COEFFICIENT MULTIPLIERS OF $H^p$ AND $B^p$ SPACES

PETER Larkin Duren and Allen Lowell Shields
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This paper describes the coefficient multipliers of $H^p(0 < p < 1)$ into $\ell^q(p \leq q \leq \infty)$ and into $H^q(1 \leq q \leq \infty)$. These multipliers are found to coincide with those of the larger space $B^p$ into $\ell^q(1 \leq q \leq \infty)$ and into $H^q(1 \leq q \leq \infty)$. The multipliers of $H^p$ and $B^p$ into $B^q(0 < p < 1, 0 < q < 1)$ are also characterized.

A function $f$ analytic in the unit disk is said to be of class $H^p(0 < p < \infty)$ if

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}$$

remains bounded as $r \to 1$. $H^\infty$ is the space of all bounded analytic functions. It was recently found ([2], [4]) that if $p < 1$, various properties of $H^p$ extend to the larger space $B^p$ consisting of all analytic functions $f$ such that

$$\int_0^1 (1 - r)^{1/p - 2} M_1(r, f) \, dr < \infty.$$

Hardy and Littlewood [8] showed that $H^p \subset B^p$.

A complex sequence $\{\lambda_n\}$ is called a multiplier of a sequence space $A$ into a sequence space $B$ if $\{\lambda_n a_n\} \in B$ whenever $\{a_n\} \in A$. A space of analytic functions can be regarded as a sequence space by identifying each function with its sequence of Taylor coefficients. In [4] we identified the multipliers of $H^p$ and $B^p(0 < p < 1)$ into $\ell^1$. We have also shown ([2], Th. 5) that the sequence $\{n^{1/q - 1/p}\}$ multiplies $B^p$ into $B^q$. We now extend these results by describing the multipliers of $H^p(0 < p < 1)$ into $\ell^q(p \leq q \leq \infty)$, of $B^p$ into $\ell^q(1 \leq q \leq \infty)$, and of both $H^p$ and $B^p$ into $B^q(0 < q < 1)$. We also extend a theorem of Hardy and Littlewood (whose proof was never published) by characterizing the multipliers of $H^p$ and $B^p$ into $H^q(0 < p < 1 \leq q \leq \infty)$. In almost every case considered, the multipliers of $B^p$ into a given space are the same as those of $H^p$.

2. Multipliers into $\ell^q$. We begin by describing the multipliers of $H^p$ and $B^p$ into $\ell^\infty$, the space of bounded complex sequences.

**Theorem 1.** For $0 < p \leq 1$, a sequence $\{\lambda_n\}$ is a multiplier of $H^p$ into $\ell^\infty$ if and only if
For $p < 1$, the condition (1) also characterizes the multipliers of $B^p$ into $\ell^\infty$.

Proof. If $f(z) = \sum a_n z^n$ is in $B^p$, then by Theorem 4 of [2],

(2) $a_n = o(n^{1/p-1})$.

If $f \in H^1$, then $a_n \to 0$ by the Riemann-Lebesgue lemma. This proves the sufficiency of (1). Conversely, suppose $\{\lambda_n\}$ is a multiplier of $H^p$ into $\ell^\infty$. Then the closed linear operator

$\Lambda: f \mapsto \{\lambda_n a_n\}$

maps $H^p$ into $\ell^\infty$. Thus $\Lambda$ is bounded, by the closed graph theorem (which applies since $H^p$ is a complete metric space with translation invariant metric; see [1], Chapter 2). In other words,

(3) $\sup_n |\lambda_n a_n| = ||\Lambda(f)|| \leq K ||f||$.

Now let

$g(z) = (1 - z)^{-1-1/p} = \sum b_n z^n,$

where $b^\infty \sim Bn^{1/p}$; and choose $f(z) = g(rz)$ for fixed $r < 1$. Then by (3)

$|\lambda_n| n^{1/p} r^n \leq C(1 - r)^{-1}$.

