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CONCERNING SEMI-STRATIFIABLE SPACES

GEOFFREY DAVID DOWNS CREEDE

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GEOFFREY D. CREEDE

In this paper, a class of spaces, called semi-stratifiable spaces is introduced. This class of spaces lies between the class of semi-metric spaces and the class of spaces in which closed sets are G_δ . This class of spaces is invariant with respect to taking countable products, closed maps, and closed unions. In a semi-stratifiable space, bicomactness and countable compactness are equivalent properties. A semi-stratifiable space is F_σ -screenable.

A T_1 -space is semi-metric if and only if it is semi-stratifiable and first countable. A completely regular space is a Moore space if and only if it is a semi-stratifiable p -space.

The concept of semi-stratifiable spaces as a generalization of semi-metric spaces (see Corollary 1.4) is due to E. A. Michael. It appears that all properties of semi-metric spaces which do not depend on first countability also hold in semi-stratifiable spaces. The class of semi-stratifiable spaces contains all stratifiable spaces [3], all cosmic spaces [13], and all spaces with a σ -locally finite [15] or σ -discrete [2] network.

Some of the results of this paper were announced in [5].

Most terms which are not defined in this paper are used as in Kelley [10].

1. Preliminaries.

DEFINITION 1.1. A topological space X is a *semi-stratifiable* space if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^\infty$ of closed subsets of X such that

- (a) $\bigcup_{n=1}^\infty U_n = U$,
- (b) $U_n \subset V_n$ whenever $U \subset V$, where $\{V_n\}_{n=1}^\infty$ is the sequence assigned to V .

A correspondence $U \rightarrow \{U_n\}_{n=1}^\infty$ is a *semi-stratification* for the space X whenever it satisfies conditions (a) and (b) of Definition 1.1.

By comparing the above definition with Definition 1.1 of [3], one can see that, if the correspondence $U \rightarrow \{U_n\}_{n=1}^\infty$ is a stratification for X , then $U \rightarrow \{U'_n\}_{n=1}^\infty$, where $U'_n = \text{Cl } U_n$, is a semi-stratification for X . In [8], Heath gives an example of a (paracompact) semi-stratifiable space which is not stratifiable.

THEOREM 1.2. *A necessary and sufficient condition for a topological space X to be semi-stratifiable is that there be a sequence*

$\{g_i\}_{i=1}^{\infty}$ of functions from X into the collection of open sets of X such that (i) $\bigcap_{i=1}^{\infty} g_i(x) = Cl\{x\}$ for each x , and (ii) if y is a point of X and $\{x_i\}_{i=1}^{\infty}$ is a sequence of points in X , with $y \in g_i(x_i)$ for all i , then $\{x_i\}_{i=1}^{\infty}$ converges to y .

Proof. Let $U \rightarrow \{U_n\}_{n=1}^{\infty}$ be a semi-stratification for X . For each i , define the function g_i by $g_i(x) = X - (X - Cl\{x\})_i$. The sequence $\{g_i\}_{i=1}^{\infty}$ satisfies conditions (i) and (ii) of the theorem.

Conversely, let $\{g_i\}_{i=1}^{\infty}$ satisfy conditions (i) and (ii) of the theorem. For each n and each open set U , let $U_n = X - \bigcup\{g_n(x) : x \in X - U\}$. Then correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a semi-stratification for X .

DEFINITION 1.3. A topological space X is *semi-metric* if there is a distance function d defined on X such that

- (1) $d(x, y) = d(y, x) \geq 0$,
- (2) $d(x, y) = 0$ if and only if $x = y$,
- (3) x is a limit point of a set M if and only if $\inf\{d(x, y) : y \in M\} = 0$.

See [7, 11].

With the aid of Theorem 3.2 of [7] we have the following relationship between semi-stratifiable spaces and semi-metric spaces.

COROLLARY 1.4. A T_1 -space is a semi-metric space if and only if it is a first countable semi-stratifiable space.

2. Properties of semi-stratifiable spaces.

THEOREM 2.1. The countable product of semi-stratifiable spaces is semi-stratifiable.

Proof. For each i , let X_i be a semi-stratifiable space and $\{g_{ij}\}_{j=1}^{\infty}$ be a sequence of functions on X_i satisfying the conditions of Theorem 1.2. Let $X = \prod_{i=1}^{\infty} X_i$ and let π_i be the projection of X onto X_i . For each i, j and each x in X , let $h_{ij}(x) = g_{ij}(\pi_i(x))$ if $j \leq i$ and $h_{ij}(x) = X_i$ if $j > i$. Now let $g_j(x) = \prod_{i=1}^{\infty} h_{ij}(x)$ for each j and x . The sequence $\{g_i\}_{i=1}^{\infty}$ satisfies the conditions of Theorem 1.2 and, hence, X is semi-stratifiable.

