $W_1 \mid E = C(E)$. A major step in the proof is a maximum principle for interpolation, Theorem 1.7. We also give a direct proof that $U_s \neq W_p$ (see Theorem 2.7) for corresponding s and p.

1. Some properties of analytic interpolation.

1.1. Let B^0 and B^1 be two Banach spaces continuously embedded in a topological vector space V such that $B^0 \cap B^1$ is dense in both B^0 and B^1. For $0 < s < 1$, let $\mathfrak{F}$, $[B^0, B^1]_s$ and $B^0 + B^1$ denote the spaces as defined in [1, § 1]. For two Banach spaces X and Y we let $O(X, Y)$ denote the Banach space of bounded linear operators from X into Y where the norm is the usual operator norm. Let $O(X)$ denote $O(X, X)$.

1.2. Assume the notation and conditions of paragraph 1.1 and for convenience let B_s denote the space $[B^0, B^1]_s, 0 < s < 1$. Let V' denote the Banach space

$$O(B^0 \cap B^1, B^0 + B^1) .$$

Let A_j be a closed subspace of $O(B^j), j = 0, 1$. By restricting the elements in A_j to $B^0 \cap B^1$ in the obvious way we may regard A_j as continuously embedded in the topological vector space V', and it is with respect to this embedding that we understand $[A_0, A_1]_s$; in particular, $[A_0, A_1]_s$ is a subspace of V'. We will assume that $A_0 \cap A_1$ is dense in A_j with respect to the norm of $A_j ,j = 0, 1$, when these spaces are embedded in V' as described. Since $B^0 \cap B^1$ is dense in B^0 and B^1, we know from [1, § 9.3] that $B^0 \cap B^1$ is dense in B_s; thus, since $B_s \subset B^0 + B^1$, the restriction of elements of $O(B_s)$ to $B^0 \cap B^1$ gives a continuous embedding of $O(B_s)$ in V' in the obvious manner. Note that each element of $A_0 \cap A_1$ is bounded with respect to the norm $\| \ \|_{B_s}$ restricted to $B^0 \cap B^1$ and is, therefore, contained in the enbedded $O(B_s)$. Let A_s denote the closure of $A_0 \cap A_1$ in $O(B_s)$ where $O(B_s)$ is regarded as embedded in V' in the manner just described. Finally, we let M_s and N_s denote the norms of the spaces A_s and $[A_0, A_1]_s$, respectively.

LEMMA 1.3. *Assuming* 1.2, $[A_0, A_1]_s \subset A_s$ *and* $M_s \leqq N_s$, $0 < s < 1$.

This lemma is an immediate consequence of [1, § 11.1].

1.4. Assume the notation and conditions of 1.1. Let J be a closed subspace of $B^0 + B^1$. We will assume that

(1.4.1) $I^j = J \cap B_j$, is closed in B^j, $j = 0, 1$. Clearly the map α defined by

$$\alpha(x + I^j) = x + J \qquad j = 0, 1$$

is a continuous one to one linear map from B^j/I^j into V/J. Let

$$D_s = [\alpha(B^0/I^0), \alpha(B^1/I^1)]_s .$$

LEMMA 1.5. *Assuming* 1.4, *if* $x \in B_s$, $0 < s < 1$, *then* $x + J \in D_s$ *and*

$$\| x + J \|_{D_s} \leqq \| x + (J \cap B_s) \|_{B_s}/(J \cap B_s) . \tag{1.5.1}$$

Proof. Let $x \in B_s$, $h \in J \cap B_s$ and $\varepsilon > 0$. Choose $f \in \mathfrak{F} = \mathfrak{F}(B^0, B^1)$ such that $f(s) = x + h$ and

$$\| f \|_{\mathfrak{F}} \leqq \varepsilon + \| x + h \|_{B_s} . \tag{1.5.2}$$

Let $g(\xi) = f(\xi) + J$ for $1 \leqq | \xi | \leqq \varepsilon$. Then it is clear that $g \in \mathfrak{F}_1$ where

$$\mathfrak{F}_1 = \mathfrak{F}(\alpha(B^0/I^0), \alpha(B^1/I^1))$$

and that

$$g(s) = x + J . \tag{1.5.3}$$

Hence, $x + J \in D_s$. Furthermore, since it is clear that

$$\| g \|_{\mathfrak{F}1} \leqq \| f \|_{\mathfrak{F}} , \tag{1.5.4}$$

(1.5.1) follows from (1.5.2), (1.5.3), (1.5.4) and the fact that h and ε were chosen arbitrarily.

The following lemma can be proved by the usual method of successive approximations.

LEMMA 1.6. *Suppose that* D_1 *is a Banach space that is continuously embedded in a Banach space* D_0 *such that* D_1 *is dense in* D_0 *with respect to the norm of* D_0. *Suppose that there exist constants* c, c_1, $c < 1$, *with the property that for each* $x \in D_1$ *there is a corresponding element* z *in* D_1 *such that*

$$|z|_1 \leqq c_1|x|_0 \quad \textit{and} \quad |x-z|_0 \leqq c|x|_0 .$$

Then $D_1 = D_0$.

We will now establish a "maximum principle" for analytic interpolation.

THEOREM 1.7. *If, in addition to the assumptions of paragraph* 1.1, $B^0 = [B^0, B^1]_s$ *for some* s $(0 < s < 1)$, *then* $B^0 = B^1$.

Proof. From the fact that B^0 and B^1 are continuously embedded in V and the closed graph theorem we conclude that the norms $|\ |_0$ and $|\ |_s$ on B^0 and $[B^0, B^1]_s$, respectively, are equivalent. In particular, there is a constant c such that

$$(1.7.1) \qquad |x|_0 \leqq c|x|_s \quad \text{for all } x \text{ in } B^0 .$$

From [1, 9.4. (ii)] we conclude that

$$(1.7.2) \qquad |x|_s \leqq |x|_0^{1-s}|x|_1^s \quad \text{for all } x \text{ in } B^0 \cap B^1 .$$

We conclude from (1.7.1) and (1.7.2) that

$$|x|_0 \leqq c^{1/s}|x|_1 \quad \text{for all } x \text{ in } B^0 \cap B^1 .$$

Thus, B_1 is continuously embedded in B^0. We shall now prove that (1.7.3) there is a constant c_1 with the property that for each x in B^1 there is a corresponding y in B^1 such that

$$|y|_1 \leqq c_1|x|_0 \quad \text{and} \quad |y-x|_0 \leqq (1/2)|x|_0 .$$

Let $x \in B^1$. In particular, $x \in [B^0, B^1]_s$ and, therefore, there exists an $f \in \mathfrak{F}(B^0, B^1)$ such that $f(s) = x$ and $|f|_{\mathfrak{F}(B^0,B^1)} \leqq 2|x|_s$. Since the norms $|\ |_0$ and $|\ |_s$ are equivalent we can choose a real number λ so that $2|u|_s e^{\lambda s} \leqq (1/2)|u|_0$ for every u in B^0. Let $g(\xi) = f(\xi)e^{-\lambda(\xi-s)}$ where $0 \leqq \operatorname{Re}\xi \leqq 1$. Then

$$(1.7.4) \qquad \begin{aligned} x = g(s) = &\int_{-\infty}^{\infty} g(it)\mu_0(s,t)dt \\ &+ \int_{-\infty}^{\infty} g(1+it)\mu_1(s,t)dt \end{aligned}$$

where μ_0 and μ_1 are the Poisson kernels for the strip $0 \leqq \operatorname{Re}\xi \leqq 1$ (see [1, 9.4]). Let y and z denote the first and second integrals, respectively, appearing in (1.7.4). Since $\int_{-\infty}^{\infty}|\mu_i(s,t)|dt \leqq 1$ $(i = 0, 1)$, $|g(it)|_0 \leqq 2|x|_s e^{\lambda s} \leqq (1/2)|x|_0$ (all real t), and

$$|g(1+it)|_1 \leqq 2|x|_s e^{-\lambda(1-s)} \leqq (1/2)e^{-\lambda}|x|_0$$

(all real t), it follows that $|x - z|_0 \leqq (1/2)|x|_0$ and $|z|_1 \leqq (1/2)e^{-\lambda}|x|_0$. This proves (1.7.3). Since B^1 is continuously embedded as a dense subspace in B^0 and (1.7.3) holds, the conclusion of Theorem 1.7 follows from Lemma 1.6.

2. The spaces W_p and U_s. Let l_p, $1 \leqq p < \infty$, denote the Banach space of complex valued functions x on the integers such that

$$\| x \|_{l_p} = (\sum | x(n) |^p)^{1/p} < \infty$$

where the sum is over all integers n. Each function α on the integers which vanishes outside some finite set determines a linear transformation T_α on l_p defined by

$$T_\alpha x(n) = \sum_{-\infty < k < \infty} x(n - k)\alpha(k) .$$

Let W_p' denote the closure of the operators T_α in $O(l_p)$. Since l_1 is a dense subspace of each space l_p, $1 \leqq p < \infty$, the restriction of elements in $O(l_p)$, $1 \leqq p \leqq 2$, to the subspace l_1 gives a one-to-one continuous linear embedding of $O(l_p)$, $1 \leqq p \leqq 2$, into the space

$$R = O(l_1, l_2) .$$

Throughout this section we will identify $O(l_p)$ with its image under this embedding without further comment. Let U_s' denote the space $[W_2', W_1']_s$ where V in 1.1 is, in this case, R.

Our immediate purpose is to define a "Fourier transform" on W_p' and to prove Lemmas 2.2 and 2.3.

If x is a complex valued function on the integers Z, let $\tau_n x(k) = x(k - n)$. Let δ_n denote the function on Z such that $\delta_n(n) = 1$ and $\delta_n(k) = 0, k \neq n$. If x and y are two complex valued function on Z let

$$x*y(m) = \sum_{n \in Z} x(m - n)y(n)$$

define the function $x*y$ provided the sum converges absolutely for each $m \in Z$. For each H in W_p' let $H^\sim$ denote the function $H(\delta_0)$ in l_p. The following lemma states the needed properties of the map $H \rightarrow H^\sim$. Note that $\tau_n x = \delta_n * x$ for each $n \in Z$ and for each complex valued function x on z.

LEMMA 2.1.

(2.1.1) *$H \rightarrow H^\sim$ is a one-to-one linear transformation from W_p' into l_p.*

(2.1.2) *$Hx = H^\sim * x$, $H \in W_p'$, $x \in l_p$.*

(2.1.3) *$(HK)^\sim = H^\sim * K^\sim$, $H, K \in W_p'$.*

Proof. The map $H \rightarrow H^\sim$ is clearly linear. Evidently, each H in

W'_p commutes with all operators τ_m, $m \in Z$, since the operators of the form T_α commute with the operators τ_m, $m \in Z$. Thus for $H \in W'_p$ and $m \in Z$, we see that

$$(2.1.4) \qquad H(\delta_m) = H(\tau_m \delta_0) = \tau_m H(\delta_0) = \tau_m H^\sim = H^\sim * \delta_m .$$

From this we see that since the linear span of the elements δ_m is dense in l_p, the map $H \to H^\sim$ is one-to-one. Obviously, $H^\sim$ is in l_p. To establish (2.1.2) we first note that since $H^\sim$ is in $l_q(q^{-1} + p^{-1} = 1)$ the map $x \to H^\sim * x$ is a continuous linear map from l_p into c_0, the space of complex valued functions on Z which tend to 0 at $\pm\infty$. The map $x \to Hx$ is also a continuous linear map from l_p into c_0. These observations together with (2.1.4) and the density property of the δ_m's noted above complete the proof of (2.1.2). To prove (2.1.3) we note that for H and K in W'_p, $K^\sim \in l_p$, so by (2.1.2) we have

$$H^\sim * K^\sim = H(K^\sim) = H(K\delta_0) = (HK)\delta_0 = (HK)^\sim .$$

This completes the proof of the lemma.

Let $L_p(1 \leqq p < \infty)$ denote the Banach space of measurable functions $g(\theta)$ on the circle (reals mod 2π) whose norm $\| g \|_{L_p}$,

$$\| g \|_{L_p} = \left((1/2\pi) \int_0^{2\pi} | g(\theta) |^{1/p} d\theta \,| \right)^{1/p} ,$$

is finite. Let L_∞ denote the space of essentially bounded measurable functions g with $\| g \|_{L_\infty}$ denoting the essential supremum of g.

Since each function $H^\sim$, $H \in W'_p$, is in l_p, which is contained in l_2, there is a unique function $H^\wedge$ in L_2 such that $\sum H^\sim(n) e^{in\theta}$ is the Fourier series of $H^\wedge$.

LEMMA 2.2. *For $1 \leqq p \leqq 2$ the map $H \to H^\wedge$ is a norm decreasing algebraic isomorphism from W'_p into L_∞.*

Proof. The fact that $H \to H^\wedge$ is a one-to-one linear map from W'_p into L_2 is clear from (2.1.1) and the fact that each function in L_2 is uniquely determined by its Fourier coefficients. For each $f \in L_1$, let $\lambda(f)$ denote the function on Z defined by:

$$\lambda(f)(n) = (1/2\pi) \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta .$$

It is clear from the Schwarz inequality that the map $(f, g) \to \lambda(f \cdot g)(n)$ is a continuous bilinear functional on $L_2 \oplus L_2$ for each integer n. On the other hand, the map

$$(f, g) \to (\lambda(f) * \lambda(g))(n)$$

is also a continuous bilinear functional on $L_2 \oplus L_2$. Since these functionals (for each n) clearly agree when f and g are trigonometric polynomials, they must agree on $L_2 \oplus L_2$. Since λ is a one-to-one map, the multiplicative property of $H \to H^\wedge$ now follows from (2.1.3). To prove that the map is norm decreasing we first note the following inequalities:

$$\|H^n\|_{W'_p} \geqq \|H^n\delta_0\|_{l_p} = \|(H^n)^\sim\|_{l_p} \geqq \|(H^n)^\sim\|_{l_2} = \|(H^n)^\wedge\|_{L_2} = \|(H^\wedge)^n\|_{L_2} .$$

It is well known that $(\|H^n\|_{W'_p})^{1/n}$ converges to the spectral radius of H, which is dominated by $\|H\|_{W'_p}$, and that $(\|(H^\wedge)^n\|_{L_2})^{1/n}$ converges to $\|H^\wedge\|_{L_\infty}$ as $n \to \infty$. This proves the lemma.

Let W_p and U_s denote the functions on the circle of the form $H^\wedge$ where $H \in W'_p$, U'_s, respectively. The following lemma is an immediate consequence of Lemma 2.2.

LEMMA 2.3. *W_p consists precisely of the functions on the circle which are the uniform limits of sequences $H^\wedge_n$ of trigonometric polynomials such that H_n is a Cuachy sequence in W'_p.*

For any subset E of the circle group $U_s \mid E$ denotes the functions on E obtained by restricting the functions of U_s to E and $C(E)$ denotes the continuous complex valued functions on E.