The choice $r = 1 - 1/n$ now gives (1). Note that $\{\lambda_n\}$ multiplies $H^p$ or $B^p$ into $\ell^\infty$ if and only if it multiplies into $c_0$ (the sequences tending to zero).

As a corollary we may show that the estimate (2) is best possible in a rather strong sense. For functions of class $H^p$, this estimate is due to Hardy and Littlewood [8]. Evgrafov [6] later showed that if $\{a_n\}$ tends monotonically to zero, then there is an $f \in H^p$ for which $a_n \neq O(\delta_n n^{1/p-1})$. A simpler proof was given in [5]. The result may be reformulated: if $a_n = O(d_n)$ for all $f \in H^p$, then $d_n n^{1-1/p}$ cannot tend monotonically to zero. We can now sharpen this statement as follows.

COROLLARY. If $\{d_n\}$ is any sequence of positive numbers such that $a_n = O(d_n)$ for every function $\sum a_n z^n$ in $H^p$, then there is an $\varepsilon > 0$ such that

$d_n n^{1-1/p} \geq \varepsilon > 0$, \hspace{1cm} n = 1, 2, \ldots .

Proof. If $a_n = O(d_n)$ for every $f \in H^p$, then $\{1/d_n\}$ multiplies $H^p$ into $\ell^\infty$. Thus $1/d_n = O(n^{1-1/p})$, as claimed.
We now turn to the multipliers of $H^p$ and $B^p$ into $\ell^q(q < \infty)$, the space of sequences $\{c_n\}$ with $\sum |c_n|^q < \infty$. The following theorem generalizes a previously known result [4] for $\ell^1$.

**Theorem 2.** Suppose $0 < p < 1$.

(i) A complex sequence $\{\lambda_n\}$ is a multiplier of $H^p$ into $\ell^q(p \leq q < \infty)$ if and only if

\begin{equation}
\sum_{n=1}^{N} n^{q/p} |\lambda_n|^q = O(N^q).
\end{equation}

(ii) If $1 \leq q < \infty$, $\{\lambda_n\}$ is a multiplier of $B^p$ into $\ell^q$ if and only if (4) holds.

(iii) If $q < p$, the condition (4) does not imply that $\{\lambda_n\}$ multiplies $H^p$ into $\ell^q$; nor does it imply that $\{\lambda_n\}$ multiplies $B^p$ into $\ell^q$ if $q < 1$.

**Proof.** (i) A summation by parts (see [4]) shows that (4) is equivalent to the condition

\begin{equation}
\sum_{n=1}^{\infty} |\lambda_n|^q = O(N^{q(1-1/p)}).
\end{equation}

Assume without loss of generality that $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n^q = 1$. Let $s_1 = 0$ and

$$s_n = 1 - \left\{ \sum_{k=1}^{n} \lambda_k^\beta \right\}^{1/\beta}, \quad n = 2, 3, \ldots,$$

where $\beta = q(1/p - 1)$. Note that $s_n$ increases to 1 as $n \to \infty$. By a theorem of Hardy and Littlewood ([8], p. 412), $f \in H^p(0 < p < 1)$ implies

\begin{equation}
\int_0^1 (1 - r)^{\beta - 1} M_i^q(r, f)dr < \infty, \quad p \leq q < \infty.
\end{equation}

Thus if $f(z) = \sum a_n z^n$ is in $H^p$ and $\{\lambda_n\}$ satisfies (4) with $p \leq q < \infty$, it follows that

