

# Pacific Journal of Mathematics

**LOCAL ISOMETRIES OF FLAT TORI**

HEINZ HELFENSTEIN

## LOCAL ISOMETRIES OF FLAT TORI

H. G. HELFENSTEIN

Let  $T_1$  and  $T_2$  be two flat tori (i.e., provided with a complete Riemannian metric of vanishing curvature). Since they are locally Euclidean each pair of points  $P_1, P_2, P_i \in T_i$ , has isometric neighborhoods. In general it is not possible, however, to join these separate isometries of neighborhoods to produce a single isometry  $T_1 \rightarrow T_2$  or  $T_2 \rightarrow T_1$ ; indeed there may not even exist a locally isometric map (of the whole surfaces). Necessary and sufficient conditions for the existence of such maps are deduced, making use of a recent conformal classification of maps between tori. As expected "ample" and nonample tori behave differently, and the determination of all local isometries leads to number-theoretic problems. Finally, for two given tori, the local isometries are compared with respect to homotopy by analyzing their effect on the fundamental groups.

Let  $\mathbf{R}^+$  denote the positive reals,  $H$  the upper  $z$ -half-plane, and  $SL(2, \mathbf{Z})$  the group of all  $2 \times 2$  unimodular matrices with integral entries acting in the usual way as hyperbolic motions on  $H$ . The set of isometry classes of complete flat tori is parametrized by the 3-dimensional manifold  $\mathbf{R}^+ \times (H/SL(2, \mathbf{Z}))$ . A point  $(r^2, \tau)$  of this space represents the isometry class of the torus  $E^2/\Gamma$ , where  $\Gamma$  is the group of Euclidean motions generated by the translations

$$t_1(z) = z + r \quad \text{and} \quad t_2(z) = z + rh,$$

with  $h \in \tau$ , (cf. [2]). Instead of "an isometry class of tori" we speak simply of "a torus". A torus  $T = (r^2, \tau)$  is called *ample* if there exists  $h \in \tau$  such that both  $\Re h$  and  $|h|^2$  are rational.

2. Riemannian covering maps. The following statements are generalizations of results obtained in [1] which can be similarly proved.

(i) For two tori  $T_i = (r_i^2, \tau_i)$  there exist conformal covering maps  $T_1 \rightarrow T_2$  if and only if two representatives  $h_i \in \tau_i$  are equivalent under the action of the group  $GL^+(2, \mathbf{Q}) =$  group of  $2 \times 2$  matrices with rational entries and positive determinant.

(ii) Lifting any conformal covering  $T_1 \rightarrow T_2$  to the universal covering planes we obtain

$$(1) \quad F(z, C, D) = Cz + D,$$

with complex constants  $C \neq 0$  and  $D$ .

(iii) For nonample  $T_i$  only

$$(2) \quad C(\kappa) = \frac{r_2}{r_1} \kappa, \quad \kappa = \pm 1, \pm 2, \dots$$

are admissible values in (1).

(iv) For ample  $T_i = (r_i^2, \tau_i)$  (2) is replaced by

$$(3) \quad C(\kappa_1, \kappa_2) = \frac{r_2}{r_1} (\kappa_1 + \kappa_2 q'' s'' h_2),$$

where  $h_2 \in \tau_2$ ,  $h_1 = ah_2$ ,  $a$  an integer,  $(\kappa_1, \kappa_2) \neq (0, 0)$  is a pair of arbitrary integers, and the integers  $q'', s''$  are determined via the following relations,

$$2\Re h_2 = \frac{p}{q}, \quad |h_2|^2 = \frac{r}{s},$$

$p, q > 0, r > 0, s > 0$  integers,

$$\begin{aligned} \text{g.c.d.}(p, q) &= \text{g.c.d.}(r, s) = 1, \\ g &= \text{g.c.d.}(q, s), \quad q' = q/g, \quad s' = s/g, \\ g' &= \text{g.c.d.}(a, q), \quad a' = a/g', \quad q'' = q/g', \\ g'' &= \text{g.c.d.}(a', s'), \quad a'' = a'/g'', \quad s'' = s'/g''. \end{aligned}$$

