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LOCAL ISOMETRIES OF FLAT TORI

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H. G. HELFENSTEIN

Let T_1 and T_2 be two flat tori (i.e., provided with a complete Riemannian metric of vanishing curvature). Since they are locally Euclidean each pair of points $P_1, P_2, P_i \in T_i$, has isometric neighborhoods. In general it is not possible, however, to join these separate isometries of neighborhoods to produce a single isometry $T_1 \rightarrow T_2$ or $T_2 \rightarrow T_1$; indeed there may not even exist a locally isometric map (of the whole surfaces). Necessary and sufficient conditions for the existence of such maps are deduced, making use of a recent conformal classification of maps between tori. As expected "ample" and nonample tori behave differently, and the determination of all local isometries leads to number-theoretic problems. Finally, for two given tori, the local isometries are compared with respect to homotopy by analyzing their effect on the fundamental groups.

Let \mathbf{R}^+ denote the positive reals, H the upper z -half-plane, and $SL(2, \mathbf{Z})$ the group of all 2×2 unimodular matrices with integral entries acting in the usual way as hyperbolic motions on H . The set of isometry classes of complete flat tori is parametrized by the 3-dimensional manifold $\mathbf{R}^+ \times (H/SL(2, \mathbf{Z}))$. A point (r^2, τ) of this space represents the isometry class of the torus E^2/Γ , where Γ is the group of Euclidean motions generated by the translations

$$t_1(z) = z + r \quad \text{and} \quad t_2(z) = z + rh,$$

with $h \in \tau$, (cf. [2]). Instead of "an isometry class of tori" we speak simply of "a torus". A torus $T = (r^2, \tau)$ is called *ample* if there exists $h \in \tau$ such that both $\Re h$ and $|h|^2$ are rational.

2. Riemannian covering maps. The following statements are generalizations of results obtained in [1] which can be similarly proved.

(i) For two tori $T_i = (r_i^2, \tau_i)$ there exist conformal covering maps $T_1 \rightarrow T_2$ if and only if two representatives $h_i \in \tau_i$ are equivalent under the action of the group $GL^+(2, \mathbf{Q}) =$ group of 2×2 matrices with rational entries and positive determinant.

(ii) Lifting any conformal covering $T_1 \rightarrow T_2$ to the universal covering planes we obtain

$$(1) \quad F(z, C, D) = Cz + D,$$

with complex constants $C \neq 0$ and D .

(iii) For nonample T_i only

$$(2) \quad C(\kappa) = \frac{r_2}{r_1} \kappa, \quad \kappa = \pm 1, \pm 2, \dots$$

are admissible values in (1).

(iv) For ample $T_i = (r_i^2, \tau_i)$ (2) is replaced by

$$(3) \quad C(\kappa_1, \kappa_2) = \frac{r_2}{r_1} (\kappa_1 + \kappa_2 q'' s'' h_2),$$

where $h_2 \in \tau_2$, $h_1 = ah_2$, a an integer, $(\kappa_1, \kappa_2) \neq (0, 0)$ is a pair of arbitrary integers, and the integers q'' , s'' are determined via the following relations,

$$2\Re h_2 = \frac{p}{q}, \quad |h_2|^2 = \frac{r}{s},$$

$p, q > 0, r > 0, s > 0$ integers,

$$\begin{aligned} \text{g.c.d.}(p, q) &= \text{g.c.d.}(r, s) = 1, \\ g &= \text{g.c.d.}(q, s), \quad q' = q/g, \quad s' = s/g, \\ g' &= \text{g.c.d.}(a, q), \quad a' = a/g', \quad q'' = q/g', \\ g'' &= \text{g.c.d.}(a', s'), \quad a'' = a'/g'', \quad s'' = s'/g''. \end{aligned}$$

The following matrices are computable from these numbers.

$$\tilde{T}_1 = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} a'ps'', & -a'q'r \\ q''s'', & 0 \end{pmatrix}$$

Our main result is

THEOREM 1. *For the existence of a local isometry $f: T_1 \rightarrow T_2$ the following conditions are necessary and sufficient:*

(1) τ_1 and τ_2 are equivalent under $GL^+(2, \mathbb{Q})$;

(2a) If T_1 is nonample, then r_1/r_2 must be an integer;

(2b) If T_1 is ample, then $(r_1^2/r_2^2)a$ must be an integer N , and N must be representable by the quadratic form

$$(4) \quad \det(\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2)$$

with suitable integers κ_1 and κ_2 .