\begin{align*}
\sum_{n=1}^{\infty} \int_{z_n}^{z_n+1} (1 - r)^{\beta - 1} M_i^q(r, f)dr \\
\geq \sum_{n=1}^{\infty} |a_n|^q \int_{z_n}^{z_n+1} (1 - r)^{\beta - 1} r^n qdr \\
\geq \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \int_{s_n}^{s_n+1} (1 - r)^{\beta - 1}dr \\
= \frac{1}{\beta} \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \left\{ (1 - s_n)^\beta - (1 - s_{n+1})^\beta \right\} \\
= \frac{1}{\beta} \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \lambda_n^q,
\end{align*}
by the definition of $s_n$. But by (5),
\[ \left\{ \sum_{k=n}^{\infty} \lambda_k \right\}^{1/\beta} \leq \frac{C}{n}, \]
which shows, by the definition of $s_n$, that
\[ (s_n)^{n_q} \geq (1 - C/n)^{n_q} \rightarrow e^{-C} > 0. \]
Since these factors $(s_n)^{n_q}$ are eventually bounded away from zero, the preceding estimates show that $\sum |a_n|^q \lambda_n < \infty$. In other words, $\{\lambda_n\}$ is a multiplier of $H^p$ into $\ell^q$ if it satisfies the condition (4).

(ii) The above proof shows that $\{\lambda_n\}$ multiplies $B^p$ into $\ell^1$ under the condition (4) with $q = 1$. (This was also shown in [4].) The more general statement (ii) now follows by showing that if $\{\lambda_n\}$ satisfies (4), then the sequence $\{\mu_n\}$ defined by
\[ \mu_n = |\lambda_n|^q n^{(1/p-1)(q-1)} \]
satisfies (4) with $q = 1$. Hence $\{\mu_n\}$ is a multiplier of $B^p$ into $\ell^1$, and in view of (2), $\{\lambda_n\}$ is a multiplier of $B^p$ into $\ell^q$. Alternatively, it can be observed that $f \in B^p$ implies (6) for $1 < q < \infty$, so that the foregoing proof applies directly. Indeed, if $f \in B^p$, then (as shown in [2], proof of Theorem 3)
\[ M_1(r, f) = O((1 - r)^{1-1/p}); \]
hence, if $1 \leq q < \infty$,
\[ \int_0^1 (1 - r)^{q(1/p-1)-1} M_1(r, f) \, dr \leq C \int_0^1 (1 - r)^{1/p-2} M_1(r, f) \, dr < \infty. \]

(iii) That (4) does not imply $\{\lambda_n\}$ multiplies $H^p$ into $\ell^q(q < p)$ or $B^p$ into $\ell^q(q < 1)$, follows from the fact [4] that the series
\[ \sum_{n=1}^{\infty} n^{q(1-1/p)-1} |a_n|^q \]
may diverge if $f \in H^p$ and $q < p$, or if $f \in B^p$ and $q < 1$.

To show the necessity of (4), we again appeal to the closed graph theorem. If $\{\lambda_n\}$ multiplies $H^p$ into $\ell^q(0 < p < \infty, 0 < q < \infty)$, then
\[ A: f \longrightarrow \{\lambda_n a_n\} \]
is a bounded operator:
\[ \left\{ \sum_{n=1}^{\infty} |\lambda_n a_n|^q \right\}^{1/q} \leq C \|f\|, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p. \]
Choosing $f(z) = g(rz)$ as in the proof of Theorem 1, we now find
and (4) follows after terminating this series at \( n = N \) and setting \( r = 1 - 1/N \). Note that the argument shows (4) is necessary even if \( p \geq 1 \) or \( q < p \).

**Corollary 1.** If \( \{n_k\} \) is a lacunary sequence of positive integers \( (n_{k+1}/n_k \geq Q > 1) \), and if \( f(z) = \sum a_n z^n \) is in \( H^p(0 < p < 1) \), then

\[
\sum_{k=1}^{\infty} n_k^{q(1-1/p)} |a_{n_k}|^q < \infty , \quad p \leq q < \infty .
\]

**Corollary 2.** If \( f(z) = \sum a_n z^n \) is in \( H^p(0 < p < 1) \), then

\[
\sum n^{p-2} |a_n|^p < \infty .
\]

The first corollary extends a theorem of Paley [13] that \( f \in H^1 \) implies \( \{a_{n_k}\} \in \mathcal{L}^2 \). The second is a theorem of Hardy and Littlewood [7]. It is interesting to ask whether the converse to Corollary 1 (with \( q = p \)) is valid. That is, if \( \{c_k\} \) is a given sequence for which

\[
\sum_{k=1}^{\infty} n_k^{p-1} |c_k|^p < \infty ,
\]

then is there a function \( f(z) = \sum a_n z^n \) in \( H^p \) with \( a_{n_k} = c_k \)? We do not know the answer.

