

Pacific Journal of Mathematics

MÖBIUS FUNCTIONS OF ORDER k

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Let k denote a fixed positive integer. We define an arithmetical function μ_k , the Möbius function of order k , as follows:

$$\begin{aligned} \mu_k(1) &= 1, \\ \mu_k(n) &= 0 \text{ if } p^{k+1} | n \text{ for some prime } p, \\ \mu_k(n) &= (-1)^r \text{ if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{a_i}, \quad 0 \leq a_i < k, \\ \mu_k(n) &= 1 \text{ otherwise.} \end{aligned}$$

In other words, $\mu_k(n)$ vanishes if n is divisible by the $(k+1)$ st power of some prime; otherwise, $\mu_k(n)$ is 1 unless the prime factorization of n contains the k th powers of exactly r distinct primes, in which case $\mu_k(n) = (-1)^r$. When $k=1$, $\mu_k(n)$ is the usual Möbius function, $\mu_1(n) = \mu(n)$.

This paper discusses some of the relations that hold among the functions μ_k for various values of k . We use these to derive an asymptotic formula for the summatory function

$$M_k(x) = \sum_{n \leq x} \mu_k(n)$$

for each $k \geq 2$. Unfortunately, the analysis sheds no light on the behavior of the function $M_1(x) = \sum_{n \leq x} \mu(n)$.

It is clear that $|\mu_k|$ is the characteristic function of the set Q_{k+1} of $(k+1)$ -free integers (positive integers whose prime factors are all of multiplicity less than $k+1$). Further relations with Q_{k+1} are given in §'s 4 and 5.

The asymptotic formula for $M_k(x)$ is given in the following theorem.

THEOREM 1. *If $k \geq 2$ we have*

$$(1) \quad \sum_{n \leq x} \mu_k(n) = A_k x + O(x^{1/k} \log x),$$

where

$$(2) \quad A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \prod_{p|n} \frac{1-p^{-1}}{1-p^{-k}}.$$

Note. In (2), $\zeta(k)$ is the Riemann zeta function. The formula for A_k can also be expressed in the form

$$(3) \quad A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(n)}{nJ_k(n)}$$

where $\varphi(n)$ and $J_k(n)$ are the totient functions of Euler and Jordan, given by

$$\varphi(n) = n \prod_{p|n} (1 - p^{-1}), \quad J_k(n) = n^k \prod_{p|n} (1 - p^{-k}).$$

We also have the Euler product representation

$$(4) \quad A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}} \right).$$

2. Lemmas. The proof of Theorem 1 is based on a number of lemmas.

LEMMA 1. *If $k \geq 1$ we have $\mu_k(n^k) = \mu(n)$.*

LEMMA 2. *Each function μ_k is multiplicative. That is,*

$$\mu_k(mn) = \mu_k(m)\mu_k(n) \quad \text{whenever } (m, n) = 1.$$

LEMMA 3. *Let f and g be multiplicative arithmetical functions and let a and b be positive integers, with $a \geq b$. Then the function h defined by the equation*

$$h(n) = \sum_{d^a|n} f\left(\frac{n}{d^a}\right)g\left(\frac{n}{d^b}\right)$$

is also multiplicative. (The sum is extended over those divisors d of n for which d^a divides n .)

The first two lemmas follow easily from the definition of the function μ_k . The proof of Lemma 3 is a straightforward exercise.

The next lemma relates μ_k to μ_{k-1} .

LEMMA 4. *If $k \geq 2$ we have*

$$\mu_k(n) = \sum_{d^{k-1}|n} \mu_{k-1}\left(\frac{n}{d^{k-1}}\right)\mu_{k-1}\left(\frac{n}{d}\right).$$

Proof. By Lemmas 2 and 3, the sum on the right is a multiplicative function of n . To complete the proof we simply verify that the sum agrees with $\mu_k(n)$ when n is a prime power.

LEMMA 5. *If $k \geq 1$ we have*

$$|\mu_k(n)| = \sum_{d^{k+1}|n} \mu(d).$$

Proof. Again we note that both members are multiplicative functions of n which agree when n is a prime power.

