

# Pacific Journal of Mathematics

**MÖBIUS FUNCTIONS OF ORDER  $k$**

TOM M. (MIKE) APOSTOL

## MÖBIUS FUNCTIONS OF ORDER $k$

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Let  $k$  denote a fixed positive integer. We define an arithmetical function  $\mu_k$ , the Möbius function of order  $k$ , as follows:

$$\begin{aligned} \mu_k(1) &= 1, \\ \mu_k(n) &= 0 \text{ if } p^{k+1} | n \text{ for some prime } p, \\ \mu_k(n) &= (-1)^r \text{ if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{a_i}, \quad 0 \leq a_i < k, \\ \mu_k(n) &= 1 \text{ otherwise.} \end{aligned}$$

In other words,  $\mu_k(n)$  vanishes if  $n$  is divisible by the  $(k+1)$ st power of some prime; otherwise,  $\mu_k(n)$  is 1 unless the prime factorization of  $n$  contains the  $k$ th powers of exactly  $r$  distinct primes, in which case  $\mu_k(n) = (-1)^r$ . When  $k=1$ ,  $\mu_k(n)$  is the usual Möbius function,  $\mu_1(n) = \mu(n)$ .

This paper discusses some of the relations that hold among the functions  $\mu_k$  for various values of  $k$ . We use these to derive an asymptotic formula for the summatory function

$$M_k(x) = \sum_{n \leq x} \mu_k(n)$$

for each  $k \geq 2$ . Unfortunately, the analysis sheds no light on the behavior of the function  $M_1(x) = \sum_{n \leq x} \mu(n)$ .

It is clear that  $|\mu_k|$  is the characteristic function of the set  $Q_{k+1}$  of  $(k+1)$ -free integers (positive integers whose prime factors are all of multiplicity less than  $k+1$ ). Further relations with  $Q_{k+1}$  are given in §'s 4 and 5.

The asymptotic formula for  $M_k(x)$  is given in the following theorem.

**THEOREM 1.** *If  $k \geq 2$  we have*

$$(1) \quad \sum_{n \leq x} \mu_k(n) = A_k x + O(x^{1/k} \log x),$$

where

$$(2) \quad A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \prod_{p|n} \frac{1-p^{-1}}{1-p^{-k}}.$$

*Note.* In (2),  $\zeta(k)$  is the Riemann zeta function. The formula for  $A_k$  can also be expressed in the form

$$(3) \quad A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(n)}{nJ_k(n)}$$

where  $\varphi(n)$  and  $J_k(n)$  are the totient functions of Euler and Jordan, given by

$$\varphi(n) = n \prod_{p|n} (1 - p^{-1}), J_k(n) = n^k \prod_{p|n} (1 - p^{-k}).$$

We also have the Euler product representation

$$(4) \quad A_k = \prod_p \left( 1 - \frac{2}{p^k} + \frac{1}{p^{k+1}} \right).$$

2. Lemmas. The proof of Theorem 1 is based on a number of lemmas.

LEMMA 1. *If  $k \geq 1$  we have  $\mu_k(n^k) = \mu(n)$ .*

LEMMA 2. *Each function  $\mu_k$  is multiplicative. That is,*

$$\mu_k(mn) = \mu_k(m)\mu_k(n) \quad \text{whenever} \quad (m, n) = 1.$$

LEMMA 3. *Let  $f$  and  $g$  be multiplicative arithmetical functions and let  $a$  and  $b$  be positive integers, with  $a \geq b$ . Then the function  $h$  defined by the equation*

$$h(n) = \sum_{d^a|n} f\left(\frac{n}{d^a}\right)g\left(\frac{n}{d^b}\right)$$

*is also multiplicative. (The sum is extended over those divisors  $d$  of  $n$  for which  $d^a$  divides  $n$ .)*

The first two lemmas follow easily from the definition of the function  $\mu_k$ . The proof of Lemma 3 is a straightforward exercise.

The next lemma relates  $\mu_k$  to  $\mu_{k-1}$ .

LEMMA 4. *If  $k \geq 2$  we have*

$$\mu_k(n) = \sum_{d^k|n} \mu_{k-1}\left(\frac{n}{d^k}\right)\mu_{k-1}\left(\frac{n}{d}\right).$$

*Proof.* By Lemmas 2 and 3, the sum on the right is a multiplicative function of  $n$ . To complete the proof we simply verify that the sum agrees with  $\mu_k(n)$  when  $n$  is a prime power.

