Pacific Journal of Mathematics

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Vol. 32, No. 1 January 1970

PACIFIC JOURNAL OF MATHEMATICS
Vol. 32, No. 1, 1970

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Let S denote the set of all infinite increasing sequences of positive integers. For all $A \cong \{a_n\}$ and $B = \{b_n\}$ in S define the metric $\rho(A, B) = 0$ if $A = B$; i.e., if $a_n = b_n$ for all n and $\rho(A, B) = 1/k$ otherwise, where k is the smallest value of n for which $a_n \neq b_n$. The main object of this note is to show that the set of points of continuity of the Schnirelmann density $d(A)$ is a residual set and that this is the best possible result of this type.

The space S and some of the properties of densities defined on it have been discussed previously [2, 3, 4]. In particular, it has been shown that the set of points of continuity of $d(A)$ is the set of all points having density zero. Let $L_a = \{A \in S \mid d(A) = a\}(0 \leqq a \leqq 1)$ denote the level sets of $d(A)$ and define $M_a = \{A \in S \mid d(A) \geqq a\}$. Then $\bar{L}_a = M_a$ so that M_a is closed and L_a is dense in M_a [4]. These results are required in the sequel. A brief and lucid account of all other necessary topological results is given in [1].

THEOREM 1. *The family of all sets of the form* $S(m, n) = \{A \in S \mid a_n = m\}$ *is a sub-basis for the topology of* S.

Proof. If $A \in S(m, n)$ and $B \notin S(m, n)$, then $\rho(A, B) \geqq 1/n$. Hence $S - S(m, n)$ is closed and $S(m, n)$ is open. Also, the spheres $S_\varepsilon(A) = \{B \in S \mid \rho(A, B) < \varepsilon\}$, $0 < \varepsilon \leqq 1$, constitute a basis for S and the desired result follows since

$$S_\varepsilon(A) = \bigcap_{n=1}^{[1/\varepsilon]} S(a_n, n) .$$

COROLLARY. *S has a countable basis.*

COROLLARY. *S is separable.*

It is also clear that S is a subspace of $\times_{n=1}^{\infty} P_n$, where P_n is the set of all positive integers with the discrete topology for each n.

THEOREM 2. *S is complete.*

Proof. Let $A_n = \{a_{n,\nu}\}_{\nu=1}^{\infty}$ and suppose that $\{A_n\}$ is a Cauchy sequence in S. Also, let n_k be the smallest positive integer such that

$\rho(A_m, A_n) < 1/k$ for all $m, n \geqq n_k$ and define $A = \{a_{n_k,k}\}_{k=1}^{\infty}$. Since all of the A_n's have the same first k terms for $n \geqq n_k$, it is clear that $A \in S$ and $\rho(A_n, A) < 1/k$ for all $n \geqq n_k$. Hence $\lim_{n\to\infty} \rho(A_n, A) = 0$ and S is complete.

The following corollaries are a consequence of the Baire category theorem and the fact that M_a is a closed subset of S.

COROLLARY. *M_a is complete.*

COROLLARY. *M_a is a set of the second category in itself.*

The following result would be of no interest for those values of a for which the second of the above corollararies fails to hold.

THEOREM 3. *L_a is residual in M_a.*

Proof. $M_a - L_a = \bigcup_{k=1}^{\infty} M_{a+1/k}$. Since $\bar{L}_a = M_a$, L_a is dense in M_a and, since $M_{a+1/k} \subset M_a$, L_a is dense in $M_{a+1/k}$. Also, since $M_{a+1/k}$ is closed, $M_{a+1/k}$ is nowhere dense in M_a and $M_a - L_a$ is a set of the first category in M_a.

Since the set of points of continuity of $d(A)$ is L_0 and $M_0 = S$, the following result ensues.

COROLLARY. *The set of points of continuity of $d(A)$ is residual in S.*

The following theorem shows that the above corollary is a best possible result in the following sense. In the true statement, $S - L_0$ is a countable union of nowhere dense sets, the word countable can not be replaced by finite.

THEOREM 4. *$\overline{M_a - L_a}$ is open if and only if $a = 0$ or 1.*

Proof. $\overline{M_1 - L_1}$ is the empty set and hence open. Also, it is easily seen that $\overline{M_0 - L_0} = S(1, 1)$ in the notation of Theorem 1 and hence open.

Suppose that $\overline{M_a - L_a}$ is open for $a > 0$. Then $\overline{M_a - L_a} \subset M_a$, since M_a is closed, and it follows that $L_0 \subset S - \overline{M_a - L_a}$. Since $\bar{L}_0 = S$ and $S - \overline{M_a - L_a}$ is closed, we have $S - \overline{M_a - L_a} = S$ and $\overline{M_a - L_a}$ is the empty set. Thus $a = 1$ and the proof is complete.

The following result is included in the preceding proof.

COROLLARY. *The support of d (A) is the set of all sequences with first term one.*

The final result concerns the asymptotic density

$$\delta(A) = \liminf A(k)/k ,$$

where $A(k)$ denotes the number of elements of A which do not exceed k.

THEOREM 5. *$\delta(A)$ is a function of Baire class two.*

Proof. Let $\delta_n(A) = \inf_{k \geqq n} A(k)/k$. Then $\delta_n(A)$ is a function of Baire class one [4, Th. 3]. Also, $\delta(A) = \lim_{n\to\infty} \delta_n(A)$. Now $\delta(A)$ is obviously everywhere discontinuous on S. Suppose $\delta(A)$ is a function of Baire class one. Then the set of points of discontinuity of $\delta(A)$ is a set of the first category [5, Th. 36]. But S is a set of the second category and the desired result follows.

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Received February 6, 1968, and in revised form July 15, 1969.

KING OF PRUSSIA GRADUATE CENTER
THE PENNSYLVANIA STATE UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 32, No. 1 January, 1970