Proof. Since f is a conformal covering we have necessarily (1) by (i). The following identity is readily verified:

$$\frac{r_1^2}{r_2^2} |C|^2 a = \begin{cases} \det(\kappa \tilde{T}_1) & \text{for } T_1 \text{ nonample} \\ \det(\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2) & \text{for } T_1 \text{ ample.} \end{cases}$$

(The right hand side gives the number N of sheets of the covering f).

Together with the condition $|C| = 1$ for local isometry it leads to (2a) and (2b). The sufficiency follows from (iii) and (iv).

In both cases we have the following consequences. A flat torus can cover a countably infinite set of tori by local isometries. For $T_1 = T_2$ a local isometry is a global isometry, since $|C| = 1$ entails $N = 1$. In general the existence of a local isometry $T_1 \rightarrow T_2$ does not imply that there is also a local isometry $T_2 \rightarrow T_1$; this occurs if and only if both $r_1 = r_2$ and condition (1) are satisfied. (Then the tori still need not be globally isometric).

3. Homotopy classes. We show how the combination $\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2$ controls also the deformation properties of our maps. If the constant D in (ii) is varied the map stays in the same homotopy class, but maps corresponding to different parameter values κ or (κ_1, κ_2) are not analytically homotopic (i.e., with analytic intermediately stages during the deformation), since the set of admissible values of C is discrete. We show that they are not even homotopic in the ordinary sense.

Since the fundamental group $\pi_1(T)$ of a torus is Abelian the set \mathcal{H} of homotopy classes of continuous maps $T_1 \rightarrow T_2$ is in one-to-one correspondence with the set of all homomorphisms $\eta: \pi_1(T_1) \rightarrow \pi_1(T_2)$. Denoting by L_i and L'_i ($i = 1, 2$) the path homotopy classes of two generating loops of $\pi_1(T_i)$, each such η is characterized by the integral matrix

$$\hat{\xi} = \begin{pmatrix} \hat{\xi}_4 & \hat{\xi}_3 \\ \hat{\xi}_2 & \hat{\xi}_1 \end{pmatrix}$$

given by

$$\eta(L_i) = L_2^{\hat{\xi}_1} L_1^{\hat{\xi}_2}, \eta(L'_i) = L_2^{\hat{\xi}_3} L_1^{\hat{\xi}_4};$$

hence \mathcal{H} is parametrized by Z^4 . The subset $\{\hat{\xi} \in Z^4: \det \hat{\xi} \neq 0\}$ contains those points of Z^4 representing monomorphisms, hence it corresponds to the homotopy classes containing covering maps.

THEOREM 2. *The subset of Z^4 corresponding to homotopy classes which contain analytic maps consists of*

- (a) 0 only if τ_1 and τ_2 are nonequivalent under $GL^+(2, \mathbb{Q})$;
- (b) the 1-dimensional sublattice spanned by \tilde{T}_1 if τ_1 and τ_2 are equivalent under $GL^+(2, \mathbb{Q})$ and both are nonample;
- (c) the 2-dimensional sublattice spanned by \tilde{T}_1 and \tilde{T}_2 if τ_1 and τ_2 are equivalent under $GL^+(2, \mathbb{Q})$ and both are ample.

Proof. We prove only (c); (a) and (b) can be handled similarly. The generators L_i, L'_i of $\pi_1(T_i)$ are represented in E_i by the segments S_i, S'_i joining the origin to r_i and $r_i h_i$ respectively. The segments S_1

and S'_1 are mapped by $F(z; C, 0)$ (cf. (ii)) into segments from the origin of E_2 to the points

$$\kappa_1 r_2 + \kappa_2 s'' q'' r_2 h_2$$

and

$$-\kappa_2 r a'' q' r_2 + (\kappa_1 a + \kappa_2 s'' p a') r_2 h_2 .$$

The former can be deformed into the two sides $\kappa_1 r_2$ and $\kappa_2 s'' q'' r_2 h_2$ of a parallelogram parallel to S_2 and S'_2 . The first side represents κ_1 circuits of L_2 , the second $\kappa_2 s'' q''$ contours of L'_2 . Similarly for S'_1 . Hence the homomorphism

$$f_*: \pi_1(T_1) \longrightarrow \pi_1(T_2)$$

induced by f is determined by

$$f_*(L_1) = L_2^{\kappa_1} L'_2{}^{\kappa_2 s'' q''}$$

and

$$f_*(L'_1) = L_2^{-\kappa_2 r a'' q'} L_2^{\kappa_1 a + \kappa_2 s'' p a'} .$$

This is equivalent to $\xi = \kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2$.

