SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN DENSITY II

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Let $S$ denote the set of all infinite increasing sequences of positive integers. For all $A = \{a_n\}$ and $B = \{b_n\}$ in $S$ define the metric $\rho(A, B) = 0$ if $A = B$; i.e., if $a_n = b_n$ for all $n$ and $\rho(A, B) = 1/k$ otherwise, where $k$ is the smallest value of $n$ for which $a_n \neq b_n$. The main object of this note is to show that the set of points of continuity of the Schnirelmann density $d(A)$ is a residual set and that this is the best possible result of this type.

The space $S$ and some of the properties of densities defined on it have been discussed previously [2, 3, 4]. In particular, it has been shown that the set of points of continuity of $d(A)$ is the set of all points having density zero. Let $L_a = \{A \in S : d(A) = \alpha\} (0 \leq \alpha \leq 1)$ denote the level sets of $d(A)$ and define $M_a = \{A \in S : d(A) \geq \alpha\}$. Then $L_a = M_a$ so that $M_a$ is closed and $L_a$ is dense in $M_a$ [4]. These results are required in the sequel. A brief and lucid account of all other necessary topological results is given in [1].

**Theorem 1.** The family of all sets of the form $S(m, n) = \{A \in S : a_n = m\}$ is a sub-basis for the topology of $S$.

**Proof.** If $A \in S(m, n)$ and $B \in S(m, n)$, then $\rho(A, B) \geq 1/n$. Hence $S - S(m, n)$ is closed and $S(m, n)$ is open. Also, the spheres $S_\varepsilon(A) = \{B \in S : \rho(A, B) < \varepsilon\}, 0 < \varepsilon \leq 1$, constitute a basis for $S$ and the desired result follows since

$$S_\varepsilon(A) = \bigcap_{n=1}^{[1/\varepsilon]} S(a_n, n).$$

**Corollary.** $S$ has a countable basis.

**Corollary.** $S$ is separable.

It is also clear that $S$ is a subspace of $\bigotimes_{n=1}^{\infty} P_n$, where $P_n$ is the set of all positive integers with the discrete topology for each $n$.

**Theorem 2.** $S$ is complete.

**Proof.** Let $A_n = \{a_n\}_{n=1}^{\infty}$ and suppose that $\{A_n\}$ is a Cauchy sequence in $S$. Also, let $n_k$ be the smallest positive integer such that
\[ \rho(A_m, A_n) < \frac{1}{k} \] for all \( m, n \geq n_k \) and define \( A = \{a_{n_k, a}\}_{k=1}^{\infty}. \) Since all of the \( A_n \)'s have the same first \( k \) terms for \( n \geq n_k \), it is clear that \( A \in S \) and \( \rho(A_n, A) < \frac{1}{k} \) for all \( n \geq n_k \). Hence \( \lim_{n \to \infty} \rho(A_n, A) = 0 \) and \( S \) is complete.

The following corollaries are a consequence of the Baire category theorem and the fact that \( M_a \) is a closed subset of \( S \).

**Corollary.** \( M_a \) is complete.

**Corollary.** \( M_a \) is a set of the second category in itself.

The following result would be of no interest for those values of \( a \) for which the second of the above corollaries fails to hold.

**Theorem 3.** \( L_a \) is residual in \( M_a \).

*Proof.* \( M_a - L_a = \bigcup_{k=1}^{\infty} M_{a+1/k} \). Since \( \bar{L}_a = M_a \), \( L_a \) is dense in \( M_a \) and, since \( M_{a+1/k} \subset M_a \), \( L_a \) is dense in \( M_{a+1/k} \). Also, since \( M_{a+1/k} \) is closed, \( M_{a+1/k} \) is nowhere dense in \( M_a \) and \( M_a - L_a \) is a set of the first category in \( M_a \).

Since the set of points of continuity of \( d(A) \) is \( L_0 \) and \( M_0 = S \), the following result ensues.

**Corollary.** The set of points of continuity of \( d(A) \) is residual in \( S \).

The following theorem shows that the above corollary is a best possible result in the following sense. In the true statement, \( S - L_0 \) is a countable union of nowhere dense sets, the word countable can not be replaced by finite.

**Theorem 4.** \( M_a - L_a \) is open if and only if \( a = 0 \) or \( 1 \).

*Proof.* \( M_i - L_i \) is the empty set and hence open. Also, it is easily seen that \( M_0 - L_0 = S(1, 1) \) in the notation of Theorem 1 and hence open.

Suppose that \( M_a - L_a \) is open for \( a > 0 \). Then \( M_a - L_a \subset M_a \), since \( M_a \) is closed, and it follows that \( L_0 \subset S - M_a - L_a \). Since \( L_0 = S \) and \( S - M_a - L_a \) is closed, we have \( S - M_a - L_a = S \) and \( M_a - L_a \) is the empty set. Thus \( a = 1 \) and the proof is complete.

The following result is included in the preceding proof.
Corollary. The support of $d(A)$ is the set of all sequences with first term one.

The final result concerns the asymptotic density

$$\delta(A) = \liminf_{k \to \infty} A(k)/k,$$

where $A(k)$ denotes the number of elements of $A$ which do not exceed $k$.

Theorem 5. $\delta(A)$ is a function of Baire class two.

Proof. Let $\delta_n(A) = \inf_{k \in \mathbb{N}} A(k)/k$. Then $\delta_n(A)$ is a function of Baire class one [4, Th. 3]. Also, $\delta(A) = \lim_{n \to \infty} \delta_n(A)$. Now $\delta(A)$ is obviously everywhere discontinuous on $S$. Suppose $\delta(A)$ is a function of Baire class one. Then the set of points of discontinuity of $\delta(A)$ is a set of the first category [5, Th. 36]. But $S$ is a set of the second category and the desired result follows.

References


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