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ON VISUAL HULLS

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The concept of visual hull has been introduced by G. H. Meisters and S. Ulam. In the following article we study a few of the problems arising from this notion and, in particular, establish (Theorem 3) a conjecture of W. A. Beyer and S. Ulam.

Let C be a set in \mathbb{R}^n and $1 \leq j \leq n-1$. Then the j^{th} visual hull $H_i(C)$ of C is defined to be the largest set whose j^{th} projections are contained in those of C. Alternatively, $H_i(C)$ is the set of points x in \mathbb{R}^n such that each (n-j)-flat through x contains a point of C. Let G_j^n denote the Grassmannian of j-subspaces in \mathbb{R}^n with $\mu_j(G_j^n) = 1$ for the usual measure μ_i associated with G_i^n regarded as a metric 0_n factorspace. (For further information about μ_i compare, for example, [3]). The j^{th} virtual hull $V_j(C)$ of C is defined to be the set of points $x \in \mathbb{R}^n$ such that almost all (with respect to μ_{n-i}) (n-j)-flats through x contain a point of C. Thus, if $n = 3, j = 2, H_2(C)(V_2(C))$ corresponds to those points in R^3 which are photographically indistinguishable (with probability one) from C. A j^{th} minimal hull of C in \mathbb{R}^n is a minimal set in R^n whose j^{th} projections coincide with those of C. In [2] the announced purpose of the paper was to disprove the conjecture that $H_i(C) - C$ is connected to C, i.e., \ni disjoint open sets U, V such that $U \supset H_i(C) - C \neq \emptyset$ and $V \supset C \neq \emptyset$. To this we remark that a simple counterexample can be obtained by considering the closed set C formed by removing the relative interiors of alternate sides of a regular hexagon inscribed in a plane circle with centre a. The first visual hull $H_1(C)$ is then $C \cup \{a\}$.

2. Visual hulls of unions of polytopes.

THEOREM 1. Let A_1, \dots, A_{j+1} be spherically convex, closed subsets (not necessarily nonempty) of the sphere S^{n-1} , such that each (n - j - 1)-subsphere of S^{n-1} has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. Then $A_1 \cap \dots \cap A_{j+1} \neq \emptyset$. (so, that, in particular, each set A_i is nonempty).

REMARK. S^{n-1} is the unit sphere of R^n and an (n-j-1)-subsphere of S^{n-1} is the intersection of an n-j subspace with S^{n-1} . A set $C \subset S^{n-1}$ is spherically convex if C is contained in an open hemisphere of S^{n-1} and, if $x, y \in C$ then C contains the minor arc on the 1-subsphere determined by x, y and 0 (the centre of S^{n-1}).

Proof. The case n = 1 is trivial. We assume inductively that

the result is true for all n' < n and it remains to prove the result for j + 1 sets on S^{n-1} . Assume on the contrary that there exist spherically convex closed subsets $A_1, \dots, A_{j+1} \subset S^{n-1}$ such that

$$T \cap (A_1 \cup \cdots \cup A_{j+1}) \neq \emptyset$$

for each (n-j-1)-subsphere T of S^{n-1} , and $A_1 \cap \cdots \cap A_{j+1} = \emptyset$. Let $A = A_1 \cap \cdots \cap A_j$. Then A, A_{j+1} are disjoint spherically convex closed subsets of S^{n-1} , and there exists an (n-2)-subsphere S' of S^{n-1} which separates A and A_{j+1} and such that $S' \cap A = \emptyset, S' \cap A_{j+1} = \emptyset$. Set $A'_i = A_i \cap S'$ $(1 \leq i \leq j)$. Then each A'_i is a spherically convex closed subset of S' and, since $A_{j+1} \cap S' = \emptyset$, each (n-j-1)-subsphere of S' has a nonempty intersection with $A'_1 \cup \cdots \cup A'_j$. Hence by the inductive assumption $A'_1 \cap \cdots \cap A'_j = A \cap S' \neq \emptyset$; contradiction.

THEOREM 2. In \mathbb{R}^n let C_1, \dots, C_{j+1} be j+1 compact convex sets. If $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then either $x \in \bigcup_{i=1}^{j+1} C_i$ or there exists a halfline l emanating from x such that $l \cap C_i \neq \emptyset$, $1 \leq i \leq j+1$.

COROLLARY. In \mathbb{R}^n let C_1, \dots, C_{j+1} be compact convex sets. Then as ufficient condition for $H_j(\bigcup_{i=1}^{j+1} C_i) = \bigcup_{i=1}^{j+1} C_i$ is that the sets do not have a common transversal.