Hardy and Littlewood [9] also proved that \( \{\lambda_n\} \) multiplies \( H^1 \) into \( H^2 \) (alias \( \mathcal{L}^2 \)) if (and only if)

\[
\sum_{n=1}^{N} n^2 |\lambda_n|^2 = O(N^2) .
\]

From this it is easy to conclude that (4) characterizes the multipliers of \( H^1 \) into \( \mathcal{L}^q, 2 \leq q < \infty \). Indeed, let \( \{\lambda_n\} \) satisfy (4) and let \( \mu_n = |\lambda_n|^{q/2} \). Then, by the Hardy-Littlewood theorem, \( \{\mu_n\} \) multiplies \( H^1 \) into \( \mathcal{L}^2 \) (see [3], p. 253). Hence \( \{\lambda_n\} \) multiplies \( H^1 \) into \( \mathcal{L}^q \). (See also Hedlund [12].)

On the other hand, the condition (4) is not sufficient if \( p = 1 \) and \( q < 2 \). This may be seen by choosing a lacunary series

\[
f(z) = \sum_{k=1}^{\infty} c_k z^{n_k} , \quad n_{k+1}/n_k \geq Q > 1 ,
\]

with \( \sum |c_k|^2 < \infty \) but \( \sum |c_k|^q = \infty \) for all \( q < 2 \). The sequence \( \{\lambda_n\} \) with \( \lambda_n = 1 \) if \( n = n_k \) and \( \lambda_n = 0 \) otherwise then satisfies (4) but does not multiply \( H^1 \) into \( \mathcal{L}^q, q < 2 \).

3. Multipliers into \( B^p \). The following theorem may be regarded
as a generalization of our previous result ([2], Th. 5) that if \( f \in B^p \), then its fractional integral of order \( (1/p - 1/q) \) is in \( B^q \). (A fractional integral of negative order is understood to be a fractional derivative.)

**Theorem 3.** Suppose \( 0 < p < 1 \) and \( 0 < q < 1 \). Let \( \nu \) be the positive integer such that \( (\nu + 1)^{-1} \leq p < \nu^{-1} \). Then \( \{\lambda_n\} \) is a multiplier of \( H^p \) or \( B^p \) into \( B^q \) if and only if \( g(z) = \sum_{n=0}^{\infty} \lambda_n z^n \) has the property

\[
M_i(r, g^{(\nu)}) = O((1 - r)^{(p-1/q-\nu)}).
\]

**Proof.** Let \( \{\lambda_n\} \) satisfy (7), let \( f(z) = \sum a_n z^n \) be in \( B^p \), and let \( h(z) = \sum \lambda_n a_n z^n \). Then

\[
h(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) g(z e^{-i\theta}) d\theta, \quad 0 < \rho < 1.
\]

Differentiation with respect to \( z \) gives

\[
\rho^\nu h^{(\nu)}(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) g^{(\nu)}(z e^{-i\theta}) e^{-i\nu \theta} d\theta.
\]

Hence

\[
\rho^\nu M_i(r \rho, h^{(\nu)}) \leq M_i(r, g^{(\nu)}) M_i(\rho, f) \leq C(1 - r)^{(p-1/q-\nu)} M_i(\rho, f),
\]

where \( r = |z| \). Taking \( r = \rho \), we now see that \( f \in B^p \) implies \( h^{(\nu)} \in B^q \), \( 1/s = 1/q + \nu \). Thus \( h \in B^p \), by Theorem 5 of [2].