The following matrices are computable from these numbers.

$$\tilde{T}_1 = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} a'ps'', & -a'q'r \\ q''s'', & 0 \end{pmatrix}$$

Our main result is

**THEOREM 1.** *For the existence of a local isometry  $f: T_1 \rightarrow T_2$  the following conditions are necessary and sufficient:*

- (1)  $\tau_1$  and  $\tau_2$  are equivalent under  $GL^+(2, \mathbb{Q})$ ;
- (2a) If  $T_1$  is nonample, then  $r_1/r_2$  must be an integer;
- (2b) If  $T_1$  is ample, then  $(r_1^2/r_2^2)a$  must be an integer  $N$ , and  $N$  must be representable by the quadratic form

$$(4) \quad \det(\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2)$$

with suitable integers  $\kappa_1$  and  $\kappa_2$ .

*Proof.* Since  $f$  is a conformal covering we have necessarily (1) by (i). The following identity is readily verified:

$$\frac{r_1^2}{r_2^2} |C|^2 a = \begin{cases} \det(\kappa \tilde{T}_1) & \text{for } T_1 \text{ nonample} \\ \det(\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2) & \text{for } T_1 \text{ ample.} \end{cases}$$

(The right hand side gives the number  $N$  of sheets of the covering  $f$ ).

Together with the condition  $|C| = 1$  for local isometry it leads to (2a) and (2b). The sufficiency follows from (iii) and (iv).

In both cases we have the following consequences. A flat torus can cover a countably infinite set of tori by local isometries. For  $T_1 = T_2$  a local isometry is a global isometry, since  $|C| = 1$  entails  $N = 1$ . In general the existence of a local isometry  $T_1 \rightarrow T_2$  does not imply that there is also a local isometry  $T_2 \rightarrow T_1$ ; this occurs if and only if both  $r_1 = r_2$  and condition (1) are satisfied. (Then the tori still need not be globally isometric).

**3. Homotopy classes.** We show how the combination  $\kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2$  controls also the deformation properties of our maps. If the constant  $D$  in (ii) is varied the map stays in the same homotopy class, but maps corresponding to different parameter values  $\kappa$  or  $(\kappa_1, \kappa_2)$  are not analytically homotopic (i.e., with analytic intermediately stages during the deformation), since the set of admissible values of  $C$  is discrete. We show that they are not even homotopic in the ordinary sense.

Since the fundamental group  $\pi_1(T)$  of a torus is Abelian the set  $\mathcal{H}$  of homotopy classes of continuous maps  $T_1 \rightarrow T_2$  is in one-to-one correspondence with the set of all homomorphisms  $\eta: \pi_1(T_1) \rightarrow \pi_1(T_2)$ . Denoting by  $L_i$  and  $L'_i$  ( $i = 1, 2$ ) the path homotopy classes of two generating loops of  $\pi_1(T_i)$ , each such  $\eta$  is characterized by the integral matrix

$$\xi = \begin{pmatrix} \xi_{41} & \xi_{31} \\ \xi_{21} & \xi_{11} \end{pmatrix}$$

given by

$$\eta(L_i) = L_2^{\xi_{i1}} L_1^{\xi_{i2}}, \eta(L'_i) = L_2^{\xi_{i3}} L_1^{\xi_{i4}};$$

hence  $\mathcal{H}$  is parametrized by  $Z^4$ . The subset  $\{\xi \in Z^4: \det \xi \neq 0\}$  contains those points of  $Z^4$  representing monomorphisms, hence it corresponds to the homotopy classes containing covering maps.