LEMMA 5. *If  $k \geq 1$  we have*

$$|\mu_k(n)| = \sum_{d^{k+1}|n} \mu(d).$$

*Proof.* Again we note that both members are multiplicative functions of  $n$  which agree when  $n$  is a prime power.

LEMMA 6. If  $k \geq 2$  and  $r \geq 1$ , let

$$F_r(x) = \sum_{n \leq x} \mu_{k-1}(n) \mu_{k-1}(r^{k-1}n).$$

Then we have the asymptotic formula

$$F_r(x) = \frac{x}{\zeta(k)} \frac{\mu(r) \varphi(r) r^{k-1}}{J_k(r)} + O(x^{1/k} \sigma_{-s}(r)),$$

where  $\sigma_\alpha(r)$  is the sum of the  $\alpha$ th powers of the divisors of  $r$ , and  $s$  is any number satisfying  $0 < s < 1/k$ . (The constant implied by the  $O$ -symbol is independent of  $r$ .)

*Proof.* In the sum defining  $F_r(x)$  the factor  $\mu_{k-1}(r^{k-1}n) = 0$  if  $r$  and  $n$  have a prime factor in common. Therefore we need consider only those  $n$  relatively prime to  $r$ . But if  $(r, n) = 1$  the multiplicative property of  $\mu_{k-1}$  gives us

$$\mu_{k-1}(n) \mu_{k-1}(r^{k-1}n) = \mu_{k-1}(n)^2 \mu_{k-1}(r^{k-1}) = |\mu_{k-1}(n)| \mu(r),$$

where in the last step we used Lemma 1. Therefore we have

$$F_r(x) = \mu(r) \sum_{\substack{n \leq x \\ (n, r) = 1}} |\mu_{k-1}(n)|.$$

Using Lemma 5 we rewrite this in the form

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{\substack{n \leq x \\ (n, r) = 1}} \sum_{d^k | n} \mu(d) = \mu(r) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \sum_{\substack{q \leq x/d^k \\ (q, r) = 1}} 1 \\ &= \mu(r) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \sum_{t|r} \mu(t) \left[ \frac{x}{td^k} \right] \\ &= \mu(r) \sum_{t|r} \mu(t) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \left[ \frac{x}{td^k} \right]. \end{aligned}$$

At this point we use the relation  $[x] = x + O(x^s)$ , valid for any fixed  $s$  satisfying  $0 \leq s < 1$ , to obtain

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{t|r} \mu(t) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \left\{ \frac{x}{td^k} + O\left(\frac{x^s}{t^s d^{ks}}\right) \right\} \\ &= x \mu(r) \sum_{t|r} \frac{\mu(t)}{t} \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \frac{\mu(d)}{d^k} + O\left(x^s \sum_{t|r} \frac{1}{t^s} \sum_{d \leq x^{1/k}} \frac{1}{d^{ks}}\right). \end{aligned}$$

If we choose  $s$  so that  $0 < ks < 1$  we have

$$\sum_{d \leq x^{1/k}} \frac{1}{d^{ks}} = O\left(\int_1^{x^{1/k}} \frac{dt}{t^{ks}}\right) = O(x^{-s+1/k}),$$

and the  $O$ -term in the last formula for  $F_r(x)$  is  $O(x^{1/k}\sigma_{-s}(r))$ . To complete the proof of Lemma 6 we use the relations

$$\sum_{t|r} \frac{\mu(t)}{t} = \frac{\varphi(r)}{r}$$

and

$$\begin{aligned} \sum_{\substack{d^k \leq x \\ (d,r)=1}} \frac{\mu(d)}{d^k} &= \sum_{\substack{d=1 \\ (d,r)=1}}^{\infty} \frac{\mu(d)}{d^k} + O\left(\sum_{d > x^{1/k}} d^{-k}\right) \\ &= \frac{1}{\zeta(k)} \prod_{p|r} \frac{1}{1-p^{-k}} + O(x^{(1-k)/k}) \\ &= \frac{1}{\zeta(k)} \frac{r^k}{J_k(r)} + O(x^{(1-k)/k}). \end{aligned}$$