THEOREM 2.4. *Suppose that E is a compact subset of the circle group and $0 < s < 1$. Then $U_s \mid E = C(E)$ if and only if $W_1 \mid E = C(E)$.*

Proof. First assume that $W_1 \mid E = C(E)$. By Lemma 1.3, $U'_s \subset W'_p$; consequently, $U_s \subset W_p$. We conclude from Lemma 2.3 that $W_p \subset C(T)$. Thus, $U_s \mid E \subset C(E)$. Since $W'_2 \supset W'_1$, it is clear from the definition of interpolation that $U'_s \supset W'_1$. Thus, $U_s \mid E \supset C(E)$.

Consider the converse and assume that $U_s \mid E = C(E)$. In 1.4 we let $B^0 = W'_2$, $B^1 = W'_1$, $V = R$ and

$$J = \{a \in W'_2 : \hat{a}(\theta) = 0,\ \theta \in E\} .$$

The assumptions on J in 1.4 are clearly satisfied since by Lemma 2.2, the maps $a \to \hat{a}$ are continuous on W'_1 and W'_2. By Theorem 1.5, if $x \in U'_s$, then $x + J$ is in the space

$$[\alpha(W'_2/J),\ \alpha(W'_1/(J \cap W'_1))]_s . \tag{2.4.1}$$

However, by hypothesis, the cosets in V of the form $x + J$,, $x \in U'_s$, are the same as the cosets $y + J$, $y \in W'_2$. Therefore, the space in (2.4.1) is $\alpha(W'_2/J)$. Since $W'_2 \supset W'_1$,

$$\alpha(W'_2/J) \supset \alpha(W'_1/(J \cap W'_1)) ;$$

therefore, we conclude from 1.7 that

$$\alpha(W_2'/J) = \alpha(W_1'/(J \cap W_1')) ;$$

or, what is the same thing, that $W_1 \mid E = C(E)$. This completes the proof.

COMMENT 2.5. It is natural to compare U_s and W_p where $[l_2, l_1]_s = l_p$, i.e., $(1 - s)/2 + s = 1/p$. In [3] we showed that Theorem 2.4 is not valid for W_p. To be exact, there is a compact subset E of the circle such that $W_p \mid E \neq C(E) = W_{4/3} \mid E, 1 \leqq p < 4/3$. We had originally used this result to show that $W_p \neq U_s$; however, the referee has suggested a direct proof which we will now give.

LEMMA 2.6. *Let h_n be a sequence in $U_s, 0 < s < 1$, such that $\| h_n \|_s \leqq M$ (here $\| \ \|_s$ is the norm in U_s) and $h_n \to h$ almost everywhere. Then h agrees with some continuous function almost everywhere.*

Proof. Since $\| h_n \|_s \leqq M$ there exist functions $f_n(\theta, \xi)$, analytic in ξ for $0 < B(\xi) < 1$ and continuouns in $0 \leqq B(\xi) \leqq 1$, such that for any real number t, $\| f_n(\theta, it) \|_0 \leqq 2M$, $\| f_n(\theta, 1 + it) \|_1 \leqq 2M$ and $f_n(\theta, s) = h_n(\theta)$. Let $g_n(\theta, \xi) = f_n(\theta, \xi)e^{+\lambda(\xi - s)}$. Then

$$\begin{aligned} h_n(\theta) = f_n(\theta, s) = g_n(\theta, s) &= \int_{-\infty}^{+\infty} g_n(\theta, it)\mu_0(s, t)dt \\ &\quad + \int_{-\infty}^{+\infty} g_n(\theta, 1 + it)\mu_1(s, t)dt \\ &= u_n(\theta) + v_n(\theta) \end{aligned}$$

where μ_0 and μ_1 are the Poisson Kernels for the strip (see [1, 9.4]). Evidently $\| u_n \|_0 \leqq 2e^{-\lambda s}M$, $\| v_n \|_1 \leqq 2e^{\lambda(1-s)}M$. Since the v_n are uniformly bounded, by taking a subsequence if necessary, we may assume that v_n converges weakly to a bounded function $v(\theta)$, that is

$$\lim_{n \to \infty} \int v_n(\theta)\varphi(\theta)d\theta = \int v(\theta)\varphi(\theta)d\theta$$

for every integrable φ. Furthermore, as is readily seen, $v(\theta)$ belongs to U_1 and therefore is continuous. Since h_n is uniformly bounded and converges almost everywhere, h_n converges weakly. Since h_n and v_n converge weakly, u_n converges weakly to some function u. From the fact that $| u_n(\theta) | \leqq \| u_n \|_0 \leqq 2e^{-\lambda s}M$, it follows that $| u(\theta) | \leqq 2e^{-\lambda s}$ almost everywhere. Since $h = u + v$ almost everywhere and λ can be taken arbitrarily large, h agrees almost everywhere with the uniform limit of continuous functions. This completes the proof of the lemma.

THEOREM 2.7. *U_s is properly contained in W_p for $1 < p < 2$.*

Proof. To prove the theorem it suffices to exhibit a sequence of functions in U_s whose norms in U_s tend to infinity and whose norms in W_p remain bounded. Let $h(e^{it}) = 1$ for $0 \leqq t \leqq \pi$ and $h(e^{it}) = 0$ for $\pi < t < 2\pi$. Then h is a multiplier for l_p (see [2]), which does not agree almost everywhere with any continuous function. Let φ_n be defined by: $\varphi_n(e^{it}) = n$ for $|t| \leqq 1/2n$, $\varphi_n(e^{it}) = 0$ otherwise, $n = 1, 2, \cdots$. Let $h_n = h*\varphi_n$, $n = 1, 2, \cdots$. Since $\int_0^{2\pi} | h_n(e^{it}) | dt = 1$, it follows that the W_p norm of h_n is the same as the W_p norm of h; thus, h_n is bounded in W_p. Since both h and φ_n belong to $L_2(0, 2\pi)$, $h_n \in W_1 \subset U_s$. Obviously, h_n converges to h almost everywhere. Since h does not agree almost everywhere with any continuous function, it follows from Lemma 2.6 that h_n is not bounded in U_s.

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Received August 26, 1968.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

SEMI-SIMPLE RADICAL CLASSES

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The purpose of this paper is to characterize all semi-simple radical classes (those classes of rings which are semi-simple classes and at the same time radical classes).

Andrunakievic has shown that the class of Boolean rings is a semi-simple radical class. More recently, Armendariz has considered such classes.

For "I is an ideal of the ring R" we shall write "$I \lhd R$".

Following Divinsky [6], but substituting classes of rings for ring properties, we define:

(i) A nonempty class of rings $\mathscr{C}$ is a *radical class* if and only if $\mathscr{C}$ satisfies the following conditions:

(A) Homomorphic images of rings in $\mathscr{C}$ are in $\mathscr{C}$.

(B) Every ring R has an ideal $\mathscr{C}(R) \in \mathscr{C}$ such that if $I \lhd R$ and $I \in \mathscr{C}$ then $I \subseteqq \mathscr{C}(R)$.

(C) The only ideal of the factor ring $R/\mathscr{C}(R)$ which is in $\mathscr{C}$ is the zero ideal.

(ii) If $\mathscr{C}$ is a radical class, a ring R is $\mathscr{C}$ *semi-simple* if and only if $\mathscr{C}(R) = (0)$.

(iii) A nonempty class of rings $\mathscr{C}$ is a *semi-simple class* if and only if $\mathscr{C}$ satisfies the following conditions:

(E) Every nonzero ideal of a ring in $\mathscr{C}$ can be homomorphically mapped onto a nonzero ring in $\mathscr{C}$.

(F) If every nonzero ideal of a ring R can be homomorphically mapped onto a nonzero ring in $\mathscr{C}$ then $R \in \mathscr{C}$.

2. Rings without nilpotent elements.

Our purpose in this section is to establish:

THEOREM 2.1.[1] *A ring R without nilpotent elements is isomorphic to a subdirect sum of rings without proper divisors of zero.*

It will be convenient to first prove:

LEMMA 2.2. *If R has no nilpotent elements and $0 \neq x \in R$ then*

(i) $x_r = \{y \in R: xy = 0\} \lhd R$ *and* $x_r = x_l = \{y \in R: yx = 0\}$,

(ii) $x \notin x_l$,

[1] The author wishes to thank the referee for pointing out that this result has also been obtained by V. Andrunakievic and Ju. M. Rjabuhin, *Rings without nilpotent elements, and completely simple ideals*, Dokl. Akad. Nauk. SSR. **180**, 9-11 (Translation, Soviet Mathematics **9** (1968), 565-568).

(iii) *if* $r \in R$ *and* $rx \in x_l$ *then* $r \in x_l$,

(iv) *the factor ring* R/x_l *has no nilpotent elements.*

Proof. Let R be a ring with no nilpotent elements and $0 \neq x \in R$. If $a \in R$ and $ax = 0$ then $(xa)^2 = 0$ so $xa = 0$. Similarily if $xa = 0$ then $ax = 0$. This establishes (i). Since $x^2 \neq 0$, (ii) is clear. If a, $b \in R$ and $ab^2 = 0$ then $(bab)^2 = 0$ so $bab = 0$, but then $(ab)^2 = 0$ so $ab = 0$. From this (iii) and (iv) follow immediately.

To prove the theorem it is sufficient to find, for each $0 \neq x \in R$, an ideal $I(x)$ of R for which $R/I(x)$ has no proper divisors of zero and $x \notin I(x)$. Let $Z(x) = \{I \lhd R: x \notin I$, if $rx \in I$ then $r \in I$, and R/I has no nilpotent elements$\}$. By 2.2 $x_l \in Z(x)$ so $Z(x) \neq \emptyset$ and it is clear that the union of an ascending chain in $Z(x)$ is also in $Z(x)$. Thus we may choose, by Zorn's Lemma, $I(x)$ maximal in $Z(x)$.

If $a \in R$ and $a \notin I(x)$ let $J = \{y \in R: ay \in I(x)\} \supseteqq I(x)$. Then $J/I(x) = (a + I(x))_r$ in $R/I(x)$ and by 2.2 (i) $(a + I(x))_l = (a + I(x))_r \lhd R/I(x)$. Since $a \notin I(x)$, $ax \notin I(x)$ so $x \notin J$. If $rx \in J$ then $arx \in I(x)$ so $ar \in I(x)$, hence $r \in J$. Finally by 2.2 (iv)$R/J \cong R/I(x)/J/I(x)$ has no nilpotent elements, so $J \in Z(x)$. Hence $J = I(x)$ so $R/I(x)$ has no proper divisors of zero.

Note 2.3. The generalized nil radical Ng of Andrunakievic [4] and Thierrin [10] (see also [6]) is the upper radical with respect to the class of rings without proper divisors of zero. A ring R is Ng semi-simple if and only if R is isomorphic to a subdirect sum of rings without proper divisors of zero. In this context, 2.1 can be restated as: A ring R is Ng semi-simple if and only if R has no nilpotent elements.

3. $\mathscr{B}_1$-rings.

If $x \in R$, let $[x]$ = the subring of R generated by x.

DEFINITION 3.1. R is a $\mathscr{B}_1$-ring $. \equiv .$ for all $x \in R$, $[x] = [x]^2$.

Let R be a ring and $x \in R$. Clearly $[x] = [x]^2$ if and only if $x \in [x]^2$ if and only if there are integers $a_2, \cdots, a_k$ such that $x = \sum_{i=2}^k a_i x^i$. Using this it is clear that homomorphic images of $\mathscr{B}_1$-rings are $\mathscr{B}_1$-rings and that if A/B and B are $\mathscr{B}_1$-rings then A is a $\mathscr{B}_1$-ring. It then easily follows that the class of $\mathscr{B}_1$-rings (which we shall denote by $\mathscr{B}_1$) is a radical class.

LEMMA 3.2. *A nonzero* $\mathscr{B}_1$*-ring without proper divisors of zero is a field of prime characteristic which is algebraic over its prime subfield.*

Proof. Let R be a nonzero $\mathscr{B}_1$-ring without proper divisors of zero. If x is a nonzero element of R there are integers $a_2, \cdots, a_k$ such that $x = \sum_{i=2}^{k} a_i x^i$, hence $e_x = \sum_{i=2}^{k} a_i x^{i-1}$ is an identity for $[x]$. Since x is not a zero divisor e_x is an identity for R. If $w \in R$, $w \neq 0$, $e_w \in [w] = [w]^2$ so $e_w \in [w] \cdot w \subseteqq Rw$ thus $R = Rw$. Since R is nonzero, R is a division ring.

Let e be the identity of R. Then $[2e] = [2e]^2 = [4e]$ so $Ne = 0$ for some positive integer N. Consequently the characteristic of R is a prime and since $e = e_w \in [w]$ for all nonzero $w \in R$, R is algebraic over its prime subfield. Therefore, by Theorem 2, page 183 of Jacobson [7] R is a field.

COROLLARY 3.3. *If R is a $\mathscr{B}_1$-ring then R is isomorphic to a subdirect sum of algebraic fields of prime characteristic. So, in particular, R is commutative.*

Proof. If $x \in R$, $x^N = 0$ and $R \in \mathscr{B}_1$, then $[x] = [x]^2 = \cdots = [x]^N = (0)$ so $x = 0$. Hence $\mathscr{B}_1$-rings do not have nilpotent elements so the corollary follows from 2.1 and 3.2.

THEOREM 3.4. *A ring R is a $\mathscr{B}_1$-ring if and only if every finitely generated subring of R is isomorphic to a finite direct sum of finite fields.*

Proof. Let $R \in \mathscr{B}_1$ and R' be a finitely generated subring of R. Then $R' \in \mathscr{B}_1$ and hence is commutative, so by the Hilbert Basis Theorem R' has maximum condition on ideals. If $P' \neq R'$ and P' is a prime ideal of R' then P' is a maximal ideal of R' since by 3.2 R'/P' is a field. Since R' is finitely generated, commutative, and $[g]$ has an identity for each generator g of R', R' has an identity. Then by Theorem 2, page 203 of [11] R' has minimum condition on ideals. But then R' is a commutative Wedderburn ring so R' is isomorphic to a finite direct sum of fields each of which must be finite since they are finitely generated, algebraic and of prime characteristic.

The converse is obvious; in fact, if $x \in R'$ and R' is isomorphic to a finite direct sum of finite fields then there is an integer $n(x) \geqq 2$ such that $x^{n(x)} = x$. Thus we have:

COROLLARY 3.5. *R is a $\mathscr{B}_1$-ring if and only if for each $x \in R$ there exists an integer $n(x) \geqq 2$ such that $x^{n(x)} = x$.*

A class of rings $\mathscr{C}$ is said to be *hereditary* if $I \lhd R \in \mathscr{C}$ implies that $I \in \mathscr{C}$. Analogously we say:

DEFINITION 3.6. A class of rings $\mathscr{C}$ is *strongly hereditary* $. \equiv$ if S is a subring of $R \in \mathscr{C}$ then $S \in \mathscr{C}$.