Proof. On S^{n-1} define j + 1 spherically convex closed subsets A_1, \dots, A_{j+1} so that $u \in A_i$ if $u \in S^{n-1}$ and the half line $\{x + \lambda u \mid \lambda \geq 0\}$ meets C_i . Then, as $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ each (n - j - 1)-subsphere of S^{n-1} has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. And so, by Theorem 1, there exists $u \in \bigcap_{i=1}^{j+1} A_i$, i.e., the halfline $\{x + \lambda u \mid \lambda \geq 0\}$ meets each of C_1, \dots, C_{j+1} .

THEOREM 3. In \mathbb{R}^n let C_1, \dots, C_{j+1} be nonempty compact convex sets. Then the number of components of $H_j(\bigcup_{i=1}^{j+1} C_i)$ is at most j+1with equality if and only if C_1, \dots, C_{j+1} are pairwise disjoint.

Proof. By Theorem 2, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i) - \bigcup_{i=1}^{j+1} C_i$, then there exists a halfline $l = \{x + \lambda u \mid \lambda \ge 0\}$ such that l meets each of

$$C_1, \cdots, C_{j+1}$$
 .

Then $x + \alpha_k u \in C_k$ for some $\alpha_k > 0$. We set $\alpha = \min\{\alpha_k \mid 1 \leq k \leq j+1\}$ and want to show that $x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for all λ with $0 \leq \lambda \leq \alpha$. Set $y = x + \lambda u$ and let P be an (n - j)-subspace. As $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ there exists i such that the (n - j)-flat x + P meets C_i at v, say. Set $z = x + \alpha_i u \in C_i$. Then, as y lies between x and z on l, there exists μ , $0 \leq \mu \leq 1$, such that $y = \mu x + (1 - \mu)z$. Then the (n - j)flat y + P through y contains the point $\mu v + (1 - \mu)z$ of C_i . As P

was arbitrary we conclude that $y \in H_i(\bigcup_{i=1}^{j+1}C_i)$ and hence that x + i $\lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for $0 \leq \lambda \leq \alpha$. Hence, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then x is connected, via a line segment in $H_i(\bigcup_{i=1}^{i=1} C_i)$, to at least one of the sets C_i . Hence $H_j(\bigcup_{i=1}^{j+1} C_i)$ has at most j+1 components with equality only if the C_i 's are disjoint. If the sets C_1, \dots, C_{j+1} are pairwise disjoint then in order to show that $H_{i}(\bigcup_{i=1}^{j+1} C_{i})$ has exactly j+1components it is enough to show that for each $k, 1 \leq k \leq j + 1$, there exist disjoint open sets U_k , V_k such that $U_k \cup V_k \supset H_j(\bigcup_{i=1}^{j+1} C_i)$ and $U_k \supset C_k, V_k \supset \{C_1 \cup \cdots \cup C_{k-1} \cup C_{k+1} \cup \cdots \cup C_{j+1}\}$. We suppose, without loss of generality, that k = 1. For $i = 2, \dots, j + 1$ let H_i denote a hyperplane which strictly separates C_1 from C_i , and let H_i^0 be the open halfspace bounded by H_i and containing C_1 . We can assume that the H_i 's are in general position. Set $U_1 = \bigcap_{i=2}^{j+1} H_i^0$, $V_1 = R^n - \overline{U}_1$. Then U_1 and V_1 are disjoint open sets, $C_1 \subset U_1, \bigcup_{i=2}^{j+1} C_i \subset V_1$. It remains to show that $H_j(\bigcup_{i=1}^{j+1} C_i) \subset U_1 \cup V_1$, and it is enough to show that $(\overline{U}_{1} \cap \overline{V}_{1}) \cap H_{i}(\bigcup_{i=1}^{q+1} C_{i}) = \emptyset$. Since the H_{i} 's are in general position, their intersection $\bigcap_{i=2}^{j+1} H_i$ is an (n-j)-dimensional flat L. Let I be the j-dimensional subspace orthogonal to L. If M is any subset of R^n we denote by proj M the set of all points $x \in I$ for which the flat L_x , which is parallel to L and contains x, has a nonempty intersection with M. $\operatorname{proj}_I U_1$ and $\operatorname{proj}_I V_1$ are two open sets in I with common boundary $\operatorname{proj}_I(\bar{U}_1 \cap \bar{V}_1)$. As $\operatorname{proj}_I C_1 \subset \operatorname{proj}_I U_1$, $\operatorname{proj}_I \bigcup_{i=2}^{j+1} C_i \subset \operatorname{proj}_I V_1$ it follows that $(\operatorname{proj}_{I}(\overline{U}_{1} \cap \overline{V}_{1})) \cap (\operatorname{proj}_{I} \bigcup_{i=1}^{j+1} C_{i}) = \emptyset$. Now, if z is an arbitrary point in $\overline{U}_1 \cap \overline{V}_1$ it follows that $L_z \cap (\bigcup_{i=1}^{j+1} C_i) = \emptyset$, and since dim $L_z = n - j$, we find, by the definition of H_j , that z does not belong to $H_i(\bigcup_{i=1}^{j+1} C_i)$. Therefore $(U_1 \cap V_1) \cap H_i(\bigcup_{i=1}^{j+1} C_i) = \emptyset$.

