LOCAL ISOMETRIES OF FLAT TORI

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Let $T_1$ and $T_2$ be two flat tori (i.e., provided with a complete Riemannian metric of vanishing curvature). Since they are locally Euclidean each pair of points $P_1, P_2, P_i \in T_i$, has isometric neighborhoods. In general it is not possible, however, to join these separate isometries of neighborhoods to produce a single isometry $T_1 \to T_2$ or $T_2 \to T_1$; indeed there may not even exist a locally isometric map (of the whole surfaces). Necessary and sufficient conditions for the existence of such maps are deduced, making use of a recent conformal classification of maps between tori. As expected “ample” and nonample tori behave differently, and the determination of all local isometries leads to number-theoretic problems. Finally, for two given tori, the local isometries are compared with respect to homotopy by analyzing their effect on the fundamental groups.

Let $\mathbb{R}^+$ denote the positive reals, $H$ the upper $z$-half-plane, and $SL(2, \mathbb{Z})$ the group of all $2 \times 2$ unimodular matrices with integral entries acting in the usual way as hyperbolic motions on $H$. The set of isometry classes of complete flat tori is parametrized by the 3-dimensional manifold $\mathbb{R}^+ \times (H/SL(2, \mathbb{Z}))$. A point $(r^2, \tau)$ of this space represents the isometry class of the torus $E^2/\Gamma$, where $\Gamma$ is the group of Euclidean motions generated by the translations

$$t_1(z) = z + r \quad \text{and} \quad t_2(z) = z + rh,$$

with $h \in \tau$, (cf. [2]). Instead of “an isometry class of tori” we speak simply of “a torus”. A torus $T = (r^2, \tau)$ is called ample if there exists $h \in \tau$ such that both $\Re h$ and $|h|^2$ are rational.

2. Riemannian covering maps. The following statements are generalizations of results obtained in [1] which can be similarly proved.

(i) For two tori $T_i = (r^i, \tau_i)$ there exist conformal covering maps $T_1 \to T_2$ if and only if two representatives $h_i \in \tau_i$ are equivalent under the action of the group $GL^+(2, \mathbb{Q}) = \text{group of } 2 \times 2 \text{ matrices with rational entries and positive determinant.}$

(ii) Lifting any conformal covering $T_1 \to T_2$ to the universal covering planes we obtain

$$F(z, C, D) = Cz + D,$$

with complex constants $C \neq 0$ and $D$.

(iii) For nonample $T_i$ only
are admissible values in (1).

(iv) For ample \( T_i = (r_i^1, \tau_i) \) (2) is replaced by

\[
C(\kappa_1, \kappa_2) = \frac{r_2}{r_1}(\kappa_1 + \kappa_2 q'' s'' h_2),
\]

where \( h_2 \in \tau_2, h_1 = a h_2, a \) an integer, \((\kappa_1, \kappa_2) \neq (0, 0)\) is a pair of arbitrary integers, and the integers \( q'', s'' \) are determined via the following relations,

\[
2\Re h_2 = \frac{p}{q}, \quad |h_2|^2 = \frac{r}{s},
\]

\( p, q > 0, r > 0, s > 0 \) integers,

\[
g = \gcd (p, q) = \gcd (r, s) = 1,
\]

\[
g' = \gcd (q, s), q' = q/g, s' = s/g,
\]

\[
g'' = \gcd (a, q), a' = a/g', q'' = q/g',
\]

\[
g''' = \gcd (a', s'), a''' = a'/g'', s''' = s'/g''.
\]

The following matrices are computable from these numbers.

\[
\bar{T}_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{T}_2 = \begin{pmatrix} a' ps'' & -a'' q' r' \\ q'' s'' & 0 \end{pmatrix}
\]

Our main result is

**THEOREM 1.** For the existence of a local isometry \( f: T_1 \rightarrow T_2 \) the following conditions are necessary and sufficient:

1. \( \tau_1 \) and \( \tau_2 \) are equivalent under \( GL^+(2, \mathbb{Q}) \);
2. (a) If \( T_1 \) is nonample, then \( r_1/r_2 \) must be an integer;
   (b) If \( T_1 \) is ample, then \( (r_1^1/r_2^1) a \) must be an integer \( N \), and \( N \) must be representable by the quadratic form

\[
\det (\kappa_1 \bar{T}_1 + \kappa_2 \bar{T}_2)
\]

with suitable integers \( \kappa_1 \) and \( \kappa_2 \).

**Proof.** Since \( f \) is a conformal covering we have necessarily (1) by (i). The following identity is readily verified:

\[
\frac{r_1^2}{r_2^2} |C|^2 a = \begin{cases} \det (\kappa_1 \bar{T}_1) & \text{for } T_1 \text{ nonample} \\
\det (\kappa_1 \bar{T}_1 + \kappa_2 \bar{T}_2) & \text{for } T_1 \text{ ample}. \end{cases}
\]

(The right hand side gives the number \( N \) of sheets of the covering \( f \).)
Together with the condition $|C| = 1$ for local isometry it leads to (2a) and (2b). The sufficiency follows from (iii) and (iv).

In both cases we have the following consequences. A flat torus can cover a countably infinite set of tori by local isometries. For $T_1 = T_2$ a local isometry is a global isometry, since $|C| = 1$ entails $N = 1$. In general the existence of a local isometry $T_1 \to T_2$ does not imply that there is also a local isometry $T_2 \to T_1$; this occurs if and only if both $r_1 = r_2$ and condition (1) are satisfied. (Then the tori still need not be globally isometric).