PROPOSITION 3.7. *If $\mathscr{F}$ is a strongly hereditary finite set of finite fields then a ring R is isomorphic to a subdirect sum of fields in $\mathscr{F}$ if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathscr{F}$.*

Proof. Since $\mathscr{F}$ is a finite set of finite fields there exists an integer $N \geqq 2$ such that $x^N = x$ for all $x \in F \in \mathscr{F}$.

Let R have ideals $I_\alpha: \alpha \in A$ such that $R/I_\alpha \cong F_\alpha \in \mathscr{F}$ and $\cap \{I_\alpha: \alpha \in A\} = (0)$. Let R' be a finitely generated subring of R. Then $R' \in \mathscr{B}_1$ since $x^N = x$ for all $x \in R \supseteqq R'$, so by 3.4 $R' \cong A_1 \oplus \cdots \oplus A_k$ and the A_i are finite fields. Choose $a_i \in R'$ such that $[a_i] \cong A_i$. Then $a_i \neq 0$ so $a_i \notin I_{\beta_i}$ for some $\beta_i \in A$ but $I_{\beta_i} \cap [a_i] \lhd [a_i]$ so $I_{\beta_i} \cap [a_i] = (0)$. Therefore $A_i \cong [a_i] \cong [a_i] + I_{\beta_i}/I_{\beta_i}$ is isomorphic to a subring of F_{β_i}. Since $\mathscr{F}$ is strongly hereditary R' is isomorphic to a finite direct sum of fields in $\mathscr{F}$.

Conversely, if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathscr{F}$, R must be a $\mathscr{B}_1$-ring since again $x^N = x$ for all $x \in R$. Thus by 3.3 there are ideals $I_\alpha: \alpha \in A$ of R such that $\cap \{I_\alpha: \alpha \in A\} = (0)$ and R/I_α is a field of prime characteristic; moreover, R/I_α must be a finite field since $x^N - x = 0 \in I_\alpha$ for all $x \in R$. Therefore, for each $\alpha \in A$, there exists $x_\alpha \in R$ such that $[x_\alpha] + I_\alpha/I_\alpha = R/I_\alpha$. But then R/I_α is a homomorphic image of $[x_\alpha]$ so R/I_α is isomorphic to a field in $\mathscr{F}$.

4. Semi-simple radical classes.

LEMMA 4.1. *If $\mathscr{C}$ is a class of rings such that subdirect sums of rings in $\mathscr{C}$ are in $\mathscr{C}$ and $\mathscr{C}$ satisfies (A) then $\mathscr{C}$ is strongly hereditary.*

Proof. Let $R \in \mathscr{C}$ and S be a subring of R.

Set $R_i = R$ for all $i \in Z^+ =$ the set of positive integers. Now the (discrete) direct sum $\sum \{R_i: i \in Z^+\}$ is an ideal of the direct product (complete direct sum) $\prod \{R_i: i \in Z^+\}$. If $s \in S$ let $\hat{s}(i) = s$ for all $i \in Z^+$. Then $S \rightarrow \Delta(S) = \{\hat{s}: s \in S\}$ is an embedding of S into $\prod \{R_i: i \in Z^+\}$. $\Delta(S) + \sum \{R_i: i \in Z^+\}$ is a subdirect sum of copies of R and hence is in $\mathscr{C}$, so

$$S \cong \Delta(S) \cong \frac{\Delta(S) + \sum \{R_i: i \in Z^+\}}{\sum \{R_i: i \in Z^+\}} \in \mathscr{C} .$$

Using a theorem of Amitsur [1] which states that every ring is a homomorphic image of a subdirect sum of total matrix rings of finite order over the ring of all integers, Armendariz in [5] proves that if a hypernilpotent radical class $\mathscr{C}$ is a semi-simple class, then $\mathscr{C}$ contains all rings. A hypernilpotent radical class is a hereditary radical class which contains all nilpotent rings.

THEOREM 4.2. *If $\mathscr{C}$ is a semi-simple radical class and $\mathscr{C} \nsubseteq \mathscr{B}_1$ then $\mathscr{C}$ consists of all rings.*

Proof. Let $\mathscr{C}$ be a semi-simple radical class. If $\mathscr{C} \nsubseteq \mathscr{B}_1$ then there is a $R \in \mathscr{C}$ and $x \in R$ such that $[x] \neq [x]^2$. In [8] Kurosh shows that for any semi-simple class $\mathscr{S}$, subdirect sums of rings in $\mathscr{S}$ are in $\mathscr{S}$. Thus, by 4.1, $[x] \in \mathscr{C}$ and since $[x]^2 \lhd [x]$, $[x]/[x]^2 \in \mathscr{C}$. Now $[x]/[x]^2$ is a zero ring on a cyclic group and since $\mathscr{C}$ satisfies (F), $C^\infty =$ the zero ring on the infinite cyclic group is in $\mathscr{C}$. This implies (see [3] and [6]) that $\mathscr{C}$ contains all nilpotent rings. Since $\mathscr{C}$ is a semi-simple class (see [2] and [6]) $\mathscr{C}$ is hereditary, hence $\mathscr{C}$ is hypernilpotent. Therefore, by [5], $\mathscr{C}$ is the class of all rings.

THEOREM 4.3. *If $\mathscr{C}$ is not the class of all rings then the following are equivalent*:

(1) *$\mathscr{C}$ is a semi-simple radical class,*

(2) *there is a strongly hereditary finite set $\mathscr{C}(F)$ of finite fields such that: $R \in \mathscr{C}$ if and only if R is isomorphic to a subdirect sum of fields in $\mathscr{C}(F)$,*

(3) *there is a strongly hereditary finite set $\mathscr{C}(F)$ of finite fields such that: $R \in \mathscr{C}$ if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathscr{C}(F)$.*

Proof. By 3.7 we have that (2) and (3) are equivalent.

Assume that $\mathscr{C}$ satisfies condition (3). Clearly $\mathscr{C}$ satisfies (A) and (E).

If $B \lhd A$ and both A/B and B are in $\mathscr{C}$ and A' is a finitely generated subring of A then $A' + B/B \cong A'/A' \cap B$ is isomorphic to a finite direct sum of fields in $\mathscr{C}(F)$. A slight modification of the proof given for Proposition 1 on page 241 of Jacobson [7] shows that $A' \cap B$ is finitely generated as a ring. Thus $A' \cap B$ is also isomorphic to a finite direct sum of fields in $\mathscr{C}(F)$ and so $A' \cong A'/A' \cap B \oplus A' \cap B$. Therefore $A \in \mathscr{C}$. From this it is easy to show that if $\mathscr{C}(R) =$ the sum of all ideals of R which are in $\mathscr{C}$ then $\mathscr{C}(R) \in \mathscr{C}$ and $\mathscr{C}(R/\mathscr{C}(R)) = (0)$. Thus, $\mathscr{C}$ satisfies (B) and (C).

If every nonzero ideal of a ring R can be homomorphically mapped onto a nonzero ring in $\mathscr{C}$ then by 3.7, every nonzero ideal of R can be homomorphically mapped onto a ring in $\mathscr{C}(F)$. Sulinski [9] (see also [6], Theorem 46) shows that this implies that R is isomorphic to a subdirect sum of rings in $\mathscr{C}(F)$ and hence by 3.7 again, $R \in \mathscr{C}$. So

$\mathscr{C}$ satisfies (F) and hence $\mathscr{C}$ is a semi-simple radical class.

Conversely, suppose $\mathscr{C}$ satisfies condition (1). Let $\mathscr{C}(F)$ = the class of all fields which are in $\mathscr{C}$ and define $A = \prod \{R: R \in \mathscr{C}(F)\}$. Since $\mathscr{C}$ is a semi-simple class subdirect sums of rings in $\mathscr{C}$ are in $\mathscr{C}$; thus $A \in \mathscr{C}$. By hypothesis, $\mathscr{C} \subseteqq \mathscr{B}_1$ so by 3.4 all elements of A must be torsion. From this it follows that there is a finite number of primes $p_1, \cdots, p_N$ such that every field in $\mathscr{C}(F)$ is of characteristic p_i for some $1 \leqq i \leqq N$. For each finite field $R \in \mathscr{C}(F)$ choose $a(R)$ such that $[a(R)] = R$ and for each infinite field $R \in \mathscr{C}(F)$ set $a(R) = 0$. Then $a = \{a(R)\}_{R \in \mathscr{C}(R)}$ is in A and by 3.5 $a^K = a$ for some integer $K \geqq 2$. Thus, for all finite fields R in $\mathscr{C}(F)$, the dimension of R over its prime subfield is $\leqq K - 1$. Hence there is only a finite number of finite fields in $\mathscr{C}(F)$. Suppose there is an infinite field $R \in \mathscr{C}(F)$. By 3.2 R is of prime characteristic and is algebraic over its prime subfield so R has an infinite number of non-isomorphic finite subfields. All these subfields are in $\mathscr{C}(F)$ since $\mathscr{C}$ is strongly hereditary by 4.1. This is impossible since there is only a finite number of finite fields in $\mathscr{C}(F)$. Therefore $\mathscr{C}(F)$ is a strongly hereditary finite set of finite fields. If $R \in \mathscr{C}$ then $R \in \mathscr{B}_1$ so by 3.3 R is isomorphic to a subdirect sum of fields all of which are in $\mathscr{C}(F)$ since $\mathscr{C}$ satisfies (A). Conversely, any ring isomorphic to a subdirect sum of rings in $\mathscr{C}(F)$ is in $\mathscr{C}$ since $\mathscr{C}$ is semi-simple class. Thus $\mathscr{C}$ satisfies (2).

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Received November 6, 1968. The author holds a National Research Council of Canada Postgraduate Scholarship.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

ON LEFT QF-3 RINGS

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In this paper the following results are proved:

(i) Three classes of left QF-3 rings are closed under taking left quotient rings respectively.

(ii) A subcategory of left modules having dominant dimensions $\geqq 2$ over a right perfect left QF-3 ring R is equivalent to a category of all left fRf-modules, where f is a suitable idempotent of R.

(iii) In case a left QF-3 ring is obtained as the endomorphism ring of a generator, dominant dimensions ($\geqq 2$) of modules are closely connected with the vanishing of Ext-functors.

(iv) Three classes of left and right QF-3 rings are identical in case of perfect rings.

Let R be an associative ring having an identity element 1 and denote by ${}_RR$ (resp. R_R) a left (resp. right) R-module R. To generalize the notion of QF-3 algebras [18] we shall make the following definitions:

(1) R is said to be left QF-3, if ${}_RR$ has a direct summand Re (e is an idempotent of R) which is a faithful, injective left ideal.

(2) R is said to be left QF-3$^+$, if the injective hull $E({}_RR)$ of ${}_RR$ is projective.

(3) R is said to be left QF-3$'$, if the injective hull $E({}_RR)$ of ${}_RR$ is torsionless in the sense of Bass [1].

Right QF-3, QF-3$^+$ and QF-3$'$ rings are defined in a similar fashion. It is obvious that the class of left QF-3$'$ rings is the most general class of the above three classes.

Our main purpose in this note is to introduce some generalizations of results for QF-3 algebras [11], [12], [15], [16], [17] and semi-primary QF-3 rings [4], [6], [13], [14] to the above generalized classes of rings.

We shall say that the dominant dimension of left (resp. right) R-module X, denoted by dom. dim ${}_RX$(resp. dom. dim X_R), is at least n, if there exists an injective resolution of X:

$$0 \longrightarrow X \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_n$$

such that all W_i, $1 \leqq i \leqq n$, are torsionless. Then it is clear that R is left (resp. right) QF-3$'$ if and only if dom. dim ${}_RR$ (resp. dom. dim R_R) $\geqq$ 1.

In §1 we shall show that each class defined as above is closed under taking quotient rings (not necessarily classical), that is, a left quotient ring S of R is left QF-3', QF-3$^+$ or QF-3 if R is left QF-3', QF-3$^+$ or QF-3 respectively. Further, S is the maximal left quotient ring of R if and only if dom. dim $_SS \geqq 2$. This is a generalization of the results for QF-3 algebras by Morita [12], Tachikawa [17] and Mochizuki [11]. Then, as an immediate consequence we have that R and the double centralizer R' of any faithful right ideal of R are contained in the same class, and dom. dim $_RR = 1$, if $R' \neq R$. Here it is to be noted that the double centralizer of a faithful left ideal of R is not necessarily left QF-3', even if R is left QF-3' (cf. §4, example 1).

In §2 we shall consider a left QF-3 ring R which has a faithful projective right ideal K and shall develop a proof in order to notice that the injectivity of K_R is not necessary to obtain some results in [14, Propositions 1.1 and 1.2]. Further, defining a special injective dimension closely connected to a fixed injective module, we shall prove that in the case R is a right perfect left QF-3 ring, there exists a suitable idempotent f of R such that the subcategory of left R-modules of dominant dimension at least two is equivalent to the category of all left fRf-modules. We shall remark also that the two characterizations for dominant dimension by Mueller [15] can be applied to an estimation of dominant dimensions of endomorphism rings of modules which are generators.

Recently, the characterization of Artinian QF-3 rings due to Wu, Mochizuki and Jans [19] suggested the notion of QF-3' rings to Colby and Rutter [4] and Kato [7]. In [4] it was proved that semi-primary[1] left QF-3' rings are not necessarily left QF-3$^+$, however "left QF-3'" implies "left QF-3" for semi-primary rings. Without the proof we shall state in §3 that the same result holds for perfect rings, since the proof in [4] is available for this case. Moreover, we shall prove by duality of modules and the result proved in the first part of §2 the notions of two sided QF-3', QF-3$^+$ and QF-3 are identical for perfect rings.

1. Quotient rings of QF-3 rings.

Let R be a ring with an identity element 1 and N a submodule of a left R-module M. M is said to be a rational extension of N in case $f(N) = 0$ implies $f = 0$ for $f \in \mathrm{Hom}_R(L, M)$, where L is any submodule of M containing N. Then, following Lambek [10], a ring S is said to be a left quotient ring of R if S contains R as a subring and if S is a rational extension of R as a left R-module. To begin with we shall prove

PROPOSITION 1.1 *Let S be a left quotient ring of R. If R is*

[1] In this case a ring R is said to be semi-primary if it contains a nilpotent ideal N with R/N semi-simple with minimum condition.

left QF-3′ (resp. QF-3⁺), *then* S *is left* QF-3′ (resp. QF-3⁺).