**THEOREM 2.** *The subset of  $Z^4$  corresponding to homotopy classes which contain analytic maps consists of*

- (a) 0 only if  $\tau_1$  and  $\tau_2$  are nonequivalent under  $GL^+(2, \mathbb{Q})$ ;
- (b) the 1-dimensional sublattice spanned by  $\tilde{T}_1$  if  $\tau_1$  and  $\tau_2$  are equivalent under  $GL^+(2, \mathbb{Q})$  and both are nonample;
- (c) the 2-dimensional sublattice spanned by  $\tilde{T}_1$  and  $\tilde{T}_2$  if  $\tau_1$  and  $\tau_2$  are equivalent under  $GL^+(2, \mathbb{Q})$  and both are ample.

*Proof.* We prove only (c); (a) and (b) can be handled similarly. The generators  $L_i, L'_i$  of  $\pi_1(T_i)$  are represented in  $E_i$  by the segments  $S_i, S'_i$  joining the origin to  $r_i$  and  $r_i h_i$  respectively. The segments  $S_i$

and  $S'_1$  are mapped by  $F(z; C, 0)$  (cf. (ii)) into segments from the origin of  $E_2$  to the points

$$\kappa_1 r_2 + \kappa_2 s'' q'' r_2 h_2$$

and

$$-\kappa_2 r a'' q' r_2 + (\kappa_1 a + \kappa_2 s'' p a') r_2 h_2 .$$

The former can be deformed into the two sides  $\kappa_1 r_2$  and  $\kappa_2 s'' q'' r_2 h_2$  of a parallelogram parallel to  $S_2$  and  $S'_2$ . The first side represents  $\kappa_1$  circuits of  $L_2$ , the second  $\kappa_2 s'' q''$  contours of  $L'_2$ . Similarly for  $S'_1$ . Hence the homomorphism

$$f_* : \pi_1(T_1) \longrightarrow \pi_1(T_2)$$

induced by  $f$  is determined by

$$f_*(L_1) = L_2^{\kappa_1} L_2'^{\kappa_2 s'' q''}$$

and

$$f_*(L'_1) = L_2^{-\kappa_2 r a'' q'} L_2'^{\kappa_1 a + \kappa_2 s'' p a'} .$$

This is equivalent to  $\xi = \kappa_1 \tilde{T}_1 + \kappa_2 \tilde{T}_2$ .

The determination of *all* local isometries for two given tori is easy for the nonample case. In the ample case it involves the number of ways in which  $N = (r_1^2/r_2^2)a$  can be represented by the quadratic form (4). Since this form is positive definite we have, in conjunction with Theorem 2:

**THEOREM 3.** *The number of homotopy classes of local isometries between two flat tori is finite.*

We obtain an upper bound for this number as follows: From (3) we find

$$\Re C = \frac{r_2}{r_1} \left( \kappa_1 + \kappa_2 s'' \frac{p}{2g'} \right) ,$$

which shows that  $\Re C$  has the form  $(r_2/r_1)(\gamma/2g')$ , with  $\gamma$  an integer. Substituting this in  $|\Re C| \leq |C| = 1$  leads to

$$(5) \quad |\gamma| \leq 2g' \frac{r_1}{r_2} .$$

From  $(\Im C)^2 = |C|^2 - (\Re C)^2$  we deduce

$$(6) \quad \kappa_2^2 Q''^2 S''^2 (\mathfrak{F}h_2)^2 = \frac{r_1^2}{r_2^2} - \frac{\gamma^2}{4g'^2}$$

and

$$(7) \quad \kappa_1 = \frac{\gamma}{2g'} - \kappa_2 S'' \frac{p}{2g'}$$

Each of the  $2[2g'(r_1/r_2)] + 1$  integers  $\gamma$  compatible with (5) leads to at most two pairs  $(\kappa_1, \kappa_2)$  compatible with (6) and (7). Thus the number of homotopically different local isometries does not exceed  $4[2g'(r_1/r_2)] + 2$ .

#### BIBLIOGRAPHY

1. H. Helfenstein, *Analytic maps between tori*, Bull. Amer. Math. Soc. Vol. **75**, No. 4, 857-859.
2. J. A. Wolf, *Spaces of constant curvature*, New York, 1967.

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