3. Homotopy classes. We show how the combination $\kappa_1 T_1 + \kappa_2 T_2$ controls also the deformation properties of our maps. If the constant $D$ in (ii) is varied the map stays in the same homotopy class, but maps corresponding to different parameter values $\kappa$ or $(\kappa_1, \kappa_2)$ are not analytically homotopic (i.e., with analytic intermediate stages during the deformation), since the set of admissible values of $C$ is discrete. We show that they are not even homotopic in the ordinary sense.

Since the fundamental group $\pi_1(T)$ of a torus is Abelian the set $\mathcal{H}$ of homotopy classes of continuous maps $T_1 \to T_2$ is in one-to-one correspondence with the set of all homomorphisms $\eta: \pi_1(T_1) \to \pi_1(T_2)$. Denoting by $L_i$ and $L'_i$ ($i = 1, 2$) the path homotopy classes of two generating loops of $\pi_1(T_i)$, each such $\eta$ is characterized by the integral matrix

$$
\hat{\eta} = \begin{pmatrix}
\hat{\xi}_1 & \hat{\xi}_2 \\
\hat{\xi}_3 & \hat{\xi}_4
\end{pmatrix}
$$

given by

$$
\eta(L_i) = L_i^{\hat{\xi}_1} L_i^{\hat{\xi}_2}; \quad \eta(L'_i) = L_i^{\hat{\xi}_3} L_i^{\hat{\xi}_4};
$$

hence $\mathcal{H}$ is parametrized by $Z^4$. The subset $\{\xi \in Z^4: \det \xi \neq 0\}$ contains those points of $Z^4$ representing monomorphisms, hence it corresponds to the homotopy classes containing covering maps.

**Theorem 2.** The subset of $Z^4$ corresponding to homotopy classes which contain analytic maps consists of

(a) 0 only if $\tau_1$ and $\tau_2$ are nonequivalent under $GL^+(2, \mathbb{Q})$;

(b) the 1-dimensional sublattice spanned by $\bar{T}_1$ if $\tau_1$ and $\tau_2$ are equivalent under $GL^+(2, \mathbb{Q})$ and both are nonample;

(c) the 2-dimensional sublattice spanned by $\bar{T}_1$ and $\bar{T}_2$ if $\tau_1$ and $\tau_2$ are equivalent under $GL^+(2, \mathbb{Q})$ and both are ample.

*Proof.* We prove only (c); (a) and (b) can be handled similarly. The generators $L_i, L'_i$ of $\pi_1(T_i)$ are represented in $E_i$ by the segments $S_i, S'_i$ joining the origin to $r_i$ and $r_i h_i$ respectively. The segments $S_i$
and $S'$ are mapped by $F(z; C, 0)$ (cf. (ii)) into segments from the origin of $E_2$ to the points
\[ \kappa_1 r_2 + \kappa_2 s'' q'' r_2 h_2 \]
and
\[ -\kappa_2 a'' q' r_2 + (\kappa_1 a + \kappa_2 s'' p a') r_2 h_2. \]

The former can be deformed into the two sides $\kappa_1 r_2$ and $\kappa_2 s'' q'' r_2 h_2$ of a parallelogram parallel to $S_2$ and $S'_2$. The first side represents $\kappa_1$ circuits of $L_2$, the second $\kappa_2 s'' q''$ contours of $L'_2$. Similarly for $S'$. Hence the homomorphism
\[ f_* : \pi_1(T_1) \longrightarrow \pi_1(T_2) \]
induced by $f$ is determined by
\[ f_*(L_i) = L_2^{\kappa_1 s'' q''} L_2' L_2^{r_2 s'' q''} \]
and
\[ f_*(L'_i) = L_2^{-\kappa_2 a'' q'} L_2' L_2^{r_2 s'' p a'}. \]
This is equivalent to $\xi = \kappa_1 T_1 + \kappa_2 T_2$.

The determination of all local isometries for two given tori is easy for the nonample case. In the ample case it involves the number of ways in which $N = (r_2/r_1)a$ can be represented by the quadratic form (4). Since this form is positive definite we have, in conjunction with Theorem 2:

**Theorem 3.** The number of homotopy classes of local isometries between two flat tori is finite.

We obtain an upper bound for this number as follows: From (3) we find
\[ \mathcal{NC} = \frac{r_2}{r_1} \left( \kappa_1 + \kappa_2 s'' \frac{p}{2g'} \right), \]
which shows that $\mathcal{NC}$ has the form $(r_2/r_1)(\gamma/2g')$, with $\gamma$ an integer. Substituting this in $|\mathcal{NC}| \leq |C| = 1$ leads to
\[ |\gamma| \leq 2g' \frac{T_1}{r_2}. \]
From $(\mathcal{N}C)^2 = |C|^2 - (\mathcal{NC})^2$ we deduce
Each of the $2[2g'(r_1/r_2)] + 1$ integers $\gamma$ compatible with (5) leads to at most two pairs $(\kappa_1, \kappa_2)$ compatible with (6) and (7). Thus the number of homotopically different local isometries does not exceed $4[2g'(r_1/r_2)] + 2$.

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