Proof[2]. Denote by I the injective hull $E({}_RR)$. Then, by the assumption $I \cong {}_RE({}_SS)$ and we have

$$I \xrightarrow{j} \textstyle\prod_\lambda R \xrightarrow{\varphi} \prod_\lambda S\,,$$

$$\left(\text{resp. } I \xrightarrow{j} \textstyle\sum_\lambda \oplus R \xrightarrow{\varphi} \sum_\lambda \oplus S\right)$$

where φ is an R-monomorphism and j is the inclusion mapping.

Suppose s be an element of S such that $\varphi j(sx) \neq s(\varphi j(x))$ for some $x \in I$. Then we have a projection p_λ of $\prod_\lambda S$(resp. $\sum \oplus S$) onto S such that the R-homomorphism

$$s \longrightarrow p_\lambda(\varphi j(sx) - s(\varphi j(x))$$

of S into S is nonzero but has kernel containing R. However, this contradicts that ${}_RS$ is a rational extention of R. Hence φj is an S-monomorphism and consequently S is left QF-3′ (resp. left QF-3⁺).

PROPOSITION 1.2. *Let S be a left quotient ring of R. If R is left* QF-3, *then S is left* QF-3.

Proof. Let Re, $e^2 = e$ be a faithful projective, injective left ideal of R. Since ${}_RS$ is an essential extension of ${}_RR$ and $e^2 = e$, ${}_RSe$ is an essential extension ${}_RRe$ so $Se = Re$.

Next, we shall prove that ${}_SSe$ is injective. Let L be a left ideal of S and φ a left S-homomorphism of L into Se. Denote by $\hat{\varphi}$ the map of $L \cap R$ into $Se(=Re)$ which is the restriction of φ. Since ${}_RRe$ is injective, it follows by Baer's Criterion that there exists an element q of Re such that $\hat{\varphi}(l) = lq$ for all $l \in L \cap R$. Then we shall define a map $\bar{\Psi}$ of L into Se by putting $\bar{\Psi}(r) = \varphi(r) - rq$, for all $r \in S$. Now we shall suppose that $\bar{\Psi}(r_1) \neq 0$ for some nonzero element r_1 of L. $\bar{\Psi}(r_1)$ and r_1 are both elements of S. Since ${}_RS$ is a rational extension, there exists an element r_0 of R such that $r_0\bar{\Psi}(r_1) \neq 0$ and $r_0r_1 \in R$. Then $r_0r_1 \in R \cap L$ and hence $r_0\bar{\Psi}(r_1) = r_0\varphi(r_1) - r_0(r_1q) = \varphi(r_0r_1) - r_0r_1q = \hat{\varphi}(r_0r_1) - (r_0r_1)q = 0$. This is a contradiction. Thus $\bar{\Psi}(r) = 0$ for all $r \in L$. Hence a left S-homomorphism Φ of S into Se defined by $\Phi(r) = rq$ for all $r \in S$ is an extension of φ and by Baer's Criterion we obtain that ${}_SSe$ is injective.

It remains to prove the faithfulness of ${}_SSe$. Let q be a nonzero element of S. Then there exists an element d of R such that $dq \in R$ and $dq \neq 0$. Then there exists an element x of Re such that $dqx \neq 0$,

[2] The author is grateful to the referee who clarified the proofs in this section.

because ${}_R Re$ is faithful. Hence $qx \neq 0$, $x \in Se$, this implies that ${}_S Se$ is faithful.

Let us denote by Q the maximal left quotient ring of R. Then we have

PROPOSITION 1.3. *If R is left* QF-3′ *and S is a left quotient ring of R, then* dom. dim. ${}_S S \geqq 2$ *if and only if* $S = Q$.

Proof. It is known [9] that S is a subring of Q and Q can be imbedded into I by a Q-monomorphism j. Lambek proved in [9] that $j(Q) = \{x \in I \mid h(j(S)) = 0$ implies $h(x) = 0$ for $h \in \mathrm{Hom}_S(I, I)\}$. If dom. dim ${}_S S \geqq 2$, $I/j(S)$ is isomorphic to a submodule of a direct product of copies of S. Hence for $x \in I$ such that $x \notin j(S)$, there exists a S-homomorphism f of I into I with $f(x) \neq 0$ and $f(j(S)) = 0$. Thus by the remark above $x \notin j(Q)$ and consequently $S = Q$.

Conversely assume $S = Q$. Since the maximal left quotient ring of Q is itself, by the same reason we have $\bigcap_{h \in H_0} \mathrm{Ker}\, h = j(Q)$, where $H_0 = \{h \in \mathrm{Hom}_Q(I, I) \mid h(j(Q)) = 0\}$.

Thus,

$$0 \longrightarrow Q \xrightarrow{j} I \xrightarrow{\varphi} \prod_{h \in H_0} I$$

is exact, where $\varphi(x) = (\cdots, h(x), \cdots)$, $x \in I$. It follows that

$$\text{dom. dim}\ {}_S S = \text{dom. dim}\ {}_Q Q \geqq 2.$$

In case R is a finite dimensional QF-3 algebra, $E({}_R R)$ is similar to the unique minimal faithful left R-module and the double centralizers of these modules are isomorphic to each other. Thus Proposition 1.3 is a generalization of the result for QF-3 algebras by Tachikawa [17] and Mochizuki [11].

COROLLARY 1.4. *Let K be a faithful right ideal of R. If R is left* QF-3 (QF-3′ *or* QF-3$^+$), *then the double centralizer R' of K_R is also left* QF-3 (QF-3′ *or* QF-3$^+$ *respectively*).

Proof. Since every left quotient ring of R can be imbedded naturally into Q, we shall prove that R' is a left quotient ring. For this purpose it is sufficient to show that for any two elements $r_1' \neq 0$ and r_2' of R', there exists an element r of R such that $rr_1' \neq 0$ and $rr_2' \in R$. However, K is faithful, hence there exists an element $k \in K \subseteqq R$ such that $kr_1' \neq 0$ and it is obvious $kr_2' \in K \subseteqq R$.

COROLLARY 1.5. *Let R' be a double centralizer of a faithful right ideal of R. If* dom. dim ${}_R R \geqq 2$, *then* $R' = R$.

2. On left QF-3 rings. Throughout this section we shall assume that Re means always a faithful, projective, injective left ideal of R and Q is the maximal left quotient ring of R.

We shall denote by K a finitely generated, projective, faithful right ideal of R, by C the ring eRe, by D $\mathrm{End}_R(K_R)$ and by U the D-C-bimodule Ke respectively. Then we have

PROPOSITION 2.1. *Let R' be the double centralizer of K_R. If R is a left* QF-3 *ring having Re as a faithful, projective, injective left ideal, then it holds*

(1) ${}_{R'}[\mathrm{Hom}_D\,({}_DK_{R'},\,{}_DU_C)]_C \cong {}_{R'}R'e_C$,

(2) ${}_DU$ is injective.

Proof. Since K_R is faithful, we shall identify a D-endomorphism of ${}_DK$ obtained by the right multiplication of an element r of R with r itself. Then it follows that $\mathrm{Hom}_D\,(K,\,U) = \mathrm{Hom}_D\,(K,\,Ke) \supseteqq Re$. On the other hand, $\mathrm{Hom}_D\,(K,\,U)$ $(= \mathrm{Hom}_D\,(K,\,Ke))$ is a subset of $R' = \mathrm{Hom}_D\,(K,\,K)$. Hence we have that $\mathrm{Hom}_D\,(K,\,U) \subseteqq R'e$. However, it is known by Proposition 1.2 that $Re = R'e$. Thus we have (1).

Next, assume that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\varphi"] & Y' \\ & U & \end{tikzcd}$$

is given, where the row is exact, Y, Y' are left D-modules and j, φ are left D-homomorphisms. Then we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & {[\mathrm{Hom}_D\,(K,\,Y)]} \arrow[r, "\mathrm{Hom}\,(1,\,j)"] \arrow[d, "\mathrm{Hom}\,(1,\,\varphi)"] & {[\mathrm{Hom}_D\,(K,\,Y')]} \arrow[dl, dashed, "\Phi"] \\ & {[\mathrm{Hom}_D\,(K,\,U)]}\,. & \end{tikzcd}$$

By (1), ${}_R[\mathrm{Hom}_D\,(K,\,U)]$ is isomorphic to Re and hence injective. Therefore there exists a dotted R-homomorphism Φ so as to make the above diagram commutative. Further, we have the next commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \underset{R}{\otimes} \mathrm{Hom}_D\,(K,\,Y) \arrow[r, "1\otimes \mathrm{Hom}\,(1,\,j)"] \arrow[d, "1\otimes \mathrm{Hom}\,(1,\,\varphi)"] & K \underset{R}{\otimes} \mathrm{Hom}_D\,(K,\,Y) \arrow[dl, "1\otimes\Phi"] \\ & K \otimes \mathrm{Hom}_D\,(K,\,U)\,. & \end{tikzcd}$$

Since K is a finitely generated, projective right R-module, the functor $K \otimes_R \mathrm{Hom}_D(K, -)$ is naturally equivalent to the identity functor on the category of left D-modules. It follows that ${}_DU$ is injective. This completes the proof.

Let ${}_R\mathscr{M}$ and ${}_D\mathscr{M}$ be categories of left R-modules and left D-modules respectively. We shall define covariant functors $S: {}_D\mathscr{M} \longrightarrow {}_R\mathscr{M}$ and $T: {}_R\mathscr{M} \longrightarrow {}_D\mathscr{M}$ by $S(Y) = {}_R[\mathrm{Hom}_D(K, Y)]$ for $Y \in {}_D\mathscr{M}$ and $T(X) = {}_D[K \otimes_R X]$ for $X \in {}_R\mathscr{M}$ respectively. Then, since K is finitely generated, projective it is well known that there exists a natural equivalence $\sigma: TS \longrightarrow 1_{D\mathscr{M}}$, where $1_{D\mathscr{M}}$ means the identity functor on ${}_D\mathscr{M}$ and $\sigma(Y)\ (r \otimes f) = f(r)$ for $r \in K$, $f \in \mathrm{Hom}_D(K, Y)$. On the other hand, there exists a natural transformation (not necessarily an equivalence) $\tau: 1_{R\mathscr{M}} \longrightarrow ST$, where $1_{R\mathscr{M}}$ means the identity functor on ${}_R\mathscr{M}$ and $[\tau(X)](r) = r \otimes x$, for $x \in X$, $r \in K$.

Assume that ${}_RW$ is isomorphic to a direct product $\prod_{\lambda \in \Lambda} Re^{(\lambda)}$, $Re^{(\lambda)} \cong Re$. Since K_R is finitely generated, projective, by (1) of Proposition 2.1 we have that

$$\begin{aligned} {}_RW &\cong \prod_{\lambda \in \Lambda} Re^{(\lambda)} \cong \prod_{\lambda \in \Lambda} \mathrm{Hom}_D(K, Ke^{(\lambda)}) \cong \mathrm{Hom}_D(K, \prod_{\lambda \in \Lambda} Ke^{(\lambda)}) \\ &\cong \mathrm{Hom}_D(K, K \otimes_R \prod_{\lambda \in \Lambda} Re^{(\lambda)}) \cong \mathrm{Hom}_D(K, K \otimes_R W) = ST(W) \end{aligned}$$

and the composite of all isomorphisms is $\tau(W)$.

Now, we shall introduce a special injective dimension closely connected with ${}_DU$. Let Y be a left D-module. Then we shall say that Y has U-injective dimention $\geqq n$ (denoted by U-inj. dim ${}_DY$), if there exists a following injective resolution of Y:

$$0 \longrightarrow Y \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n$$

such that all V_i, $1 \leqq i \leqq n$, are isomorphic to direct products of copies of U. It is to be noted that this notion can be defined for any injective module.

Then we have

PROPOSITION 2.2. *Let $\mathscr{A}$ be the category consisting of left R-modules X such that* dom. dim ${}_RX \geqq 2$ *and $\mathscr{B}$ the category of left D-modules Y such that* U-inj. dim ${}_DY \geqq 2$. *Then $\mathscr{A}$ and $\mathscr{B}$ are equivalent by S and T.*

Proof. If $X \in \mathscr{A}$, then we have a injective resolution of X:

$$0 \longrightarrow X \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots, \quad \text{where} \quad W_i,\ i = 1, 2,$$

are torsionless and injective. Since ${}_RRe$ is faithful, ${}_RR$ is imbedded into a direct product of copies of Re. On the other hand, every torsionless, injective left R-module is imbedded into a direct product of copies of ${}_R R$. Thus, every torsionless injective left R-module is imbedded into a direct product of copies of Re. Hence, without loss of generality we can assume that W_i are isomorphic to $\prod_{\lambda \in \Lambda_i} Re^{(\lambda)}$, $i = 1, 2$, $Re^{(\lambda)} \cong Re$. It follows by the finitely generated projectivity and the faithfulness of K_R that U-inj. $\dim {}_D T(X) \geqq 2$, if $X \in \mathscr{A}$. Conversely dom. $\dim {}_R S(Y) \geqq 2$, if $Y \in \mathscr{B}$, by Proposition 2.1 (1). Now, by the exposition preceding this proposition, it is enough to prove that the restriction of $\tau: 1_{{}_R\mathscr{M}} \longrightarrow ST$ to $\mathscr{A}$ is an equivalence. However, from the above remark and the exposition, in the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & W_1 & \longrightarrow & W_2 \\ & & \downarrow{\scriptstyle \tau(X)} & & \downarrow{\scriptstyle \tau(W_1)} & & \downarrow{\scriptstyle \tau(W_2)} \\ 0 & \longrightarrow & ST(X) & \longrightarrow & ST(W_1) & \longrightarrow & ST(W_2) \end{array}$$

we know that $\tau(W_1)$ and $\tau(W_2)$ are left R-isomorphisms. Hence by the Five lemma it follows that $\tau(X)$ is a left R-isomorphism.

THEOREM 2.3. *Let R be a right perfect ring. If R is left* QF-3, *then there exists an idempotent f of R such that fR_R is faithful, and the category $\mathscr{A}$ consisting of all left R-modules of dominant dimension at least two is equivalent to the category ${}_{fRf}\mathscr{M}$ of all left fRf-module.*

Proof. Since R is a right perfect ring, every nonzero left R-module has nonzero socle and there exists at most a finite number of non-isomorphic irreducible left R-modules. Therefore we can assume that the socle S of Re is a direct sum of finite number of non-isomorphic irreducible left R-module. Then there is an idempotent f of R such that Rf/Nf is isomorphic to ${}_R S$, where N is the Jacobson radical of R.

Suppose that fR_R is not faithful. Then there exists an nonzero element r of R such that $fRr = 0$. Since Re is faithful, there exists an element x of Re such that $rx \neq 0$. Hence we have a nonzero submodule Rrx of Re. It follows that $Rrx \cap S = 0$, for Re is an essential extension of S. Therefore $fRrx \neq 0$. Thus we have that and $fRr \neq 0$, but this is a contradiction.