THEOREM 2.2. A semi-stratifiable space is hereditarily semi-stratifiable.

Theorem 2.2 can be proved by taking the natural restriction to the subspace of a semi-stratification of the larger space. In the case of closed subspaces, all semi-stratifications on the subspace can be

constructed in this manner.

THEOREM 2.3. *If Y is a closed subspace of a semi-stratifiable space X and $U \rightarrow \{U_n\}_{n=1}^\infty$ is a semi-stratification for Y , then there is a semi-stratification $V \rightarrow \{V_n\}_{n=1}^\infty$ for X such that $(V \cap Y)_n = (V_n \cap Y)$.*

Proof. If $W \rightarrow \{W_n\}_{n=1}^\infty$ is any semi-stratification for X , then let $V_n = (W \cap Y)_n \cup (W - Y)_n$. The correspondence $V \rightarrow \{V_n\}_{n=1}^\infty$ is a semi-stratification for X satisfying $(V \cap Y)_n = V_n \cap Y$.

By applying Theorem 2.3 with respect to the common subspace, we obtain the following theorem:

THEOREM 2.4. *The union of two closed (in the union) semi-stratifiable spaces is semi-stratifiable.*

DEFINITION 2.5. A topological space is F_σ -screenable if every open cover has a σ -discrete closed refinement which covers the space.

Theorem 2.6 generalizes McAulely's Lemma 1 of [12].

Theorem 2.6. *A semi-stratifiable space is F_σ -screenable.*

Proof. Let X be a semi-stratifiable space with a semi-stratification $U \rightarrow \{U_n\}_{n=1}^\infty$. Let $\{O_\alpha: \alpha \in I\}$ be an open cover of X and let I be well-ordered. For each natural number n , define: $H_{1_n} = (O_1)_n$ and, for each $\alpha > 1$, $H_{\alpha_n} = (O_\alpha)_n - \cup \{O_\beta: \beta \in I, \beta < \alpha\}$. For each natural number n , let $\mathcal{H}_n = \{H_{\alpha_n}: \alpha \in I\}$. Then \mathcal{H}_n is a discrete collection of closed sets. By the well-ordering on I , $\mathcal{H} = \cup_{n=1}^\infty \mathcal{H}_n$ covers X .

DEFINITION 2.7. A topological space is \aleph_1 -compact if every uncountable subset has a limit point.

THEOREM 2.8. *In a semi-stratifiable T_1 -space X , the following are equivalent (1) X is Lindelöf, (2) X is hereditarily separable, and (3) X is \aleph_1 -compact.*

Proof. (1) \Rightarrow (2) Let X be a Lindelöf semi-stratifiable space. Since a Lindelöf space in which open sets are F_σ is hereditarily Lindelöf, it is sufficient to show that X is separable. Let $\{g_i\}_{i=1}^\infty$ be a sequence of functions satisfying the conditions of Theorem 1.2. For each i , $\{g_i(x): x \in X\}$ is an open cover of X and, since X is Lindelöf, there is a countable subset D_i of X such that $\{g_i(x): x \in D_i\}$ is an open cover of X . The set $D = \cup_{i=1}^\infty D_i$ is a countable dense subset of X .

(2) \Rightarrow (3) The proof of this part is well-known.

(3) \Rightarrow (1) Let X be an \aleph_1 -compact semi-stratifiable T_1 -space. Let \mathcal{G} be an open cover of X and suppose that \mathcal{G} has no countable subcover. By Theorem 2.6, \mathcal{G} has a closed refinement $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ where each \mathcal{H}_n is discrete. Since \mathcal{G} has no countable subcover, there is an n such that \mathcal{H}_n is uncountable. Let X' be a subset of X consisting of exactly one point of each nonempty element of \mathcal{H}_n . The set X' is uncountable and has no limit point.

Theorem 2.8 cannot be strengthened by replacing hereditarily separable by separable. The example of a Moore space which is not metrizable due to R. L. Moore (see [9]) is an example of a separable semi-stratifiable space which is not Lindelöf.

Since, in a Lindelöf space, bicomact is equivalent to countably compact, Theorem 2.8 has the following corollary:

COROLLARY 2.9. *In a semi-stratifiable T_1 -space, bicomact is equivalent to countably compact.*

3. Mappings. It is a well-known theorem [17] that the closed compact image of a separable metric space is a separable metric space. However, there are closed images of separable metric spaces which are not first countable. The following theorem gives a property of metric spaces which is preserved by closed maps.