LEMMA 6. *If $k \geq 2$ and $r \geq 1$, let*

$$F_r(x) = \sum_{n \leq x} \mu_{k-1}(n) \mu_{k-1}(r^{k-1}n).$$

Then we have the asymptotic formula

$$F_r(x) = \frac{x}{\zeta(k)} \frac{\mu(r) \varphi(r) r^{k-1}}{J_k(r)} + O(x^{1/k} \sigma_{-s}(r)),$$

where $\sigma_\alpha(r)$ is the sum of the α th powers of the divisors of r , and s is any number satisfying $0 < s < 1/k$. (The constant implied by the O -symbol is independent of r .)

Proof. In the sum defining $F_r(x)$ the factor $\mu_{k-1}(r^{k-1}n) = 0$ if r and n have a prime factor in common. Therefore we need consider only those n relatively prime to r . But if $(r, n) = 1$ the multiplicative property of μ_{k-1} gives us

$$\mu_{k-1}(n) \mu_{k-1}(r^{k-1}n) = \mu_{k-1}(n)^2 \mu_{k-1}(r^{k-1}) = |\mu_{k-1}(n)| \mu(r),$$

where in the last step we used Lemma 1. Therefore we have

$$F_r(x) = \mu(r) \sum_{\substack{n \leq x \\ (n, r) = 1}} |\mu_{k-1}(n)|.$$

Using Lemma 5 we rewrite this in the form

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{\substack{n \leq x \\ (n, r) = 1}} \sum_{d|n} \mu(d) = \mu(r) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \sum_{\substack{q \leq x/d^k \\ (q, r) = 1}} 1 \\ &= \mu(r) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \sum_{t|r} \mu(t) \left[\frac{x}{td^k} \right] \\ &= \mu(r) \sum_{t|r} \mu(t) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \left[\frac{x}{td^k} \right]. \end{aligned}$$

At this point we use the relation $[x] = x + O(x^s)$, valid for any fixed s satisfying $0 \leq s < 1$, to obtain

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{t|r} \mu(t) \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \mu(d) \left\{ \frac{x}{td^k} + O\left(\frac{x^s}{t^s d^{ks}}\right) \right\} \\ &= x \mu(r) \sum_{t|r} \frac{\mu(t)}{t} \sum_{\substack{d^k \leq x \\ (d, r) = 1}} \frac{\mu(d)}{d^k} + O\left(x^s \sum_{t|r} \frac{1}{t^s} \sum_{d \leq x^{1/k}} \frac{1}{d^{ks}}\right). \end{aligned}$$

If we choose s so that $0 < ks < 1$ we have

$$\sum_{d \leq x^{1/k}} \frac{1}{d^{ks}} = O\left(\int_1^{x^{1/k}} \frac{dt}{t^{ks}}\right) = O(x^{-s+1/k}),$$

and the O -term in the last formula for $F_r(x)$ is $O(x^{1/k}\sigma_{-s}(r))$. To complete the proof of Lemma 6 we use the relations

$$\sum_{t|r} \frac{\mu(t)}{t} = \frac{\varphi(r)}{r}$$

and

$$\begin{aligned} \sum_{\substack{d^k \leq x \\ (d,r)=1}} \frac{\mu(d)}{d^k} &= \sum_{\substack{d=1 \\ (d,r)=1}}^{\infty} \frac{\mu(d)}{d^k} + O\left(\sum_{d > x^{1/k}} d^{-k}\right) \\ &= \frac{1}{\zeta(k)} \prod_{p|r} \frac{1}{1-p^{-k}} + O(x^{(1-k)/k}) \\ &= \frac{1}{\zeta(k)} \frac{r^k}{J_k(r)} + O(x^{(1-k)/k}). \end{aligned}$$

3. Proof of Theorem 1. In the sum defining $M_k(x)$ we use Lemma 4 to write

$$\begin{aligned} M_k(x) &= \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x} \sum_{d|n} \mu_{k-1}\left(\frac{n}{d^k}\right) \mu_{k-1}\left(\frac{n}{d}\right) \\ &= \sum_{d^k \leq x} \sum_{m \leq x/d^k} \mu_{k-1}(m) \mu_{k-1}(d^{k-1}m) \\ &= \sum_{d^k \leq x} F_d(x/d^k) = \sum_{r \leq x^{1/k}} F_r(x/r^k). \end{aligned}$$