Now, it is known that we can set fR as K in Proposition 2.2 and fRf as D. It follows that ${}_{fRf}fRe$ is injective. Further, the condition that $Rf/Nf \cong {}_R S$ insure that ${}_{fRf}fRe$ is cogenerator. Hence every left fRf-module has fRe-injective dimension $\geqq 2$ (in fact $= \infty$). This completes the proof by Proposition 2.2.

By similar proofs as in [14, Proposition 2.6 and Theorem 2.4] and [13, Corollary 5.3] we have further

THEOREM 2.4. *Under the same assumption as in Theorem* 2.3, *the following facts hold*:

(1) *Let X be a left R-module.* $\mathrm{dom.\,dim}\,_RX \geqq 2$ *if and only if* $X \cong {}_R\mathrm{Hom}_D(fR, Y)$ *for a left D-module Y where $D = fRf$.*

(2) *If* $[\mathrm{End}_D(fR)]^\circ = R'$, *then* $R' = Q$ (= *the double centralizer of* $E({}_RR)$)[3].

(3) *If* $\mathrm{dom.\,dim}\,_RX = 1$, *then there exists a left R-module X' such that* $X' \supset X$ *and such that* $\mathrm{dom.\,dim}\,_RX' \geqq 2$. *Further,* $\mathrm{dom.\,dim}\,_RX'' = 1$ *and* $\mathrm{dom.\,dim}\,_RX'/X'' = 0$, *if* $X' \supset X'' \supset X$.

(4) *The following two conditions are equivalent* ($n \geqq 1$).

I. $\mathrm{dom.\,dim}\,_RX \geqq n+1$

II. (a) ${}_R[\mathrm{Hom}_D(fR, fR \otimes_R X)] \cong {}_RX$,

(b) $\mathrm{Ext}_D^k(fR, fR\otimes_R X) = 0$ *for* $1 \leqq k \leqq n-1$.

Now for given left modules M and U over a ring D, we shall say that U is a M-cogenerator if for a nonzero left D-homomorphism f of M into itself there exists a left D-homomorphism φ of M into U such that the composite of f and φ is a nonzero left D-homomorphism. Then we prove

PROPOSITION 2.5. *Let M be a left module over a ring D such that* (i) ${}_DM$ *is a generator and* (ii) *M has an injective, M-cogenerator submodule* ${}_DU$. *Then the inverse D-endomorphism ring R of* ${}_DM$ *is left* QF-3. *Conversely, every left* QF-3 *ring is obtained in the above way for a suitable ring D and* ${}_DM$ *which satisfy* (i) *and* (ii).

Proof. Assume that ${}_DM$ satisfies the conditions (i) and (ii). Since ${}_DM$ is a generator, M_R is finitely generated, projective and hence ${}_R[\mathrm{Hom}_D({}_DM_R, {}_DU)]$ is injective. On the other hand, if we denote by e the projection of M onto U, ${}_R[\mathrm{Hom}_D({}_DM_R, {}_DU)]$ is Re and hence is projective. Let r be a nonzero element of R. By (ii) there exists an element $r' \in Re$ such that $rr' \neq 0$. Thus Re is faithful and R is left QF-3.

To prove the converse, we have only to take R as D, R as M and Re as U respectively.

In view of Proposition 2.5, it seems of interest to obtain a method by which we can calculate $\mathrm{dom.\,dim}\,_RR$ in case $R = [\mathrm{End}_D(M)]^0$ and

[3] (2) follows from (1) and Theorem 1.5 by putting $Y = fR$ and (2) implies that every right perfect left QF-3 ring R' of dominant dimension ≥ 2 is obtained as an inverse fRf-endomorphism ring of a generator-cogenerator fRf-module. (cf. Kato [8].)

${}_DM$ is a generator. However it was done already by Morita [13], when D is an Artinian ring. We shall remark here that his condition concerned with the self injective dimension can be replaced by another condition related to Ext-functors.

PROPOSITION 2.6. *Assume that D, M and R retain their meanings in Proposition* 2.5. *Then* I *implies* II, *where* I *and* II *are the following two conditions for a left R-module X*:

I. dom. dim ${}_RX \geqq n+1$ $(n \geqq 1)$.

II. (a) ${}_R[\mathrm{Hom}_D({}_DM_R, {}_DM \otimes_R X)] \cong {}_RX$

(b) $\mathrm{Ext}_D^k(P \oplus M, M \otimes_R X) = 0$ *for* $1 \leqq k \leqq n-1$ *and for every maximal left ideal P of D such that D/P is not isomorphic to any submodule of U.*

The two conditions are equivalent, if D is a right perfect ring. Similarly, if ${}_DU$ is a cogenerator, the two conditions are equivalent, provided we replace $P \oplus M$ in II (b) *by M.*

Proof. Let $S = \mathrm{Hom}_D({}_DM, -)$ and $T = M \otimes_R -$ be two covariant functors. Assume I. There exists an exact sequence

$$(1) \qquad 0 \longrightarrow X \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{n+1}$$

such that all W_i are direct products of $S(U)$[4]. Then, in the following commutative diagram:

$$(2) \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & X & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & \cdots & \longrightarrow & W_{n+1} \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & & & \downarrow{\scriptstyle \tau_{n+1}} \\ 0 & \longrightarrow & ST(X) & \longrightarrow & ST(W_1) & \longrightarrow & ST(W_2) & \longrightarrow & \cdots & \longrightarrow & ST(W_{n+1}) \end{array}$$

$\tau_1, \tau_2, \cdots, \tau_{n+1}$ are isomorphisms, becacuse M_R is finitely generated, projective. Hence τ is an isomorphism and this implies

II. (a) $\mathrm{Hom}_D({}_DM, {}_DM \otimes_R X) \cong {}_RX$.

On the other hand, we have the following exact sequence

$$(3) \qquad 0 \longrightarrow T(X) \longrightarrow T(W_1) \longrightarrow T(W_2) \longrightarrow \cdots \longrightarrow T(W_{n+1})$$

where all $T(W_i)$ are isomorphic to direct products of ${}_DU(= TS(U))$. Since $T(W_i)$ are injective, the sequence (3) can be consider as an injective resolution of $T(X)$. Hence by the exactness of the bottom sequence in (2) it follows that

[4] By Proposition 2.5 we know $Re \cong S(U)$.

(4) $\operatorname{Ext}_D^k(M, T(X)) = \operatorname{Ext}_D^k(M, M \underset{R}{\otimes} X) = 0, \qquad 1 \leqq k \leqq n-1.$

Let

(5) $$0 \longrightarrow T(X) \xrightarrow{\sigma_0} V_1 \xrightarrow{\sigma_1} V_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_{n+1}$$

be the minimal injective resolution of $T(X)$. Then, by [15, Lemma 1] V_i is isomorphic to a direct summand of a direct product $T(W_i)$ of U. Hence any irreducible submodule of V_i is isomorphic to a submodule of U. Thus by [15, Lemma 7] we have that $\operatorname{Ext}_D^k(B, T(X)) = \operatorname{Ext}_D^k(B, M \otimes_R X) = 0$ for all $0 \leqq k \leqq n$ and for all irreducible left D-modules B which are not isomorphic to submodules of U.

Now, let P be a maximal left ideal of D such that $D/P \cong B$. Then,

(6) $\operatorname{Ext}_D^k(P, M \underset{R}{\otimes} X) = \operatorname{Ext}_D^{k+1}(B, M \underset{R}{\otimes} X) = 0, \qquad 1 \leqq k \leqq n-1\,.$

Hence form (4) and (6) we obtain

II. (b) $\operatorname{Ext}_D^k(P \oplus M, M \otimes_R X) = 0, \qquad 1 \leqq k \leqq n-1.$

Conversely, assume II. Clearly II (b) implies (4) and (6). Since S is left exact, it follows from II (a) and (4) that

$$0 \longrightarrow X \xrightarrow{S(\sigma_0)\theta} S(V_1) \xrightarrow{S(\sigma_1)} S(V_2) \xrightarrow{S(\sigma_2)} \cdots \xrightarrow{S(\sigma_n)} S(V_{n+1})$$

is an injective resolution of X, where θ is a given isomorphism: $X \longrightarrow \operatorname{Hom}_D(M, M \otimes_R X)(= ST(X))$ in II (a).

Now, we shall assume that D is right perfect. Since every nonzero left D-module has a nonzero socle, each V_i is the injective hull of its socle. However, by [15, Lemma 7] it follows from (6) that B is not isomorphic to a submodule of V_i. Hence V_i is imbedded into a direct product of U and consequently $S(V_i)$ is imbedded into a direct product of $S(U)$. Thus dom. dim ${}_RX \geqq n+1$ by [15, Lemma 1].

In case ${}_DU$ is a cogenerator, V_i is imbedded into a direct product of U and the converse also holds[5] by [15, Lemma 1].

Especially, dom. dim ${}_RR$ is characterized only by the vanishing of $\operatorname{Ext}_D^k(P \oplus M, M)$, $k = 1, 2, \cdots$.

3. Perfect QF-3 rings. Following Thrall's paper [18] we shall say that R has a minimal faithful left module L if L is a faithful left R-module and if L appears as a direct summand of every faithful left R-module. It is clear that L is projective, and injective, and is isomorphic to some left ideal direct summand of R. Jans

[5] This proof can be regarded as a proof of Theorem 2.8, (4) for the case $R = [\operatorname{End}_D(fR)]^0$, since fRf is right perfect and ${}_DfR$ is a cogenerator.

proved in [5] that semi-primary ring has a minimal faithful left module, if $E({}_RR)$ is projective.

The next propositions show the equivalence between notions of left QF-3 and left QF-3′ for right perfect rings and show a necessary and sufficient condition for perfect, left QF-3′ rings to be left QF-3$^+$.

PROPOSITION 3.1. *If R is a right perfect ring, then the following conditions are equivalent:*

(1) *$E({}_RR)$ is torsionless, i.e., R is left* QF-3′.

(2) *R has a minimal faithful left module.*

(3) R *has a faithful, projective, injective ideal, i.e., R is left* QF-3.

PROPOSITION 3.2. *Let R be a perfect left* QF-3′ *ring. Then R is left* QF-3$^+$ *if and only if the socle of ${}_RR$ is finitely generated.*

For the proofs we shall refer to that of Proposition 2 and Theorem 2 in [4], which are known to be valid, if we consider that for right perfect rings, every nonzero left module has nonzero socle and for left perfect rings, every projective module is isomorphic to a direct sum of primitive left ideals.

PROPOSITION 3.3. *Let R be a perfect ring. If R is left and right* QF-3′, *then R is left and right* QF-3$^+$.

Proof. By Proposition 3.1 we may assume that R has faithful, projective, injective left ideal Re and right ideal fR, where e and f are idempotents of R. Then it is seen by Proposition 2.1, (2) fRe is an injective left fRf-module as well as an injective right eRe-module. Further, by Proposition 2.3, without loss of generality we can assume that fRe is a cogenerator in the category of all left fRf-modules and the category of all right eRe-modules respectively. By Proposition 2.1, (1) we have that $\mathrm{Hom}_{fRf}(fRe, fRe) \cong eRe^\circ{}_{eRe}$ and $\mathrm{Hom}_{eRe}(fRe, fRe) \cong {}_{fRf}fRf$ and hence $\mathrm{End}_{fRf}(fRe) \cong eRe^\circ$ and $\mathrm{End}_{eRe}(fRe) \cong fRf$. Therefore the fRe-duality between categories of finitely generated left fRf-modules and finitely generated right eRe-modules holds, and by Proposition 2.1, (1) we have that $\mathrm{Hom}_{fRf}(fR, fRe) \cong Re$ and $\mathrm{Hom}_{eRe}(Re, fRe) \cong fR$ which implies fR and Re are both fRe-reflexive in the sense of Cohn [3]. It follows by [12, Lemma 2.2] that the socle of fR is reflexive. Since fRf is a perfect ring, the socle of ${}_{fRf}fR$ is a nonzero submodule and hence it is isomorphic to a direct sum of a finite number of irreducible fRf-modules. Thus, for an integer n, we have an exact sequence:

$$0 \longrightarrow {}_{fRf}fR \longrightarrow \sum_{i=1}^{n} \oplus [fRe]. \quad \text{Hence}$$

$$\begin{array}{ccc} 0 \longrightarrow \operatorname{Hom}_{fRf}(fR, fR) & \longrightarrow & \sum_{i=1}^{n} \oplus [\operatorname{Hom}_{fRf}(fR, fRe)] \\ \| & & \wr\| \\ R' & & \sum_{i=1}^{n} \oplus [Re] \end{array}$$

is exact and consequently, $E({}_RR)(=E({}_RR'))$ can be imbedded into a direct sum of finite number of copies of Re. Thus $E({}_RR)$ is projective. By symmetry it can be proved that $E(R_R)$ is projective.

4. Examples. 1. As was remarked in the introduction, we shall give a left QF-3′ ring R such that the double centralizer of a faithful left ideal of R is not left QF-3′. For this purpose first we shall refer to

PROPOSITION 4.1. *Let R be a left primitive ring with a minimal ideal M. If R is left* QF-3′, *then R is left* QF-3 *and M is a faithful, projective, injective ideal.*

Proof. Since ${}_RM$ is imbedded into $E({}_RR)$, the injective hull $E({}_RM)$ of M is imbeded into $E({}_RR)$. Hence we have an exact sequence:

$$0 \longrightarrow E({}_RM) \xrightarrow{\varphi} \prod_{\nu \in \Lambda} R^{(\nu)} ,$$

where ${}_RR^{(\nu)} \cong R$, $\bigcap_{\nu \in \Lambda} \operatorname{Ker} \varphi_\nu = 0$, provided φ_ν are defined by $\varphi(x) = (\cdots, \varphi_\nu(x), \cdots)$, $x \in E({}_RM)$. Now, suppose $\operatorname{Ker} \varphi_\nu \neq 0$ for all φ_ν, $\nu \in \Lambda$. Since M is essential in $E({}_RM)$, $\operatorname{Ker} \varphi_\nu \cap M \neq 0$. It follows that $\operatorname{Ker} \varphi_\nu \supseteqq M$, because M is minimal. This implies $\bigcap_{\nu \in \Lambda} \operatorname{Ker} \varphi_\nu \supseteqq M \neq 0$, and this is a contradiction. Therefore $E({}_RM)$ can be imbedded into R, and hence $E({}_RM)$ is projective. Since R has a faithful, projective, injective ideal $E({}_RM)$, R is left QF-3.

