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**SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN  
DENSITY. II**

R. L. DUNCAN

## SOME CONTINUITY PROPERTIES OF THE SCHNIRELMANN DENSITY II

R. L. DUNCAN

Let  $S$  denote the set of all infinite increasing sequences of positive integers. For all  $A \cong \{a_n\}$  and  $B = \{b_n\}$  in  $S$  define the metric  $\rho(A, B) = 0$  if  $A = B$ ; i.e., if  $a_n = b_n$  for all  $n$  and  $\rho(A, B) = 1/k$  otherwise, where  $k$  is the smallest value of  $n$  for which  $a_n \neq b_n$ . The main object of this note is to show that the set of points of continuity of the Schnirelmann density  $d(A)$  is a residual set and that this is the best possible result of this type.

The space  $S$  and some of the properties of densities defined on it have been discussed previously [2, 3, 4]. In particular, it has been shown that the set of points of continuity of  $d(A)$  is the set of all points having density zero. Let  $L_a = \{A \in S \mid d(A) = a\}$  ( $0 \leq a \leq 1$ ) denote the level sets of  $d(A)$  and define  $M_a = \{A \in S \mid d(A) \geq a\}$ . Then  $\bar{L}_a = M_a$  so that  $M_a$  is closed and  $L_a$  is dense in  $M_a$  [4]. These results are required in the sequel. A brief and lucid account of all other necessary topological results is given in [1].

**THEOREM 1.** *The family of all sets of the form  $S(m, n) = \{A \in S \mid a_n = m\}$  is a sub-basis for the topology of  $S$ .*

*Proof.* If  $A \in S(m, n)$  and  $B \in S(m, n)$ , then  $\rho(A, B) \geq 1/n$ . Hence  $S - S(m, n)$  is closed and  $S(m, n)$  is open. Also, the spheres  $S_\varepsilon(A) = \{B \in S \mid \rho(A, B) < \varepsilon\}$ ,  $0 < \varepsilon \leq 1$ , constitute a basis for  $S$  and the desired result follows since

$$S_\varepsilon(A) = \bigcap_{n=1}^{[1/\varepsilon]} S(a_n, n).$$

**COROLLARY.**  *$S$  has a countable basis.*

**COROLLARY.**  *$S$  is separable.*

It is also clear that  $S$  is a subspace of  $\prod_{n=1}^{\infty} P_n$ , where  $P_n$  is the set of all positive integers with the discrete topology for each  $n$ .

**THEOREM 2.**  *$S$  is complete.*

*Proof.* Let  $A_n = \{a_{n,\nu}\}_{\nu=1}^{\infty}$  and suppose that  $\{A_n\}$  is a Cauchy sequence in  $S$ . Also, let  $n_k$  be the smallest positive integer such that

$\rho(A_m, A_n) < 1/k$  for all  $m, n \geq n_k$  and define  $A = \{a_{n_k, k}\}_{k=1}^{\infty}$ . Since all of the  $A_n$ 's have the same first  $k$  terms for  $n \geq n_k$ , it is clear that  $A \in S$  and  $\rho(A_n, A) < 1/k$  for all  $n \geq n_k$ . Hence  $\lim_{n \rightarrow \infty} \rho(A_n, A) = 0$  and  $S$  is complete.

The following corollaries are a consequence of the Baire category theorem and the fact that  $M_a$  is a closed subset of  $S$ .

COROLLARY.  $M_a$  is complete.

COROLLARY.  $M_a$  is a set of the second category in itself.

The following result would be of no interest for those values of  $a$  for which the second of the above corollaries fails to hold.

THEOREM 3.  $L_a$  is residual in  $M_a$ .

*Proof.*  $M_a - L_a = \bigcup_{k=1}^{\infty} M_{a+1/k}$ . Since  $\bar{L}_a = M_a$ ,  $L_a$  is dense in  $M_a$  and, since  $M_{a+1/k} \subset M_a$ ,  $L_a$  is dense in  $M_{a+1/k}$ . Also, since  $M_{a+1/k}$  is closed,  $M_{a+1/k}$  is nowhere dense in  $M_a$  and  $M_a - L_a$  is a set of the first category in  $M_a$ .

Since the set of points of continuity of  $d(A)$  is  $L_0$  and  $M_0 = S$ , the following result ensues.

COROLLARY. The set of points of continuity of  $d(A)$  is residual in  $S$ .

The following theorem shows that the above corollary is a best possible result in the following sense. In the true statement,  $S - L_0$  is a countable union of nowhere dense sets, the word countable can not be replaced by finite.