The proof of Theorem 3 also shows that any component Remarks. of $H_i(\bigcup_{i=1}^{j+1} C_i)$ has the property that any two points of it can be joined by a broken line in it, consisting of at most 3 segments. Hence it is natural to ask: When are these components convex? (supposing now that the C_i 's are disjoint). In [1] W. A. Beyer has shown an example of three (nondisjoint) polytopes C_i in R^3 such that $H_2(C_1 \cup C_2 \cup C_3)$ is not a polyhedron. We don't know whether a similar construction would be possible with disjoint polytopes. Let us mention here a few more technical terms. If M is any subset of \mathbb{R}^n , we denote by aff M the affine hull of M and by conv M the convex hull of M. relint M means the interior of M with respect to the natural topology in aff M. By the dimension $\dim M$ of M we understand the algebraic dimension of the flat aff M. A polytope is the convex hull of some finite set. If $P \subset E^n$ is a convex set we denote by ext P the set of extreme points of P and by exp P the set of its exposed points. For an exact definition of these terms the reader may compare, for example, the introductory chapters of [4].

THEOREM 4. (i) In \mathbb{R}^n let C_1, C_2 be compact convex sets. Then $H_1(C_1 \cup C_2)$ is the union of at most two convex components which are polytopes whenever C_1 and C_2 are polytopes.

(ii) There exist in \mathbb{R}^3 three disjoint polytopes such that one of the components of the second visual hull of their union is not convex.

LEMMA 1. Let C_1, C_2 be n-dimensional polytopes in \mathbb{R}^n . If $a \notin H_1(C_1 \cup C_2)$ there exists a hyperplane H such that

- (1) $a \in H, H$ separates a from C_1
- (2) $H \cap C_i = \emptyset$ or H supports C_i (i = 1, 2)
- $(3) \quad \text{aff} (H \cap (C_1 \cup C_2)) = H.$

Proof of Lemma 1. The case n = 1 is trivial, and we assume $n \ge 2$. If there exists a hyperplane P through a which does not meet $C_1 \cup C_2$ and does not separate C_1 and C_2 then conv $(C_1 \cup C_2)$ is an n-dimensional polytope not containing a, and the lemma follows from standard results on polytopes. Hence it can be supposed that there is a hyperplane H for which (1) and also (2'): H separates C_1 and C_2 holds. We choose H in the set \mathfrak{F} of hyperplanes for which (1) and (2') holds. We assume that $h = \dim \operatorname{aff} T$ is maximal, where $T = H \cap (C_1 \cup C_2)$. Obviously $h \ge 0$. If h < n - 1, let $F \subset H$ be an (n-2)-dimensional hyperplane in H containing T, and denote by π : $\mathbb{R}^n \to E$ the projection along F onto a 2-dimensional flat E orthogonal to F. It is easy to see that there is a line L in E such that: (α) : the singleton $\pi(T)$ is contained in L. $(\beta): \pi(a) \notin L$, L separates $\pi(a)$ from the polygon $\pi(C_1)(\gamma): L$ separates $\pi(C_1)$ and $\pi(C_2)$.

$$(\delta) \operatorname{aff} \left(L \cap \left(\pi(C_1) \cup \pi(C_2) \right) = L$$
 .

(Notice that the conditions $(\alpha) - (\gamma)$ are fulfilled by $\pi(H)$). The hyperplane $\pi^{-1}(L)$ of E^n intersects $C_1 \cup C_2$ in a set S with dimaff S = h + 1. Since $S \in \mathfrak{H}$ this contradicts the maximality of h. Hence the lemma is established.