On the other hand, ${}_RM$ is faithful and hence R can be imbedded into a direct product of copies of M. Hence, similarly as above we can prove that $E({}_RM)$ can be imbedded into M. However M is minimal and hence M is isomorphic to $E({}_RM)$. Thus M is projective and injective.

Let K be a field and V an infinite dimensional left K-vector space. Denote by F the inverse K-endomorphism ring of ${}_KV$. Then F is a primitive ring and has a pair of projective, minimal ideals M and N such that ${}_FM \cong {}_FV^* = \operatorname{Hom}_K({}_KV_F, {}_KK)$, $N_F \cong V_F$. Since N_F is projective, ${}_FM$ is injective and hence F is left QF-3. However F is not right QF-3′. For otherwise, it would follow by Proposition 4.1 that F is right QF-3 and N is injective. Then, by [6, Th. 1]

$$V_F \cong N \cong M^* = \mathrm{Hom}_K({}_F M_K, K_K) = V_F^{**},$$

but it is impossible, because $(V:K) = \infty$.

Now, consider the double centralizer F' of ${}_F M$. Since $[\mathrm{End}_F(M)]^0 = K$, it follows $F' = \mathrm{End}_K(M)$. Then, similarly as above we can prove that F' is not left QF-3′. Hence the double centralizer F' of a faithful left ideal M of a left QF-3 ring F need not be left QF-3′.

2. The following example shows that perferct left QF-3 rings are not necessarily semi-primary. Let K be a field, ${}_K V_s$ s-dimensional vector spaces over K, $s = 1, 2, \cdots$. Denote by M the direct sum of all V_s. Then every element of $[\mathrm{End}_K(M)]^0$ can be considered as a row finite matrix. Let A be a subring of $[\mathrm{End}_K M)]^0$ such that each element of A has the following matrix representation $\lambda E + \sum_s T_s$, where $\lambda \in K$, E is the identity matrix, T_s is the zero matrix for almost all s and T_s is written by

$$\sum_{\substack{p-1 \geqq i \geqq q+1 \\ p-2 \geqq j \geqq q \\ i > j}} \tau_{ij} C_{ij}, \quad \text{where} \quad p = (s+1)(s+2)/2,\ q = s(s+1)/2 \quad \text{and} \quad C_{ij}$$

means the matrix with 1 in the (i, j) position and 0's elsewhere and $\tau_{ij} \in K$. Then, T_s is a lower triangular matrix and every element of the Jacobson radical N of A is a sum of T_s's.

Now, consider a ring $R = Kc_{11} + M^* c_{21} + Kc_{31} + Ac_{22} + Mc_{32} + Kc_{33}$, where $M^* = {}_A[\mathrm{Hom}_K({}_K M_A, {}_K K_K)]_K$ and $+$ means the direct sum as K-modules, and the multiplication of c_{ij}, $1 \leqq i, j \leqq 3$, is same as that of matrix units and $mf = f(m)$ for $f \in M^*$, $m \in M$.

Then primitive idempotents of R are $c_{11} = e_1$, $c_{22} = e_2$ and $c_{33} = e_3$, and the Jacobson radical J of R is $M^* c_{21} + Kc_{31} + Nc_{22} + Mc_{32}$. Since $R/J = K \oplus K \oplus K$ and J is left and right T-nilpotent, R is perfect. However R is not semi-primary, for J is not nilpotent. Since $Re_1 \cong {}_R[\mathrm{Hom}_K({}_K e_3 R, {}_K K)]$, Re_1 is a faithful, projective, injective left ideal of R and hence R is left QF-3. On the other hand, by Proposition 3.2 R is not left QF-3$^+$, because the socle of ${}_R R$ is not finitely generated.

3. The next example shows that every right Artinian QF-3 ring is not necessarily left Artinian,[6] while every Artinian self-injective ring (i.e., quasi-Frobenius ring) is left Artinian.

Let Q and P be skewfields such that P is a subfield of Q and the right dimension of Q over P is finite and the left dimension of Q over P is infinite. The existence of such skewfields was proved by Cohn [2]. Similarly as in Example 2, consider a ring R such that

[6] It was proved by K. Morita [**13**, Th. 1.1] that for left or right Artinian rings "left QF-3" implies "right QF-3".

$Qc_{11}+Qc_{21}+Qc_{31}+Pc_{22}+Qc_{32}+Qc_{33}$, that is to say, R is a subring of the matrix ring Q_3. Then it is clear that R is right Artinian but not left Artinian. On the other hand, R is semi-primary and Rc_{11} and $c_{33}R$ are faithful, projective, injective, left and right ideals respectively.

4. The next example shows a non-perfect QF-3 ring.

Let K be the field of rational numbers and Z the ring of rational integers. Consider a subring R of the matrix ring K_3 such that $Kc_{11}+Kc_{21}+Kc_{31}+Zc_{22}+Kc_{32}+Kc_{33}$. It is clear that $E({}_RR)$(resp. $E(R_R)$) is isomorphic to the direct sum of 3-copies of a projective, injective ideal Rc_{11}(resp. $c_{33}R$). Hence R is QF-3, while R is not perfect.

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Received November 6, 1968.

TOKYO UNIVERSITY OF EDUCATION

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

PRODUCT INTEGRAL REPRESENTATION OF TIME DEPENDENT NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

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The object of this paper is to use the method of product integration to treat the time dependent evolution equation $u'(t) = A(t)(u(t))$, $t \geqq 0$, where u is a function from $[0, \infty)$ to a Banach space S and A is a function from $[0, \infty)$ to the set of mappings (possibly nonlinear) on S. The basic requirements made on A are that for each $t \geqq 0$ $A(t)$ is the infinitesimal generator of a semi-group of nonlinear nonexpansive transformations on S and a continuity condition on $A(t)$ as a function of t.

The product integration method has been used by T. Kato in [5] to treat evolution equations in which $A(t)$ is the infinitesimal generator of a semi-group of linear contraction operators. In [6] Kato treats the nonlinear evolution equation in which $A(t)$ is m-monotone and the Banach space S is uniformly convex. For other investigations of nonlinear evolution equations one should see P. Sobolevski [9], F. Browder [1], J. Neuberger [8], and J. Dorroh [3].

1. Definitions and theorems.

In this section definitions and theorems will be stated. For examples satisfying the definitions and theorems below, one should see § 4. Let S denote a real Banach space.

DEFINITION 1.1. The function T from $[0, \infty)$ to the set of mappings (possibly nonlinear) on S will be said to be a $\mathscr{C}$-semi-groups of mappings on S provided that the following are true:

(1) $T(x + y) = T(x)T(y)$ for $x, y \geqq 0$.

(2) $T(x)$ is nonexpansive for $x \geqq 0$.

(3) If $p \in S$ and $g_p(x)$ is defined as $T(x)p$ for $x \geqq 0$ then g_p is continuous and $g_p(0) = p$.

(4) The infinitesimal generator A of T is defined on a dense subset D_A of S (i.e., if $p \in D_A g_p'^{+}(0)$ exists and $Ap = g_p'^{+}(0)$) and if $p \in D_A g_p'^{+}(x) = Ag_p(x)$ for $x \geqq 0$, $g_p(x) = p + \int_0^x Ag_p(u)du$ for $x \geqq 0$, $g_p'^{+}$ is continuous from the right on $[0, \infty)$, and $\| g_p'^{+} \|$ is nonincreasing on $[0, \infty)$.

DEFINITION 1.2. The mapping A from a subset of S to S will be said to be a $\mathscr{C}$-mapping on S provided that the following are true:

(1) The domain D_A of A is dense in S.

(2) A is monotone on S, i.e., if $\varepsilon > 0$ and

$$p, q \in D_A \, \| (I - \varepsilon A)p - (I - \varepsilon A)q \| \geqq \| p - q \| .$$

(3) A is m-monotone on S, i.e. A is monotone on S and if $\varepsilon > 0$ then Range $(I - \varepsilon A) = S$.

(4) A is the infinitesimal generator of a $\mathscr{C}$-semi-group of mappings on S.

DEFINITION 1.3. Let each of m and n be a nonnegative integer and for each integer i in $[m, n]$ let K_i be a mapping from S to S. If $m > n$ define $\prod_{i=m}^{n} K_i = I$. If $m \leqq n$ define $\prod_{i=m}^{m} K_i = K_m$ and if $m + 1 \leqq j \leqq n$ define $\prod_{i=m}^{j} K_i = K_j \prod_{i=m}^{j-1} K_i$. Define $\prod_{n}^{i=m} K_i = \prod_{i=m}^{n} K_{n+m-i}$. If each of a and b is a nonnegative number then a chain $\{s_i\}_{i=0}^{2m}$ from a to b is a nondecreasing or nonincreasing number-sequence such that $s_0 = a$ and $s_{2m} = b$. The norm of $\{s_i\}_{i=0}^{2m}$ is $\max \{ | s_{2i} - s_{2i-2} | \mid i \in [1, m] \}$.

DEFINITION 1.4. Let F be a function from $[0, \infty) \times [0, \infty)$ to the set of mappings on S. Suppose that $p \in S$, $a, b \geqq 0$, and u is a point in S such that if $\varepsilon > 0$ there exists a chain $\{s_i\}_{i=0}^{2m}$ from a to b such that if $\{t_i\}_{i=0}^{2n}$ is a refinement of $\{s_i\}_{i=0}^{2m}$ then

$$\left\| u - \prod_{i=1}^{n} F(t_{2i-1}, | t_{2i} - t_{2i-2} |) p \right\| < \varepsilon .$$

Then u is said to be the product integral of F from a to b with respect to p and is denoted by $\prod_a^b F(I, dI)p$.

REMARK 1.1. Let A be a $\mathscr{C}$-mapping on S and define the function F from $[0, \infty) \times [0, \infty)$ to the set of mappings on S by $F(u, v) = (I - vA)^{-1}$ for $u, v \geqq 0$ (Note that $(I - vA)^{-1}$ exists and has domain S by virtue of the m-monotonicity of A). The following result in [10] will be used in the theorems below:

If A is a $\mathscr{C}$-mapping on S, T is the $\mathscr{C}$-semi-group generated by A, and F is defined as above, then for $p \in S$ and $x \geqq 0$ $T(x)p = \prod_0^x F(I, dI)p$.

In this case let $T(x)$ be denoted by $\exp(xA)$ for $x \geqq 0$.

Let A be a function from $[0, \infty)$ to the set of mappings on S such that the following are true:

(I) For each $t \geqq 0$ $A(t)$ is a $\mathscr{C}$-mapping on S

(II) There is a dense subset D of S such that if $t \geqq 0$ the domain of $A(t)$ is D

(III) A is continuous in the following sense: If $a, b \geqq 0$, M is a bounded subset of D, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$ and $| u - v | < \delta$ then $\| A(u)z - A(v)z \| < \varepsilon$ for each $z \in M$.

THEOREM 1. *Let A satisfy conditions* (I), (II) *and* (III). *If $p \in S$ and $a, b \geqq 0$ the following are true*:

(1) *If $T(u, v) = \exp(vA(u))$ for $u, v \geqq 0$, then $\prod_a^b T(I, dI)p$ exists.*

(2) *If $L(u, v) = (I - vA(u))^{-1}$ for $u, v \geqq 0$, then $\prod_a^b L(I, dI)p$ exists and $\prod_a^b L(I, dI)p = \prod_a^b T(L, dI)p$.*

THEOREM 2. *Let A satisfy conditions* (I), (II) *and* (III) *and define $U(b, a)p = \prod_a^b T(I, dI)p$ for $p \in S$ and $a, b \geqq 0$. The following are true*:

(1) *$U(b, a)$ is nonexpansive for $a, b \geqq 0$.*

(2) *$U(b, c)U(c, a) = U(b, a)$ for $a, b \geqq 0$ and $c \in [a, b]$ and $U(a, a) = I$ for $a \geqq 0$.*

(3) *If $p \in S$ and $a \geqq 0$ then $U(a, t)p$ is continuous in t*

(4) *If $p \in S, 0 \leqq a \leqq t$, and $U(t, a)p \in D$, then $\partial^+ U(t, a)p/\partial t = A(t)U(t, a)p$ and if $p \in S, 0 < s \leqq b$, and $U(s, b)p \in D$, then*

$$\partial^- U(s, b)p/\partial s = -A(s)U(s, b)p .$$

2. **Product integral representations.** In this section, Theorems 1 and 2 will be proved. Before proving part (1) of Theorem 1 three lemmas will be proved each under the hypothesis of Theorem 1.

LEMMA 1.1. *If $p \in D, a, b \geqq 0$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to b then*

$$\left\|\prod_{i=1}^{m} T(s_{2i-1}, |s_{2i} - s_{2i-2}|)p - p\right\| \leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \, \| A(s_{2i-1})p \| .$$

Proof.

$$\begin{aligned}
&\left\|\prod_{i=1}^{m} T(s_{2i-1}, |s_{2i} - s_{2i-2}|)p - p\right\| \\
&\leqq \sum_{i=1}^{m}\left\|\prod_{j=i}^{m} T(s_{2j-1}, |s_{2j} - s_{2j-2}|)p - \prod_{j=i+1}^{m} T(_{2j-1}, |s_{2j} - s_{2j-2}|)p\right\| \\
&\leqq \sum_{i=1}^{m} \| T(s_{2i-1}, |s_{2i} - s_{2i-2}|)p - p \| \\
&= \sum_{i=1}^{m}\left\|\int_0^{|s_{2i}-s_{2i-2}|} A(s_{2i-1})T(s_{2i-1}, t)pdt\right\| \\
&\leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \cdot \| A(s_{2i-1})p \| .
\end{aligned}$$

LEMMA 1.2. *If $p \in D, a, b \geqq 0$, $\{s_i\}_{i=0}^{2m}$ is a chain from a to b, and $\{s_i'\}_{i=1}^{m}$ is a sequence in $[a, b]$, then*

$$\left\|\prod_{i=1}^{m} L(s_i', |s_{2i} - s_{2i-2}|)p - p\right\| \leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \, \| A(s_i')p \| .$$

Proof.

$$\left\|\prod_{i=1}^{m} L(s_i', |s_{2i} - s_{2i-2}|)p - p\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|\prod_{j=i}^{m} L(s_j', |s_{2j} - s_{2j-2}|)p - \prod_{j=i+1}^{m} L(s_j', |s_{2j} - s_{2j-2}|)p\right\|$$
$$\leqq \sum_{i=1}^{m} \| L(s_i', |s_{2i} - s_{2i-2}|)p - p \|$$
$$= \sum_{i=1}^{m} \| L(s_i', |s_{2i} - s_{2i-2}|)p$$
$$- L(s_i', |s_{2i} - s_{2i-2}|)(I - |s_{2i} - s_{2i-2}| A(s_i'))p \|$$
$$\leqq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \cdot \| A(s_i')p \| .$$