**3. Proof of Theorem 1.** In the sum defining  $M_k(x)$  we use Lemma 4 to write

$$\begin{aligned} M_k(x) &= \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x} \sum_{d^k | n} \mu_{k-1}\left(\frac{n}{d^k}\right) \mu_{k-1}\left(\frac{n}{d}\right) \\ &= \sum_{d^k \leq x} \sum_{m \leq x/d^k} \mu_{k-1}(m) \mu_{k-1}(d^{k-1}m) \\ &= \sum_{d^k \leq x} F_d(x/d^k) = \sum_{r \leq x^{1/k}} F_r(x/r^k). \end{aligned}$$

Using Lemma 6 we obtain

$$(5) \quad M_k(x) = \frac{x}{\zeta(k)} \sum_{r \leq x^{1/k}} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O\left(x^{1/k} \sum_{r \leq x^{1/k}} \frac{\sigma_{-s}(r)}{r}\right).$$

The sum in the first term is equal to

$$\begin{aligned} \sum_{r \leq x^{1/k}} \frac{\mu(r)}{r^k} \prod_{p|r} \frac{1-p^{-1}}{1-p^{-k}} &= \sum_{r=1}^{\infty} \frac{\mu(r)}{r^k} \prod_{p|r} \frac{1-p^{-1}}{1-p^{-k}} + O\left(\sum_{r > x^{1/k}} \frac{1}{r^k}\right) \\ &= \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{(1-k)/k}). \end{aligned}$$

The sum in the  $O$ -term in (5) is equal to

$$\begin{aligned} \sum_{r \leq x^{1/k}} \frac{\sigma_{-s}(r)}{r} &= \sum_{r \leq x^{1/k}} r^{-1} \sum_{d \delta = r} d^{-s} = \sum_{\delta \leq x^{1/k}} \delta^{-1} \sum_{d \leq x^{1/k}/\delta} d^{-1-s} \\ &= O\left(\sum_{\delta \leq x^{1/k}} \delta^{-1}\right) = O(\log x). \end{aligned}$$

Therefore (5) becomes

$$M_k(x) = \frac{x}{\zeta(k)} \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{1/k} \log x),$$

which completes the proof of Theorem 1.

To deduce (4) from (2) we note that (2) has the form

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} f(n)$$

where  $f(n)$  is multiplicative and  $f(p^a) = 0$  for  $a \geq 2$ . Hence we have the Euler product decomposition: [see 3, Th. 286]

$$\begin{aligned} A_k &= \frac{1}{\zeta(k)} \prod_p \{1 + f(p)\} = \prod_p (1 - p^{-k}) \prod_p \left\{1 - \frac{1}{p^k} \frac{1 - p^{-1}}{1 - p^{-k}}\right\} \\ &= \prod_p \left\{1 - p^{-k} - \frac{1 - p^{-1}}{p^k}\right\} = \prod_p \left\{1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right\}. \end{aligned}$$

4. Relations to  $k$ -free integers. Let  $Q_k$  denote the set of  $k$ -free integers (positive integers whose prime factors are all of multiplicity less than  $k$ ), and let  $q_k$  denote the characteristic function of  $Q_k$ :

$$q_k(n) = \begin{cases} 1 & \text{if } n \in Q_k, \\ 0 & \text{otherwise.} \end{cases}$$

Gegenbauer [2, p. 47] has proved that the number of  $k$ -free integers  $\leq x$  is given by

$$(6) \quad \sum_{n \leq x} q_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k}), \quad (k \geq 2).$$

From the definition of  $\mu_k$  it follows that  $q_{k+1}(n) = |\mu_k(n)|$ , so Gegenbauer's theorem implies the asymptotic formula

$$(7) \quad \sum_{n \leq x} |\mu_k(n)| = \frac{x}{\zeta(k+1)} + O(x^{1/(k+1)}), \quad (k \geq 1).$$

From our Theorem 1 we have

$$(8) \quad \sum_{n \leq x} \mu_k(n) = A_k x + O(x^{1/k} \log x) \quad (k > 1).$$

The two formulas (7) and (8) show that among the  $(k+1)$ -free integers,  $k > 1$ , those for which  $\mu_k(n) = 1$  occur asymptotically more frequently than those for which  $\mu_k(n) = -1$ ; in particular, these two sets of integers have, respectively, the densities

$$\frac{1}{2} \left( \frac{1}{\zeta(k+1)} + A_k \right) \quad \text{and} \quad \frac{1}{2} \left( \frac{1}{\zeta(k+1)} - A_k \right).$$

This is in contrast to the case  $k = 1$  for which it is known that

$$\sum_{n \leq x} |\mu(n)| \approx \frac{x}{\zeta(2)} + O(x^{1/2}), \quad \text{but} \quad \sum_{n \leq x} \mu(n) = o(x),$$

so the square-free integers with  $\mu(n) = 1$  occur with the same asymptotic frequency as those with  $\mu(n) = -1$  [see 3, p. 270].