Conversely, let \( \{\lambda_n\} \) multiply \( H^p \) into \( B^q \). Then by the closed graph theorem,

\[
A: \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n
\]

is a bounded operator from \( H^p \) to \( B^q \). If \( (\nu + 1)^{-1} \leq p < \nu^{-1} \), let

\[
f(z) = \nu! z^\nu (1 - z)^{-\nu - 1} = \sum_{n=0}^{\infty} a_n z^n,
\]

where \( a_n = n!/(n - \nu)! \), and observe that

\[
h(z) = \sum_{n=0}^{\infty} \lambda_n a_n z^n = z^\nu g^{(\nu)}(z).
\]

Let \( f_r(z) = f(rz) \) and \( h_r(z) = h(rz) \). Since \( A \) is bounded, there is a constant \( C \) independent of \( r \) such that

\[
\|h_r\|_{B^q} = \|A(f_r)\| \leq C \|f_r\|_{H^p}.
\]

In other words,
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\[ \int_0^1 (1 - t)^{1/q - 2} M_t(r, h) dt \leq CM_p(r, f) \]
\[ = O((1 - r)^{1/p - 1}) . \]

It follows that
\[ M_x(r, h) \int_0^1 (1 - t)^{1/q - 2} dt = O((1 - r)^{1/p - 1}) , \]
or
\[ M_x(r^2, h) = O((1 - r)^{1/p - 1/1 - q}) . \]
But in view of (9), this proves (7).

**COROLLARY.** The sequence $\{\lambda_n\}$ multiplies $B^p$ into $B^p$ if and only if

\[ M_x(r, g') = O\left( \frac{1}{1 - r} \right) . \]

**Proof.** If $p = q$, the condition (10) is equivalent to (7). (see [8], p. 435.) This corollary is essentially the same as a result of Zygmund ([14], Th. 1), who found the multipliers of the Lipschitz space $A_\alpha$ or $\lambda_\alpha$ into itself. Because of the duality between these spaces and $B^p$ (see [2], §§3, 4), the multipliers from $A_\alpha$ to $A_\alpha$ and from $\lambda_\alpha$ to $\lambda_\alpha$ ($0 < \alpha < 1$) are the same as those from $B^p$ to $B^p$. Similar remarks apply to the spaces $A^*$ and $\lambda^*$, also considered in [14].

4. Multipliers into $H^q$. By combining Theorem 3 with the simple fact that $f' \in B^{1/2}$ implies $f \in H^1$, it is possible to obtain a sufficient condition for $\{\lambda_n\}$ to multiply $H^p$ into $H^q$, $0 < p < 1 \leq q \leq \infty$. However, this method leads to a sharp result only in the case $q = 1$. The following theorem provides the complete answer.

**THEOREM 4.** Suppose $0 < p < 1 \leq q \leq \infty$, and let $(\nu + 1)^{-1} \leq p < \nu^{-1}$, $\nu = 1, 2, \ldots$. Then $\{\lambda_n\}$ is a multiplier of $H^p$ or $B^p$ into $H^q$ if and only if $g(z) = \sum_{n=0}^\infty \lambda_n z^n$ has the property

\[ M_q(r, g^{(\nu + 1)}) = O((1 - r)^{1/p - \nu/2 - 2}) . \]

Hardy and Littlewood ([9], [10]) stated in different terminology that (11) implies $\{\lambda_n\}$ is a multiplier of $H^p$ into $H^q(0 < p < 1 \leq q < \infty)$, but they never published the proof. Our proof will make use of the following lemma.

**LEMMA.** Let $f$ be analytic in the unit disk, and suppose
\[ \int_0^1 (1 - r)^\alpha M_q(r, f')dr < \infty , \]

where \( \alpha > 0 \) and \( 1 \leq q \leq \infty \). Then
\[ \int_0^1 (1 - r)^{\alpha-1}M_q(r, f)dr < \infty . \]

**Proof of Lemma.** Without loss of generality, assume \( f(0) = 0 \), so that

\[ f(re^{i\theta}) = \int_0^r f'(se^{i\theta})e^{i\theta}ds . \]

The continuous form of Minkowski’s inequality now gives

\[ M_q(r, f) \leq \int_0^r M_q(s, f')ds . \]

Hence an interchange of the order of integration shows that

\[ \int_0^1 (1 - r)^{\alpha-1}M_q(r, f)dr \leq \frac{1}{\alpha} \int_0^1 (1 - s)^\alpha M_q(s, f')ds , \]

which proves the lemma.