LEMMA 1.3. *If M is a bounded subset of D, $a, b \geqq 0$, $\gamma > 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$, $|u - v| < \delta$, $0 \leqq x < \gamma$, and $z \in M$, then $\| T(u, x)z - T(v, x)z \| \leqq x \cdot \varepsilon$.*

Proof. Let $M' = \{\prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \mid z \in M, v \in [a, b], 0 \leqq x < \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from 0 to $x\}$. Let $z_0 \in M$, let $z \in M$, let $v \in [a, b]$, let $0 \leqq x < \gamma$, and let $\{s_i\}_{i=0}^{2m}$ be a chain from 0 to x. Then,

$$\left\|\prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z - \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z_0\right\| \leqq \| z - z_0 \| .$$

Further, by Lemma 1.2,

$$\left\|\prod_{i=1}^{m} L(v, s_{2i} - s_{2i-i})z_0 - z_0\right\| \leqq x \cdot \max_{u \in [0,x]} \| A(u)z_0 \| .$$

Then, $\| \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \| \leqq \| z - z_0 \| + \| z_0 \| + x \cdot \max_{u \in [0,\gamma]} \| A(u)z_0 \|$ and so M' is bounded. There exists $\delta > 0$ such that if $u, v \in [a, b]$, $|u - v| < \delta$, and $z \in M'$, then $\| A(u)z - A(v)z \| < \varepsilon$. Then if $0 \leqq x < \gamma$, $z \in M$, $\{s_i\}_{i=0}^{2m}$ is a chain from 0 to x, $u, v \in [a, b]$, and $|u - v| < \delta$,

$$\left\|\prod_{i=1}^{m} L(u, s_{2i} - s_{2i-2})z - \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|\prod_{j=i}^{m} L(u, s_{2j} - s_{2j-2}) \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z\right.$$
$$\left. - \prod_{j=i+1}^{m} L(u, s_{2j} - s_{2j-2}) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|L(u, s_{2i} - s_{2i-2}) \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z\right.$$
$$\left. - \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z\right\|$$
$$\leqq \sum_{i=1}^{m}\left\|\prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z\right.$$
$$\left. - (I - (s_{2i} - s_{2i-2})A(u)) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z\right\|$$

$$\begin{aligned}
&= \sum_{i=1}^{m} (s_{2i} - s_{2i-2})\Big\| A(v) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z \\
&\quad - A(u) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2})z \Big\| \\
&\quad < \sum_{i=1}^{m} (s_{2i} - s_{2i-2}) \cdot \varepsilon \\
&= x \cdot \varepsilon \, .
\end{aligned}$$

Then, since $T(u, x)z = \prod_0^x L(u, dI)z$ and $T(v, x)z = \prod_0^x L(v, dI)z$ (see Remark 1.1), $\| T(u, x)z - T(v, x)z \| \leqq x \cdot \varepsilon$.

Proof of Part (1) *of Theorem* 1. Let $p \in D$, let $a, b \geqq 0$, and let $\varepsilon > 0$. Let $M = \{\prod_{i=1}^m T(r_{2i-1}, | r_{2i} - r_{2i-2} |)p \mid x \in [a, b]$ and $\{r_i\}_{i=0}^{2m}$ is a chain from a to $x\}$. Then M is a bounded subset of D by Lemma 1.1. There exists $\delta > 0$ such that if $u, v \in [a, b]$, $| u - v | < \delta$, $0 \leqq x \leqq 1$ and $z \in M$, then $\| T(u, x)z - T(v, x)z \| \leqq \varepsilon \cdot x$. Let $\{s_i\}_{i=0}^{2m}$ be a chain from a to b with norm $< \min\{\delta, 1\}$ and let $\{t_i\}_{i=0}^{2n}$ be a refinement of $\{s_i\}_{i=0}^{2m}$, i.e., there is an increasing sequence u such that $u_0 = 0$, $u_m = n$, and if $1 \leqq i \leqq m$ $s_{2i} = t_{2u_i}$. For $1 \leqq i \leqq m$ let $K_i = T(s_{2i-1}, | s_{2i} - s_{2i-2} |)$ and let $J_i = \prod_{j=u_{i-1}+1}^{u_i} T(t_{2j-1}, | t_{2j} - t_{2j-2} |)$. Then,

$$\begin{aligned}
&\Big\| \prod_{i=1}^{m} T(t_{2i-1}, | t_{2i} - t_{2i-2} |)p - \prod_{i=1}^{m} T(s_{2i-1}, | s_{2i} - s_{2i-2} |)p \Big\| \\
&\quad = \Big\| \prod_{i=1}^{n} J_i p - \prod_{i=1}^{m} K_i p \Big\| \\
&\quad \leqq \sum_{i=1}^{m} \Big\| \prod_{j=i}^{m} J_j \prod_{k=1}^{i-1} K_k p - \prod_{j=i+1}^{m} J_j \prod_{k=1}^{i} K_k p \Big\| \\
&\quad \leqq \sum_{i=1}^{m} \Big\| J_i \prod_{k=1}^{i-1} K_k p - K_i \prod_{k=1}^{i-1} K_k p \Big\| \\
&\quad = \sum_{i=1}^{m} \Big\| \prod_{j=u_{i-1}+1}^{u_i} T(t_{2j-1}, | t_{2j} - t_{2j-2} |) \prod_{k=1}^{i-1} K_k p \\
&\qquad - \prod_{j=u_{i-1}+1}^{u_i} T(s_{2i-1}, | t_{2j} - t_{2j-2} |) \prod_{k=1}^{i-1} K_k p \Big\| \\
&\quad \leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} \Big\| \prod_{r=j}^{u_i} T(s_{2i-1}, | t_{2r} - t_{2r-2} |) \prod_{h=u_{i-1}+1}^{j-1} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p \\
&\qquad - \prod_{r=j+1}^{u_i} T(s_{2i-1}, | t_{2r} - t_{2r-2} |) \prod_{h=u_{i-1}+1}^{j} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p \Big\| \\
&\quad \leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} \Big\| T(s_{2i-1}, | t_{2j} - t_{2j-2} |) \prod_{h=u_{i-1}+1}^{j-1} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p \\
&\qquad - T(t_{2j-1}, | t_{2j} - t_{2j-2} |) \prod_{h=u_{i-1}+1}^{j-1} T(t_{2h-1}, | t_{2h} - t_{2h-2} |) \prod_{k=1}^{i-1} K_k p \Big\| \\
&\quad \leqq \sum_{i=1}^{m} \sum_{j=u_{i-1}+1}^{u_i} | t_{2j} - t_{2j-2} | \cdot \varepsilon = | b - a | \cdot \varepsilon \, .
\end{aligned}$$

Hence, $\prod_a^b T(I, dI)p$ exists. Further, using the fact that D is dense in S and $T(u, x)$ is nonexpansive for $u, x \geqq 0$ one sees that if $p \in S$, $a, b \geqq 0$, then $\prod_a^b T(I, dI)p$ exists and thus part (1) of Theorem 1 is proved.

Before proving part (2) of Theorem 1 three lemmas will be proved each under the hypothesis of Theorem 1.

LEMMA 1.4. *If $p, q \in S$, $a, c \geqq 0$, and $b \in [a, c]$, then the following are true*:

(i) $\| \prod_a^b T(I, dI)p - \prod_a^b T(I, dI)q \| \leqq \| p - q \|$.

(ii) $\prod_b^c T(I, dI) \prod_a^b T(I, dI)p = \prod_a^c T(I, dI)p$.

(iii) *If $p \in D$ then* $\| \prod_a^b T(I, dI)p - p \| \leqq | b - a | \cdot \max_{u \in [a,b]} \| A(u)p \|$.

Proof. Parts (i) and (ii) follow from the nonexpansive property of $T(u, x)$, $u, x \geqq 0$. Part (iii) follows from Lemma 1.1.

LEMMA 1.5. *If M is a bounded subset of D, $a, b \geqq 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$, $| v - u | < \delta$, $w \in [u, v]$, and $z \in M$, then*

$$\left\| \prod_u^v T(I, dI)z - T(w, | v - u |)z \right\| \leqq | v - u | \cdot \varepsilon .$$

Proof. Let $M' = \{ \prod_{i=1}^m T(s_{2i-1}, | s_{2i} - s_{2i-2} |)z \mid z \in M, x, y \in [a, b], \{s_i\}_{i=0}^{2m}$ is a chain from y to $x\}$. Then M' is a bounded subset of D by Lemma 1.1. By Lemma 1.3 there exists $\delta > 0$ such that if $u, v \in [a, b]$, $| u - v | < \delta$, $z \in M'$ and $0 \leqq x \leqq 1$, then $\| T(u, x)z - T(v, x)z \| \leqq x \cdot \varepsilon$. Let $u, v \in [a, b]$, $| v - u | < \min \{\delta, 1\}$, $w \in [u, v]$, $z \in M$, and let $\{s_i\}_{i=0}^{2m}$ be a chain from u to v. Then,

$$\begin{aligned}
&\left\| \prod_{i=1}^m T(s_{2i-1}, | s_{2i} - s_{2i-2} |)z - T(w, | v - u |)z \right\| \\
&= \left\| \prod_{i=1}^m T(s_{2i-1}, | s_{2i} - s_{2i-2} |)z - \prod_{i=1}^m T(w, | s_{2i} - s_{2i-2} |)z \right\| \\
&\leqq \sum_{i=1}^m \left\| T(s_{2i-1}, | s_{2i} - s_{2i-2} |) \prod_{j=1}^{i-1} T(s_{2j-1}, | s_{2j} - s_{2j-2} |)z \right. \\
&\quad \left. - T(w, | s_{2i} - s_{2i-2} |) \prod_{j=1}^{i-1} T(s_{2j-1}, | s_{2j} - s_{j-2} |)z \right\| \\
&\leqq \sum_{i=1}^m | s_{2i} - s_{2i-2} | \cdot \varepsilon \\
&= | v - u | \cdot \varepsilon .
\end{aligned}$$

Thus, $\| \prod_u^v T(I, dI)z - T(w, | v - u |)z \| \leqq | v - u | \cdot \varepsilon$.

LEMMA 1.6. *If M is a bounded subset of D, $a, b \geqq 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a, b]$, $w \in [u, v]$, $| v - u | < \delta$, $z \in M$,* *and* $\{s_i\}_{i=0}^{2m}$ *is a chain from* u *to* v, *then*

$$\left\|\prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^{m} L(w, |s_{2i} - s_{2i-2}|)z\right\| \leqq |v - u|\cdot\varepsilon .$$

Proof. An argument similar to the one in Lemma 1.3 proves Lemma 1.6.

Proof of Part (2) *of Theorem* 1. Let $p \in D$, $a, b \geqq 0$, and $\varepsilon > 0$. Let $M = \{\prod_a^x T(I, dI)p \mid x \in [a, b]\}$. Then M is a bounded subset of D by Lemma 1.4. By Lemmas 1.5 and 1.6 there exists $\delta > 0$ such that if $u, v \in [a, b]$, $w \in [u, v]$, $|u - v| < \delta$, $z \in M$, and $\{s_i\}_{i=0}^{2m}$ is a chain from u to v, then

$$\left\|\prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^{m} L(w, |s_{2i} - s_{2i-2}|)z\right\| \leqq |v - u|\cdot\varepsilon/3|b - a|$$

and $\|\prod_u^v T(I, dI)z - T(w, |v - u|)z\| \leqq |v - u|\cdot\varepsilon/3|b - a|$. Let $\{r_i\}_{i=0}^{2q}$ be a chain from a to b with norm $< \delta$. Let $\{s_i\}_{i=0}^{2m}$ be a refinement of $\{r_i\}_{i=0}^{2q}$ such that there exists an increasing sequence u such that $u_0 = 0, u_q = m$, if $1 \leqq i \leqq q$ $r_{2i} = s_{2u_i}$, and if $1 \leqq i \leqq q$ and $\{t_k\}_{k=0}^{2n}$ is a refinement of $\{s_j\}_{j=2u_{i-1}}^{2u_i}$, then

$$\left\|\prod_{k=1}^{n} L(r_{2i-1}, |t_{2k} - t_{2k-2}|) \prod_a^{r_{2i-2}} T(I, dI)p \right.$$
$$\left. - T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_a^{r_{2i-2}} T(I, dI)p\right\| \leqq |r_{2i} - r_{2i-2}|\cdot\varepsilon/3|b - a| .$$

(Note that if

$$1 \leqq i \leqq q \ T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_a^{r_{2i-2}} T(I, dI)p$$
$$= \prod_{r_{2i-2}}^{r_{2i}} L(r_{2i-1}, dI) \prod_a^{r_{2i-2}} T(I, dI)p = \prod_{r_{2i}}^{r_{2i-2}} L(r_{2i-1}, dI) \prod_a^{r_{2i-2}} T(I, dI)p$$

—see Remark 1.1). Let $\{t_i\}_{i=0}^{2n}$ be a refinement of $\{s_i\}_{i=0}^{2m}$ and let v be an increasing sequence such that $v_0 = 0, v_m = n$, and if $1 \leqq i \leqq m$ $s_{2i} = t_{2v_i}$. Then,

$$\left\|\prod_{i=1}^{n} L(t_{2i-1}, |t_{2i} - t_{2i-2}|)p - \prod_a^b T(I, dI)p\right\|$$
$$= \left\|\prod_{i=1}^{q} \prod_{j=u_{i-1}+1}^{u_i} \prod_{k=v_{j-1}+1}^{v_j} L(t_{2k-1}, |t_{2k} - t_{2k-2}|)p \right.$$
$$\left. - \prod_{i=1}^{q} \prod_{r_{2i-2}}^{r_{2i}} T(I, dI)p\right\|$$
$$\leqq \sum_{i=1}^{q}\left\| \prod_{j=u_{i-1}+1}^{u_i} \prod_{k=v_{j-1}+1}^{v_j} L(t_{2k-1}, |t_{2k} - t_{2k-2}|) \prod_a^{r_{2i-2}} T(I, dI)p \right.$$
$$\left. - \prod_{r_{2i-2}}^{r_{2i}} T(I, dI) \prod_a^{r_{2i-2}} T(I, dI)p\right\|$$

$$\leq \sum_{i=1}^{q} |r_{2i} - r_{2i-2}| \cdot \varepsilon/3|b - a|$$

$$+ \sum_{i=1}^{q} \left\| \prod_{j=u_{i-1}+1}^{u_i} \prod_{k=v_{j-1}+1}^{v_j} L(r_{2i-1}, |t_{2k} - t_{2k-2}|) \prod_{a}^{r_{2i-2}} T(I, dI)p \right.$$

$$\left. - T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_{a}^{r_{2i-2}} T(I, dI)p \right\|$$

$$+ \sum_{i=1}^{q} |r_{2i} - r_{2i-2}| \cdot \varepsilon/3|b - a|$$

$$\leq \varepsilon .$$

Thus, $\prod_a^b L(I, dI)p$ exists and is $\prod_a^b T(I, dI)p$ for $p \in D$. Further, using the fact that D is dense in S and $L(u, x)$ is nonexpansive for $u, x \geqq 0$ one sees that $\prod_a^b L(I, dI)p = \prod_a^b T(I, dI)p$ for all $p \in S$.