Using Lemma 6 we obtain

$$(5) \quad M_k(x) = \frac{x}{\zeta(k)} \sum_{r \leq x^{1/k}} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O\left(x^{1/k} \sum_{r \leq x^{1/k}} \frac{\sigma_{-s}(r)}{r}\right).$$

The sum in the first term is equal to

$$\begin{aligned} \sum_{r \leq x^{1/k}} \frac{\mu(r)}{r^k} \prod_{p|r} \frac{1-p^{-1}}{1-p^{-k}} &= \sum_{r=1}^{\infty} \frac{\mu(r)}{r^k} \prod_{p|r} \frac{1-p^{-1}}{1-p^{-k}} + O\left(\sum_{r > x^{1/k}} \frac{1}{r^k}\right) \\ &= \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{(1-k)/k}). \end{aligned}$$

The sum in the O -term in (5) is equal to

$$\begin{aligned} \sum_{r \leq x^{1/k}} \frac{\sigma_{-s}(r)}{r} &= \sum_{r \leq x^{1/k}} r^{-1} \sum_{d|r} d^{-s} = \sum_{\delta \leq x^{1/k}} \delta^{-1} \sum_{d \leq x^{1/k}/\delta} d^{-1-s} \\ &= O\left(\sum_{\delta \leq x^{1/k}} \delta^{-1}\right) = O(\log x). \end{aligned}$$

Therefore (5) becomes

$$M_k(x) = \frac{x}{\zeta(k)} \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{1/k} \log x),$$

which completes the proof of Theorem 1.

To deduce (4) from (2) we note that (2) has the form

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} f(n)$$

where $f(n)$ is multiplicative and $f(p^a) = 0$ for $a \geq 2$. Hence we have the Euler product decomposition: [see 3, Th. 286]

$$\begin{aligned} A_k &= \frac{1}{\zeta(k)} \prod_p \{1 + f(p)\} = \prod_p (1 - p^{-k}) \prod_p \left\{1 - \frac{1}{p^k} \frac{1 - p^{-1}}{1 - p^{-k}}\right\} \\ &= \prod_p \left\{1 - p^{-k} - \frac{1 - p^{-1}}{p^k}\right\} = \prod_p \left\{1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right\}. \end{aligned}$$

4. Relations to k -free integers. Let Q_k denote the set of k -free integers (positive integers whose prime factors are all of multiplicity less than k), and let q_k denote the characteristic function of Q_k :

$$q_k(n) = \begin{cases} 1 & \text{if } n \in Q_k, \\ 0 & \text{otherwise.} \end{cases}$$

Gegenbauer [2, p. 47] has proved that the number of k -free integers $\leq x$ is given by

$$(6) \quad \sum_{n \leq x} q_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k}), \quad (k \geq 2).$$

From the definition of μ_k it follows that $q_{k+1}(n) = |\mu_k(n)|$, so Gegenbauer's theorem implies the asymptotic formula

$$(7) \quad \sum_{n \leq x} |\mu_k(n)| = \frac{x}{\zeta(k+1)} + O(x^{1/(k+1)}), \quad (k \geq 1).$$

From our Theorem 1 we have

$$(8) \quad \sum_{n \leq x} \mu_k(n) = A_k x + O(x^{1/k} \log x) \quad (k > 1).$$

The two formulas (7) and (8) show that among the $(k+1)$ -free integers, $k > 1$, those for which $\mu_k(n) = 1$ occur asymptotically more frequently than those for which $\mu_k(n) = -1$; in particular, these two sets of integers have, respectively, the densities

$$\frac{1}{2} \left(\frac{1}{\zeta(k+1)} + A_k \right) \quad \text{and} \quad \frac{1}{2} \left(\frac{1}{\zeta(k+1)} - A_k \right).$$

This is in contrast to the case $k = 1$ for which it is known that

$$\sum_{n \leq x} |\mu(n)| = \frac{x}{\zeta(2)} + O(x^{1/2}), \quad \text{but} \quad \sum_{n \leq x} \mu(n) = o(x),$$

so the square-free integers with $\mu(n) = 1$ occur with the same asymptotic frequency as those with $\mu(n) = -1$ [see 3, p. 270].