**Proof of Theorem 4.** Suppose first that \( \{\lambda_n\} \) satisfies (11). Given \( f(z) = \sum a_n z^n \) in \( B^p \), we are to show that \( h(z) = \sum \lambda_n a_n z^n \) belongs to \( H^q \). By (8), with \( \nu \) replaced by \( (\nu + 1) \), we have

\[ \rho^{\nu+1} |h^{(\nu+1)}(\rho z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\xi})| |g^{(\nu+1)}(\rho e^{-i\xi})| \, d\xi . \]

Since \( q \geq 1 \), it follows from Jensen’s inequality ([11], §6.14) that

\[ \rho^{\nu+1} M_q(\rho, h^{(\nu+1)}) \leq M_q(\rho, f)M_q(\rho, g^{(\nu+1)}) \leq C(1 - \rho)^{1/p - \nu - 2}M_q(\rho, f) , \]

where \( r = |z| \) and (11) has been used. Now set \( r = \rho \) and use the hypothesis \( f \in B^p \) to conclude that

\[ \int_0^1 (1 - r)^\nu M_q(r, h^{(\nu+1)})dr < \infty . \]

But by successive applications of the lemma, this implies

\[ \int_0^1 M_q(r, h')dr < \infty . \]

Thus, in view of the inequality (12), it follows that \( h \in H^q \), which was to be shown.
Conversely, suppose \( \{\lambda_n\} \) is a multiplier of \( H^p \) into \( H^q \) for arbitrary \( q(0 < q \leq \infty) \). Then by the closed graph theorem,

\[
\Lambda: \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n
\]

is a bounded operator from \( H^p \) to \( H^q \). An argument similar to that used in the proof of Theorem 3 now leads to the estimate (11).

**COROLLARY.** If \( 0 < p < 1 \leq q \leq \infty \) and \( f \in B^p \), then its fractional integral \( f_\alpha \in H^q \), where \( \alpha = 1/p - 1/q \). This is false if \( q < 1 \).

This corollary can also be proved directly. Indeed, since ([2], Th. 5) the fractional integral of order \((1/p - 1/s)\) of a \( B^p \) function is in \( B^s \) \((0 < s < 1)\), and since ([8], p. 415) the fractional integral of order \((1 - 1/q)\) of an \( H^1 \) function is in \( H^q(1 \leq q \leq \infty) \), it suffices to show that \( f' \in B^{1/2} \) implies \( f \in H^1 \). But this is easy; it follows from (12) with \( q = 1 \). That the corollary is false for \( q < 1 \) is a consequence of the fact ([2], Th. 5) that the fractional derivative of order \((1/p - 1/q)\) of every \( B^q \) function is in \( B^p \).

The converse is also false. That is, if \( f \in H^q \), its fractional derivative of order \((1/p - 1/q)\) need not be in \( B^p(0 < p < 1 \leq q \leq \infty) \). As before, this reduces to showing that \( f \in H^1 \) does not imply \( f' \in B^{1/2} \). To see this, let \( f(z) = \sum c_k z^{n_k} \), where \( \{n_k\} \) is lacunary, \( \{c_k\} \in \ell^2 \), and \( \{c_k\} \in \ell^1 \). Then \( f \in H^2 \subset H^1 \), but \( f' \in B^{1/2} \), since it was shown in [4] (Th. 3, Corollary 2) that

\[
\sum_{k=1}^{\infty} |n_k|^{1/p} |a_{n_k}| < \infty
\]

whenever \( \sum a_n z^n \in B^p \) and \( \{n_k\} \) is a lacunary sequence.

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