THEOREM 4.  $\overline{M_a - L_a}$  is open if and only if  $a = 0$  or  $1$ .

*Proof.*  $\overline{M_1 - L_1}$  is the empty set and hence open. Also, it is easily seen that  $\overline{M_0 - L_0} = S(1, 1)$  in the notation of Theorem 1 and hence open.

Suppose that  $\overline{M_a - L_a}$  is open for  $a > 0$ . Then  $\overline{M_a - L_a} \subset M_a$ , since  $M_a$  is closed, and it follows that  $L_0 \subset S - \overline{M_a - L_a}$ . Since  $\bar{L}_0 = S$  and  $S - \overline{M_a - L_a}$  is closed, we have  $S - \overline{M_a - L_a} = S$  and  $\overline{M_a - L_a}$  is the empty set. Thus  $a = 1$  and the proof is complete.

The following result is included in the preceding proof.

COROLLARY. *The support of  $d(A)$  is the set of all sequences with first term one.*

The final result concerns the asymptotic density

$$\delta(A) = \liminf A(k)/k,$$

where  $A(k)$  denotes the number of elements of  $A$  which do not exceed  $k$ .

THEOREM 5.  *$\delta(A)$  is a function of Baire class two.*

*Proof.* Let  $\delta_n(A) = \inf_{k \geq n} A(k)/k$ . Then  $\delta_n(A)$  is a function of Baire class one [4, Th. 3]. Also,  $\delta(A) = \lim_{n \rightarrow \infty} \delta_n(A)$ . Now  $\delta(A)$  is obviously everywhere discontinuous on  $S$ . Suppose  $\delta(A)$  is a function of Baire class one. Then the set of points of discontinuity of  $\delta(A)$  is a set of the first category [5, Th. 36]. But  $S$  is a set of the second category and the desired result follows.

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Robert Alexander Adams, <i>Compact Sobolev imbeddings for unbounded domains</i> . . . . .	1
Bernhard Amberg, <i>Groups with maximum conditions</i> . . . . .	9
Tom M. (Mike) Apostol, <i>Möbius functions of order <math>k</math></i> . . . . .	21
Stefan Bergman, <i>On an initial value problem in the theory of two-dimensional transonic flow patterns</i> . . . . .	29
Geoffrey David Downs Creede, <i>Concerning semi-stratifiable spaces</i> . . . . .	47
Edmond Dale Dixon, <i>Matric polynomials which are higher commutators</i> . . . . .	55
R. L. Duncan, <i>Some continuity properties of the Schnirelmann density. II</i> . . . . .	65
Peter Larkin Duren and Allen Lowell Shields, <i>Coefficient multipliers of <math>H^p</math> and <math>B^p</math> spaces</i> . . . . .	69
Hector O. Fattorini, <i>On a class of differential equations for vector-valued distributions</i> . . . . .	79
Charles Hallahan, <i>Stability theorems for Lie algebras of derivations</i> . . . . .	105
Heinz Helfenstein, <i>Local isometries of flat tori</i> . . . . .	113
Gerald J. Janusz, <i>Some remarks on Clifford's theorem and the Schur index</i> . . . . .	119
Joe W. Jenkins, <i>Symmetry and nonsymmetry in the group algebras of discrete groups</i> . . . . .	131
Herbert Frederick Kreimer, Jr., <i>Outer Galois theory for separable algebras</i> . . . . .	147
D. G. Larman and P. Mani, <i>On visual hulls</i> . . . . .	157
R. Robert Laxton, <i>On groups of linear recurrences. II. Elements of finite order</i> . . . . .	173
Dong Hoon Lee, <i>The adjoint group of Lie groups</i> . . . . .	181
James B. Lucke, <i>Commutativity in locally compact rings</i> . . . . .	187
Charles Harris Scanlon, <i>Rings of functions with certain Lipschitz properties</i> . . . . .	197
Binyamin Schwarz, <i>Totally positive differential systems</i> . . . . .	203
James McLean Sloss, <i>The bending of space curves into piecewise helical curves</i> . . . . .	231
James D. Stafney, <i>Analytic interpolation of certain multiplier spaces</i> . . . . .	241
Patrick Noble Stewart, <i>Semi-simple radical classes</i> . . . . .	249
Hiroyuki Tachikawa, <i>On left QF - 3 rings</i> . . . . .	255
Glenn Francis Webb, <i>Product integral representation of time dependent nonlinear evolution equations in Banach spaces</i> . . . . .	269