Our Theorem 1 can also be derived very simply from an asymptotic formula of Cohen [1, Th. 4.2]. Following the notation of Cohen, let Q_k^* denote the set of positive integers n with the property that the multiplicity of each prime divisor of n is not a multiple of k . Let q_k^* denote the characteristic function of Q_k^* . Then $q_k^*(1) = 1$, and for $n > 1$ we have

$$q_k^*(n) = \begin{cases} 1 & \text{if } n = \prod_{i=1}^r p_i^{a_i}, \text{ with each } a_i \not\equiv 0 \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

The functions q_k^* and μ_k are related by the following identity:

$$(9) \quad q_k^*(n) = \sum_{d^{k|n}} \mu_k\left(\frac{n}{d^k}\right).$$

This is easily verified by noting that both members are multiplicative functions of n that agree when n is a prime power, or by equating coefficients in the Dirichlet series identity (14) given below in § 5. Inversion of (9) gives us

$$(10) \quad \mu_k(n) = \sum_{d^{k|n}} \mu(d) q_k^*\left(\frac{n}{d^k}\right).$$

Cohen's asymptotic formula states that for $k \geq 2$ we have

$$(11) \quad \sum_{n \leq x} q_k^*(n) = A_k \zeta(k)x + O(x^{1/k}),$$

where A_k is the same constant that appears in our Theorem 1. To deduce Theorem 1 from (11) we use (10) to obtain

$$\begin{aligned} \sum_{n \leq x} \mu_k(n) &= \sum_{n \leq x} \sum_{d^{k|n}} \mu(d) q_k^*\left(\frac{n}{d^k}\right) = \sum_{d^{k \leq x}} \mu(d) \sum_{m \leq x/d^k} q_k^*(m) \\ &= \sum_{d^{k \leq x}} \mu(d) \left\{ A_k \zeta(k) \frac{x}{d^k} + O\left(\frac{x^{1/k}}{d}\right) \right\} \\ &= A_k \zeta(k)x \sum_{d \leq x^{1/k}} \frac{\mu(d)}{d^k} + O\left(x^{1/k} \sum_{d^{k \leq x}} \frac{1}{d}\right) \\ &= A_k \zeta(k)x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} + O\left(\sum_{d > x^{1/k}} d^{-k}\right) + O(x^{1/k} \log x) \\ &= A_k x + O(x^{1/k} \log x). \end{aligned}$$

Conversely, if we start with equation (9) and use Theorem 1 we can deduce Cohen's asymptotic formula (11) but with an error term $O(x^{1/k} \log x)$ in place of $O(x^{1/k})$.

5. **Generating functions.** The generating function for the k -free integers is known to be given by the Dirichlet series

$$(12) \quad \sum_{n=1}^{\infty} \frac{q_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)} \quad (s > 1)$$

[see 3, Th. 303, p. 255]. It is not difficult to determine the generating functions for the functions μ_k and q_k^* as well. Straightforward calculations with Euler products show that we have

$$(13) \quad \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \zeta(s) \prod_p \left\{ 1 - \frac{2}{p^{ks}} + \frac{1}{p^{(k+1)s}} \right\}$$

and

$$(14) \quad \sum_{n=1}^{\infty} \frac{q_k^*(n)}{n^s} = \zeta(ks) \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s}$$

for $s > 1$. Equation (14) is also equivalent to equations (9) and (10). From (12) and (14) we obtain the following identity relating μ_k , q_k , and q_k^* :

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{q_k(n)}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{q_k^*(n)}{n^s} \right).$$

This shows [see 3, §17.1] that the numerical integral of μ_k is the Dirichlet convolution of q_k and q_k^* :

$$\sum_{d|n} \mu_k(d) = \sum_{d|n} q_k(d) q_k^*\left(\frac{n}{d}\right).$$

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Received April 11, 1969.

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