THEOREM 3.1. *The closed image of a semi-stratifiable space is semi-stratifiable.*

Proof. Let f be a closed continuous function from a semi-stratifiable space X onto a topological space Y . Let $U \rightarrow \{U_n\}_{n=1}^{\infty}$ be a semi-stratification for X . For each open set V of Y and each natural number n , let $V_n = f((f^{-1}[V])_n)$. The correspondence $V \rightarrow \{V_n\}_{n=1}^{\infty}$ is a semi-stratification for Y .

Theorem 3.1 does not remain true if closed is replaced by open.

Theorem 3.1 and Corollary 1.4 imply that the closed image of a semi-metric space is semi-stratifiable. However, it can be shown that the subspace of βN (the Stone-Čech compactification of the natural numbers) consisting of N together with one point of $\beta N - N$ is a semi-stratifiable space which cannot be the closed image of a semi-metric space. It is an open question whether the spaces which are closed images of semi-metric spaces are precisely the semi-stratifiable Fréchet-Urysohn spaces [2].

4. Moore spaces. In this section, we wish to give necessary and sufficient conditions for a semi-stratifiable space to be a Moore space.

DEFINITION 4.1. A sequence $\{\mathcal{S}_n\}_{n=1}^\infty$ of open covers of a topological space X is a *development* for X if (1) \mathcal{S}_{i+1} is a refinement for \mathcal{S}_i and (2), if x is a point of X and U is an open set in X containing x , then there is a natural number k such that $\text{St}(x, \mathcal{S}_k) \subset U$. A *Moore space* is a regular T_1 -space which has a development. See [7, 14].

A Moore space is semi-metric and, hence, is semi-stratifiable. The following example, due to McAuley [11], shows that this implication cannot be reversed.

EXAMPLE 4.2. Let X be the x -axis of the Cartesian plane E^2 . Let d denote the usual distance function in E^2 and, if $p \neq q$, let $\alpha(p, q)$ denote the nonobtuse angle (in radians) formed by X and the line through p and q . Define a distance function D on E^2 as follows: $D(p, p) = 0$ and, if $p \neq q$, $D(p, q) = d(p, q) + \alpha(p, q)$. A basis for the topology on E^2 is $\{U_\varepsilon(p) : p \in E^2, \varepsilon > 0\}$ where $U_\varepsilon(p) = \{q : D(p, q) < \varepsilon\}$. Let S denote E^2 with this topology. If S were a Moore space, it would be second countable, since it is Lindelöf. But S is not second countable since any basis contains uncountably many elements.

DEFINITION 4.3. A T_1 -space X is said to be *quasi-complete* provided that there is a sequence $\{\mathcal{B}_n\}_{n=1}^\infty$ of open covers of X with the following property: if $\{A_n\}_{n=1}^\infty$ is a decreasing sequence of nonempty closed subsets of X and if there exists an element $x_0 \in X$ such that, for each n , there is a $B_n \in \mathcal{B}_n$ with $A_n \cup \{x_0\} \subset B_n$, then $\bigcap_{n=1}^\infty A_n \neq \emptyset$.

DEFINITION 4.4 (Borges [4]). A T_1 -space X is a *w Δ -space* if there exists a sequence $\{\mathcal{B}_n\}_{n=1}^\infty$ of open covers of X such that, if $\{A_n\}_{n=1}^\infty$ is a decreasing sequence of nonempty closed subsets of X and there exists $x_0 \in X$ for which $A_n \subset \text{St}(x_0, \mathcal{B}_n)$ for all n , then $\bigcap_{n=1}^\infty A_n \neq \emptyset$.

Definition 4.3 is at least formally weaker than Definition 4.4. It is an open question whether all quasi-complete spaces are *w Δ -spaces*.

Theorem 4.5, due to Heath [7], gives a sufficient condition for a space to be a Moore space.

THEOREM 4.5. A regular T_1 -space X is a Moore space provided that there is a sequence $\{g_i\}_{i=1}^\infty$ of functions from X into the topology on X with the following properties: (A) For each x in X , $\{g_i(x)\}_{i=1}^\infty$ is a decreasing local base at x . (B) If y is a point of X and $\{x_i\}_{i=1}^\infty$ is a sequence in X with $y \in g_i(x_i)$ for each i , then $\{x_i\}_{i=1}^\infty$ converges to y . (C) If y is a point of X , U is an open subset of X containing y , and $\{x_i\}_{i=1}^\infty$ is a sequence in X such that, for each n , $y \in g_n(x_n)$ and there

is a natural number k with $\text{Cl}[g_{n+k}(x_{n+k})] \subset g_n(x_n)$, then there is a natural number m with $g_m(x_m) \subset U$.