Our Theorem 1 can also be derived very simply from an asymptotic formula of Cohen [1, Th. 4.2]. Following the notation of Cohen, let  $Q_k^*$  denote the set of positive integers  $n$  with the property that the multiplicity of each prime divisor of  $n$  is not a multiple of  $k$ . Let  $q_k^*$  denote the characteristic function of  $Q_k^*$ . Then  $q_k^*(1) = 1$ , and for  $n > 1$  we have

$$q_k^*(n) = \begin{cases} 1 & \text{if } n = \prod_{i=1}^r p_i^{a_i}, \text{ with each } a_i \not\equiv 0 \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

The functions  $q_k^*$  and  $\mu_k$  are related by the following identity:

$$(9) \quad q_k^*(n) = \sum_{d^k | n} \mu_k\left(\frac{n}{d^k}\right).$$

This is easily verified by noting that both members are multiplicative functions of  $n$  that agree when  $n$  is a prime power, or by equating coefficients in the Dirichlet series identity (14) given below in § 5. Inversion of (9) gives us

$$(10) \quad \mu_k(n) = \sum_{d^k | n} \mu(d) q_k^*\left(\frac{n}{d^k}\right).$$

Cohen's asymptotic formula states that for  $k \geq 2$  we have

$$(11) \quad \sum_{n \leq x} q_k^*(n) = A_k \zeta(k) x + O(x^{1/k}),$$

where  $A_k$  is the same constant that appears in our Theorem 1. To deduce Theorem 1 from (11) we use (10) to obtain

$$\begin{aligned} \sum_{n \leq x} \mu_k(n) &= \sum_{n \leq x} \sum_{d^k | n} \mu(d) q_k^*\left(\frac{n}{d^k}\right) = \sum_{d^k \leq x} \mu(d) \sum_{m \leq x/d^k} q_k^*(m) \\ &= \sum_{d^k \leq x} \mu(d) \left\{ A_k \zeta(k) \frac{x}{d^k} + O\left(\frac{x^{1/k}}{d}\right) \right\} \\ &= A_k \zeta(k) x \sum_{d \leq x^{1/k}} \frac{\mu(d)}{d^k} + O\left(x^{1/k} \sum_{d^k \leq x} \frac{1}{d}\right) \\ &= A_k \zeta(k) x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} + O\left(\sum_{d > x^{1/k}} d^{-k}\right) + O(x^{1/k} \log x) \\ &= A_k x + O(x^{1/k} \log x). \end{aligned}$$

Conversely, if we start with equation (9) and use Theorem 1 we can deduce Cohen's asymptotic formula (11) but with an error term  $O(x^{1/k} \log x)$  in place of  $O(x^{1/k})$ .

5. **Generating functions.** The generating function for the  $k$ -free integers is known to be given by the Dirichlet series

$$(12) \quad \sum_{n=1}^{\infty} \frac{q_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)} \quad (s > 1)$$

[see 3, Th. 303, p. 255]. It is not difficult to determine the generating functions for the functions  $\mu_k$  and  $q_k^*$  as well. Straightforward calculations with Euler products show that we have

$$(13) \quad \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \zeta(s) \prod_p \left\{ 1 - \frac{2}{p^{ks}} + \frac{1}{p^{(k+1)s}} \right\}$$

and

$$(14) \quad \sum_{n=1}^{\infty} \frac{q_k^*(n)}{n^s} = \zeta(ks) \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s}$$

for  $s > 1$ . Equation (14) is also equivalent to equations (9) and (10). From (12) and (14) we obtain the following identity relating  $\mu_k$ ,  $q_k$ , and  $q_k^*$ :

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \left( \sum_{n=1}^{\infty} \frac{q_k(n)}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{q_k^*(n)}{n^s} \right).$$

This shows [see 3, §17.1] that the numerical integral of  $\mu_k$  is the Dirichlet convolution of  $q_k$  and  $q_k^*$ :

$$\sum_{d|n} \mu_k(d) = \sum_{d|n} q_k(d) q_k^*\left(\frac{n}{d}\right).$$

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