Define $U(b, a)p = \prod_a^b T(I, dI)p$ for $p \in S$ and $a, b \geqq 0$.

Proof of Theorem 2. Parts (1), (2), and (3) of Theorem 2 follow from Lemma 1.4. Suppose that $p \in S, 0 \leqq a \leqq t$, and $U(t, a)p \in D$. Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < h < \delta_1$

$$\| A(t)T(t, h)U(t, a)p - A(t)U(t, a)p \| < \varepsilon/2$$

(see Definition 1.1, part (4)). By Lemma 1.5 there exists $\delta_2 > 0$ such that if $0 < h < \delta_2 \| U(t + h, t)U(t, a)p - T(t, h)U(t, a)p \| < h \cdot \varepsilon/2$. Then, if $0 < h < \min\{\delta_1, \delta_2\}$,

$$\begin{aligned}
&\| (1/h)(U(t + h, a)p - U(t, a)p) - A(t)U(t, a)p \| \\
&\quad = \| (1/h)(U(t + h,t)U(t, a)p - U(t, a)p) - A(t)U(t, a)p \| \\
&\qquad < \varepsilon/2 + \| (1/h)(T(t, h)U(t, a)p - U(t, a)p) - A(t)U(t, a)p \| \\
&\quad = \varepsilon/2 + \left\| 1/h \int_0^h [A(t)T(t, u)U(t, a)p - A(t)U(t, a)p]du \right\| < \varepsilon .
\end{aligned}$$

Hence, $\partial^+ U(t, a)p/\partial t = A(t)U(t, a)p$. Suppose that $p \in S, 0 < s \leqq b$, and $U(s, b)p \in D$. Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < h < \delta_1$ then $0 \leqq s - h$ and $\| A(s)T(s, h)U(s, b)p - A(s)U(s, b)p \| < \varepsilon/2$. By Lemma 1.5 there exists $\delta_2 > 0$ such that if $0 < h < \delta_2$

$$\| U(s - h, s)U(s, b)p - T(s, h)U(s, b)p \| < h \cdot \varepsilon/2 .$$

Then, if $0 < h < \min\{\delta_1, \delta_2\}$

$$\begin{aligned}
&\| (1/-h)(U(s - h, b)p - U(s, b)p) - (-A(s)U(s, b)p) \| \\
&\quad = \| (1/h)(U(s - h, s)U(s, b)p - U(s, b)p) - A(s)U(s, b)p \| \\
&\qquad < \varepsilon/2 + \| (1/h)(T(s, h)U(s, b)p - U(s, b)p) - A(s)U(s, b)p \| \\
&\quad = \varepsilon/2 + \left\| 1/h \int_0^h [A(s)T(s, u)U(s, b)p - A(s)U(s, b)p]du \right\| < \varepsilon .
\end{aligned}$$

Hence, $\partial^- U(s, b)p/\partial s = -A(s)U(s, b)p$.

3. Product integral representation in the uniform case. For Theorem 3 A is required to satisfy, in addition to conditions (I), (II), (III) of § 1, the following:

(IV) For each $t \geqq 0$ $A(t)$ has domain all of S.

(V) If $0 \leqq a \leqq b$, M is a bounded subset of S, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \in [a, b]$, $z, w \in M$, and $\|z - w\| < \delta$, then

$$\|A(u)z - A(u)w\| < \varepsilon .$$

THEOREM 3. *Let A satisfy conditions* (I)—(V) *and define*

$$M(u, v) = (I + vA(u))$$

for $u, v \geqq 0$. If $p \in S$ and $a, b \geqq 0$, then $\prod_a^b M(I, dI)p = U(b, a)p$.

Before proving Theorem 3, three lemmas will be proved each under the hypothesis of Theorem 3.

LEMMA 3.1. *Let $p \in S$ and let $a, b \geqq 0$. There is a neighborhood $N_{p,\delta}$ about p, a positive number γ, and a positive number K such that if $q \in N_{p,\delta}$, $x, y \in [a, b]$, $|y - x| < \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from x to y, then*

$$\left\|\prod_{i=1}^{m} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - q\right\| \leqq |y - x| \cdot K .$$

Proof. There exists a positive number K such that if $u \in [a, b]$ and $q \in N_{p,1}$ then $\|A(u)q\| \leqq K$. Let $\delta = 1/2$ and let $\gamma = 1/2K$. Let $q \in N_{p,\delta}$, $x, y \in [a, b]$, $|y - x| < \gamma$, $\{s_i\}_{i=0}^{2m}$ a chain from x to y, $1 \leqq j \leqq m - 1$, and suppose that $\|\prod_{i=1}^{j} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - q\| \leqq |s_{2j} - s_0| \cdot K$. Then, $\prod_{i=1}^{j} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q \in N_{p,1}$ and so

$$\begin{aligned}
&\left\|\prod_{i=1}^{j+1} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - q\right\| \\
&\leqq \left\|\prod_{i=1}^{j} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - q\right\| \\
&\qquad + |s_{2j+2} - s_{2j}| \cdot \left\|A(s_{2j+1}) \prod_{i=1}^{j} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q\right\| \\
&\leqq |s_{2j+2} - s_0| \cdot K .
\end{aligned}$$

LEMMA 3.2. *If $p \in S$ and $a \geqq 0$ then $U(t, a)p$ is continuous in t.*

Proof. Let $p \in S$ and $a, b \geqq 0$. In a manner similar to Lemma 3.1 one proves the following: There is a neighborhood $N_{q,\delta}$ about $q = U(b, a)p$, $\gamma > 0$, and $K > 0$ such that if $z \in N_{q,\delta}$, $x, y \in [a, b]$, $|y - x| < \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from x to y then

$$\left\|\prod_{m}^{i=1}(I - |s_{2i} - s_{2i-2}|A(s_{2i-1}))z - z\right\| \leqq |y - x|\cdot K .$$

Let $\varepsilon > 0$, let $x \in [a, b]$ such that $|x - b| < \gamma$, let $\{s_i\}_{i=0}^{2m}$ be a chain from a to b and $k \leqq m$ an integer such that $s_{2k} = x$ and

$$\left\|U(b, a)p - \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p\right\| < \min\{\varepsilon, \delta\}$$

and

$$\left\|U(x, a)p - \prod_{i=1}^{k} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p\right\| < \varepsilon .$$

Then,

$$\begin{aligned} &\| U(x, a)p - U(b, a)p \| \\ &\quad < 2\varepsilon + \left\|\prod_{m}^{i=k+1}(I - |s_{2i} - s_{2i-2}|A(s_{2i-1}))\prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p \right.\\ &\quad \left. - \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p\right\| \\ &\quad < 2\varepsilon + |b - x|\cdot K . \end{aligned}$$

Then, $\lim_{x\to b} U(x, a)p = U(b, a)p$ for $x \in [a, b]$. Further, by Lemma 1.4 $\lim_{x\to b} U(x, a)p = U(b, a)p$ for $x \notin [a, b]$.

LEMMA 3.3. *Let $p \in S$ and $a \geqq 0$. There exists a neighborhood $N_{p,\delta}$ about p and $\gamma > 0$ such that the following are true:*

(1) *If $\varepsilon > 0$ there exists $\alpha > 0$ such that if $q \in N_{p,\delta}$, $a \leqq x \leqq a + \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to x with* norm $< \alpha$, *then*

$$\left\|\prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - U(x, a)q\right\| < \varepsilon .$$

and

(2) *If $\varepsilon > 0$ there exists $\alpha > 0$ such that if $q \in N_{p,\delta}$, $\max\{0, a-\gamma\} \leqq x \leqq a$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to x with* norm $< \alpha$, *then*

$$\left\|\prod_{i=1}^{m} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - U(x, a)q\right\| < \varepsilon .$$

Proof. By Lemma 3.1 there exists $\delta > 0$ and $\gamma > 0$ such that if $q \in N_{p,\delta}$, $a \leqq x \leqq a + \gamma$, and $\{s_i\}_{i=0}^{2m}$ is a chain from a to x then

$$\prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q \in N_{p,2\delta} .$$

Let $\varepsilon > 0$. By Lemma 1.5 there exists $\alpha_1 > 0$ such that if

$$u, v \in [a, a + \gamma], 0 \leqq v - u < \alpha_1, u \leqq w \leqq v ,$$

and $q \in N_{p,2\delta}$, then $\| U(v, u)q - T(w, v-u)q \| \leqq (v-u)\cdot\varepsilon/2\gamma$. There exists $\alpha_2 > 0$ such that if $q \in N_{p,2\delta}$, $u \in [a, a+\gamma]$, and $0 \leqq x < \alpha_2$, then $\| A(u)T(u, x)q - A(u)q \| < \varepsilon/2\gamma$ (Note that

$$\| T(u, x)q - q \| = \left\| \int_0^x A(u)T(u, t)q dt \right\| \leqq x \cdot \| A(u)q \| \leqq x \cdot \\ \times (\max \| A(t)z \|, t \in [a, a+\gamma], z \in N_{p,2\delta})) .$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, let $q \in N_{p,\delta}$, let $a \leqq x \leqq a + \gamma$, and let $\{s_i\}_{i=0}^{2m}$ be a chain from a to x with norm $< \alpha$. Then,

$$\begin{aligned}
&\left\| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - U(x, a)q \right\| \\
&= \left\| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - \prod_{i=1}^{m} U(s_{2i}, s_{2i-2})q \right\| \\
&\leqq \sum_{i=1}^{m} \left\| U(s_{2i}, s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right. \\
&\quad \left. - M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right\| \\
&< \varepsilon/2 + \sum_{i=1}^{m} \left\| T(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right. \\
&\quad \left. - M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right\| \\
&= \varepsilon/2 + \sum_{i=1}^{m} \left\| \int_0^{s_{2i}-s_{2i-2}} [A(s_{2i-1})T(s_{2i-1}, t) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right. \\
&\quad \left. - A(s_{2i-1}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q]dt \right\| \\
&< \varepsilon/2 + \sum_{i=1}^{m} (s_{2i} - s_{2i-2})\cdot\varepsilon/2\gamma < \varepsilon .
\end{aligned}$$

A similar argument proves part (2) of the lemma.

Proof of Theorem 3. Let $p \in S$ and $0 \leqq a < b$. Suppose that if $a \leqq x < b$ $\prod_a^x M(I, dI)p$ exists and is $U(x, a)p$. Let $a \leqq x < b$, let $\{s_i\}_{i=0}^{2m}$ be a chain from a to b, and let $j < m$ such that $s_{2j} = x$. One uses the inequality

$$\begin{aligned}
&\left\| U(b, a)p - \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\
&\leqq \left\| U(b, a)p - \prod_a^x M(I, dI)p \right\| \\
&\quad + \left\| \prod_a^x M(I, dI)p - \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\
&\quad + \left\| \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right. \\
&\quad \left. - \prod_{i=j+1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\|
\end{aligned}$$

and Lemmas 3.1 and 3.2 to show $\prod_a^b M(I, dI)p$ exists and is $U(b, a)p$. Suppose now that for $a \leqq x \leqq b$ $\prod_a^x M(I, dI)p = U(x, a)p$. Let $b < x$, let $\{s_i\}_{i=0}^{2m}$ be a chain from a to x, and let $j < m$ such that $s_{2j} = b$. One uses the inequality

$$\begin{aligned}
&\left\| U(x, a)p - \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\
&\leqq \left\| U(x, b)U(b, a)p - U(x, b) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\| \\
&\quad + \left\| U(x, b) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right. \\
&\quad \left. - \prod_{i=j+1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\|
\end{aligned}$$

and Lemma 3.3 to show that there exists $\gamma > 0$ such that if $b \leqq x < b + \gamma$ then $\prod_a^x M(I, dI)p$ exists and is $U(x, a)p$. Thus, if $p \in S$ and $0 \leqq a \leqq b$ then $\prod_a^b M(I, dI)p$ exists and is $U(b, a)p$. With a similar argument one shows that for $p \in S$ and $0 \leqq a \leqq b$ $\prod_b^a M(I, dI)p$ exists and is $U(a, b)p$.

4. Examples.

In conclusion two examples will be given.

EXAMPLE 1. Let S be the Hilbert space and let A be densely defined and m-monotone on S (Definition 1.2). In M. Crandall and A. Pazy [2] and in T. Kato [6], it is shown that B is the infinitesimal generator of a $\mathscr{C}$-semi-group on S (Definition 1.1). Let X be a function from $[0, \infty)$ to S such that X is continuous. Define $A(t)p = Bp + X(t)$ for $p \in$ Domain (B) and $t \geqq 0$. Then A satisfies conditions (I)—(III).

EXAMPLE 2. Let S be a Banach space and let B be a mapping from S to S such that B is m-monotone S and uniformly continuous on bounded subsets of S. In [11] it is shown that B is the infinitesimal generator of a $\mathscr{C}$-semi-group of mappings on S. Let C be a continuous mapping from $[0, \infty)$ to $[0, \infty)$, let D be a continuous mapping from $[0, \infty)$ to $(0, \infty)$, and let each of E and F be a continuous mapping from $[0, \infty)$ to S. Define $A(t)p = C(t)\cdot B(D(t)\cdot p + E(t)) + F(t)$ for $t \geqq 0$ and $p \in S$. Suppose $t \geqq 0$, $\varepsilon > 0$, and $p, q \in S$. Then,

$$\begin{aligned}
&\| (I - \varepsilon A(t))p - (I - \varepsilon A(t))q \| \\
&= (1/D(t)) \| (I - \varepsilon C(t)D(t)B)(D(t)p + E(t)) \\
&\quad - (I - \varepsilon C(t)D(t)B)(D(t)q + E(t)) \| \\
&\geqq (1/D(t)) \| (D(t)p + E(t)) - (D(t)q + E(t)) \| \\
&= \| p - q \|
\end{aligned}$$

and so $A(t)$ is monotone for $t \geqq 0$. Suppose $t \geqq 0$, $\varepsilon > 0$, and $p \in S$. Let q' be in S such that $(I - \varepsilon C(t)D(t)B)q' = D(t)p + E(t) + \varepsilon D(t)F(t)$.

Let $q = (1/D(t))(q' - E(t))$. Then $(I - \varepsilon A(t))q = p$ and so $A(t)$ is m-monotone. Then A satisfies conditions (I)—(V).

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Received May 16, 1969.

VANDERBILT UNIVERSITY

Pacific Journal of Mathematics

Vol. 32, No. 1 January, 1970