THEOREM 4.6. *A regular T_1 -space is a Moore space if it is a quasi-complete semi-stratifiable space.*

Proof. Let X be a regular quasi-complete semi-stratifiable T_1 -space. Let $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ be a sequence satisfying the conditions of Definition 4.3 and let $\{h_n\}_{n=1}^{\infty}$ be a sequence satisfying the conditions of Theorem 1.2. For each x in X , let $B_n(x)$ be a member of \mathfrak{B}_n containing x . For each x , let $g_1(x)$ be an open subset of X containing x such that $\text{Cl} g_1(x) \subset B_1(x) \cap h_1(x)$ and let $g_{n+1}(x)$ be an open subset of X containing x such that $\text{Cl} g_{n+1}(x) \subset B_{n+1}(x) \cap h_{n+1}(x) \cap g_n(x)$. The sequence $\{g_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 4.5 and, hence, X is a Moore space.

By Proposition 2.8 of [4], we have the following corollary:

COROLLARY 4.7. *If X is a regular T_1 -space, then the following are equivalent:*

- (1) X is a Moore space.
- (2) X is a semi-stratifiable $w\Delta$ -space.
- (3) X is a semi-stratifiable quasi-complete space.

If X is a completely regular T_1 -space, let βX denote its Stone-Ćech compactification. The following definition is due to Arhangel'skii [1, 2].

DEFINITION 4.8. A completely regular T_1 -space X is a p -space provided that there is a sequence $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ of collections of open subsets of βX such that each \mathfrak{B}_n covers X and $\bigcap_{n=1}^{\infty} \text{St}(x, \mathfrak{B}_n) \subset X$ for each point x in X .

LEMMA 4.9. *A p -space is quasi-complete.*

Proof. Let X be a p -space and let $\{\mathfrak{B}_n\}_{n=1}^{\infty}$ satisfy the conditions of Definition 4.8. For each n , let \mathfrak{B}'_n be an open cover of X such that, if $B \in \mathfrak{B}'_n$, then $\text{Cl}_{\beta X} B$ is contained in some member of \mathfrak{B}_n . Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed subsets of X and x be a point of X such that there is a $B_n \in \mathfrak{B}'_n$ with $A_n \cup \{x\} \subset B_n$ for each n . Since $\text{Cl}_{\beta X} A_n$ is compact, $\bigcap_{n=1}^{\infty} \text{Cl}_{\beta X} A_n \neq \emptyset$. But $\text{Cl}_{\beta X} A_n \subset \text{St}(x, \mathfrak{B}_n)$ and $\bigcap_{n=1}^{\infty} \text{Cl}_{\beta X} A_n \subset X$. Thus, $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \text{Cl}_{\beta X} A_n$. Hence, X is quasi-complete.

It can be seen that, in completely regular spaces, the concepts of $w\mathcal{A}$ -spaces, p -spaces, and quasi-complete spaces are related. The exact relationship between these three concepts is an open problem.

THEOREM 4.10. *A completely regular T_1 -space is a Moore space if and only if it is a semi-stratifiable p -space.*

Proof. Lemma 4.9 and Theorem 4.6 show that a semi-stratifiable p -space is a Moore space.

Conversely, let X be a completely regular Moore space and let $\{\mathcal{C}_n\}_{n=1}^\infty$ be a development for X . By the remark following Definition 4.1, X is semi-stratifiable. For each n , let

$$\mathcal{B}_n = \{\beta X - \text{Cl}_{\beta X}(X - G) : G \in \mathcal{C}_n\}.$$

The sequence $\{\mathcal{B}_n\}_{n=1}^\infty$ satisfies the conditions of Definition 4.8. Since $G \subset \beta X - \text{Cl}_{\beta X}(X - G)$, \mathcal{B}_n covers X . If $x \in X$ and $y \in \beta X - X$, let U and V be disjoint open subsets of βX containing x and y , respectively. There is a k where $\text{St}(x, \mathcal{C}_k) \subset U \cap X$. Then $y \notin \text{St}(x, \mathcal{B}_k)$. Hence, X is a p -space.

Since a locally compact Hausdorff space is a p -space, we have the following corollary:

COROLLARY 4.11. *A locally compact semi-stratifiable Hausdorff space is a Moore space.*

In Theorem 4.10, the condition of complete regularity can be replaced with regularity by using the Wallman compactification [6, 16] instead of the Stone-Ćech compactification. Appropriate changes will also have to be made in Definition 4.8.

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