The determination of *all* local isometries for two given tori is easy for the nonample case. In the ample case it involves the number of ways in which $N = (r_1^2/r_2^2)a$ can be represented by the quadratic form (4). Since this form is positive definite we have, in conjunction with Theorem 2:

THEOREM 3. *The number of homotopy classes of local isometries between two flat tori is finite.*

We obtain an upper bound for this number as follows: From (3) we find

$$\Re C = \frac{r_2}{r_1} \left(\kappa_1 + \kappa_2 s'' \frac{p}{2g'} \right),$$

which shows that $\Re C$ has the form $(r_2/r_1)(\gamma/2g')$, with γ an integer. Substituting this in $|\Re C| \leq |C| = 1$ leads to

$$(5) \quad |\gamma| \leq 2g' \frac{r_1}{r_2} .$$

From $(\Im C)^2 = |C|^2 - (\Re C)^2$ we deduce

$$(6) \quad \kappa_2^2 g'^{r_2} s'^{r_2} (\mathfrak{I}h_2)^2 = \frac{r_1^2}{r_2^2} - \frac{\gamma^2}{4g'^2}$$

and

$$(7) \quad \kappa_1 = \frac{\gamma}{2g'} - \kappa_2 s'' \frac{p}{2g'}$$

Each of the $2[2g'(r_1/r_2)] + 1$ integers γ compatible with (5) leads to at most two pairs (κ_1, κ_2) compatible with (6) and (7). Thus the number of homotopically different local isometries does not exceed $4[2g'(r_1/r_2)] + 2$.

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Robert Alexander Adams, <i>Compact Sobolev imbeddings for unbounded domains</i>	1
Bernhard Amberg, <i>Groups with maximum conditions</i>	9
Tom M. (Mike) Apostol, <i>Möbius functions of order k</i>	21
Stefan Bergman, <i>On an initial value problem in the theory of two-dimensional transonic flow patterns</i>	29
Geoffrey David Downs Creede, <i>Concerning semi-stratifiable spaces</i>	47
Edmond Dale Dixon, <i>Matric polynomials which are higher commutators</i>	55
R. L. Duncan, <i>Some continuity properties of the Schnirelmann density. II</i>	65
Peter Larkin Duren and Allen Lowell Shields, <i>Coefficient multipliers of H^p and B^p spaces</i>	69
Hector O. Fattorini, <i>On a class of differential equations for vector-valued distributions</i>	79
Charles Hallahan, <i>Stability theorems for Lie algebras of derivations</i>	105
Heinz Helfenstein, <i>Local isometries of flat tori</i>	113
Gerald J. Janusz, <i>Some remarks on Clifford's theorem and the Schur index</i>	119
Joe W. Jenkins, <i>Symmetry and nonsymmetry in the group algebras of discrete groups</i>	131
Herbert Frederick Kreimer, Jr., <i>Outer Galois theory for separable algebras</i>	147
D. G. Larman and P. Mani, <i>On visual hulls</i>	157
R. Robert Laxton, <i>On groups of linear recurrences. II. Elements of finite order</i>	173
Dong Hoon Lee, <i>The adjoint group of Lie groups</i>	181
James B. Lucke, <i>Commutativity in locally compact rings</i>	187
Charles Harris Scanlon, <i>Rings of functions with certain Lipschitz properties</i>	197
Binyamin Schwarz, <i>Totally positive differential systems</i>	203
James McLean Sloss, <i>The bending of space curves into piecewise helical curves</i>	231
James D. Stafney, <i>Analytic interpolation of certain multiplier spaces</i>	241
Patrick Noble Stewart, <i>Semi-simple radical classes</i>	249
Hiroyuki Tachikawa, <i>On left QF - 3 rings</i>	255
Glenn Francis Webb, <i>Product integral representation of time dependent nonlinear evolution equations in Banach spaces</i>	269