Proof of Theorem 4. (i) We first prove the result when C_1, C_2 are n-dimensional polytopes. If $C_1 \cap C_2 \neq \emptyset$ then

$$H_{\scriptscriptstyle 1}(C_{\scriptscriptstyle 1}\cup C_{\scriptscriptstyle 2})=\operatorname{conv}\left(C_{\scriptscriptstyle 1}\cup C_{\scriptscriptstyle 2}
ight)$$
 ,

which is a polytope. We suppose therefore that $C_1 \cap C_2 = \emptyset$. Let $\{H_i\}_{i=1}^m$ be the finite set of those hyperplanes which do not contain an interior of C_j (j = 1, 2) and for which dim $(H_i \cap (C_1 \cup C_2)) = n - 1$. By C_j^* we denote the (finite) intersection of those closed half spaces which contain C_j and whose bounding hyperplane is amongst $\{H_i\}_{i=1}^m$, j = 1, 2. Then C_j^* is polyhedral and, since C_1, C_2 are compact, C_j^* is a polytope,

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j = 1, 2. We show that $H_1(C_1 \cup C_2) = C_1^* \cup C_2^*$. Suppose that $x^* \notin C_1^* \cup C_2^*$. Then there exist closed halfspaces H_1^*, H_2^* with bounding hyperplanes H_1, H_2 amongst $\{H_i\}_{i=1}^m$ such that $x^* \notin H_1^* \supset C_1, x^* \notin H_2^* \supset C_2$. If

$$x^* \in H_1(C_1 \cup C_2), H_1 \quad \text{and} \quad H_2$$

must separate C_1 and C_2 . Consider H_1 and the two disjoint compact sets $H_1 \cap C_1, H_1 \cap C_2$ in H_1 . There exists an n-2 dimensional flat Lin H_1 which strictly separates $H_1 \cap C_1$ and $H_1 \cap C_2$. By slightly rotating H_1 about L in the appropriate direction we obtain a hyperplane H'_1 which strictly separates C_1 and C_2 as well as x^* and C_1 . Similarly we can obtain a hyperplane H'_2 which strictly separates C_1 and C_2 , and x^* and C_2 . We may suppose that H'_1, H'_2 are not parallel and so $H'_1 \cap H'_2$ is an n-2 flat. Suppose, without loss of generality, that $H'_1 = \{x \mid \langle x, \xi \rangle = \alpha > 0\}, H_2 = \{x \mid \langle x, \eta \rangle = \beta > 0\}$. Then

$$egin{aligned} &C_{\scriptscriptstyle 1} \subset \{x \, | \, ig\langle x, \, \xi ig
angle > lpha \} \cap \{x \, | \, ig\langle x, \, \eta ig
angle > eta \} \ &C_{\scriptscriptstyle 2} \subset \{x \, | \, ig\langle x, \, \eta ig
angle < lpha \} \cap \{x \, | \, ig\langle x, \, \eta ig
angle < eta \} \;. \end{aligned}$$

Consider the hyperplane $H: \{x \mid \langle x, \lambda \xi + (1 - \lambda)\eta \rangle = 0\}$, where $\lambda \alpha + (1 - \lambda)\beta = 0$ and $0 < \lambda < 1$. Then $x^* \in H$ and, using the above inequalities, $C_i \cap H = \emptyset$, i = 1, 2. Hence x^* is not in $H_1(C_1 \cup C_2)$, and we have $H_1(C_1 \cup C_2) \subset C_1^* \cup C_2^*$. Conversely, if $x^* \in C_1^* \cup C_2^* - H_1(C_1 \cup C_2)$, suppose without loss of generality that $x^* \in C_1^*$. Then, by Lemma 1, there exists a hyperplane H amongst $\{H_i\}_{i=1}^m$ which does not contain x^* and which separates x^* from C_1 . Then, if H^* donotes the closed halfspace containing C_1 whose bounding hyperplane is $H, x^* \notin H^*$ and so $x^* \in C_1^*$; contradiction. And so $H_1(C_1 \cup C_2) = C_1^* \cup C_2^*$, which is the union of two polytopes. If C_1, C_2 are compact convex sets we choose decreasing sequences $\{P_1^n\}_{n=1}^\infty, \{P_2^n\}_{n=1}^\infty$ of polytopes such that $C_i = \bigcap_{n=1}^\infty P_i^n$, i = 1, 2. Then, using the above notation,

$$H_1(C_1 \cup C_2) = \bigcap_{n=1}^{\infty} P_1^{n*} \cap \bigcap_{n=1}^{\infty} P_2^{n*}$$
 .

(ii) Let W be the cube $\{x = (x_1, x_2, x_3) \mid -1 \leq x_i \leq 1, i = 1, 2, 3\}$ in R^3 , and denote by W_i the facet of W defined by $x_i = 1$. Set $C_1 = W_1$, $C_2 = 2W_2$, $C_3 = 3W_3$. Let $B_i(1 \leq i \leq 3)$ be the components of $H_2(\bigcup_{i=1}^3 C_i)$, where the indices are chosen such that, for all i, $C_i \subset B_i$. Clearly $(0, 0, 0) \in B_1$ as does, of course, the point $(1, -1, -1) \in B_1 \cap C_1$. However we show that the line segment $m: \{x = \lambda(1, -1, -1) \mid 0 < \lambda < 1\}$ is not in B_1 . Now $C_1 \cup C_2$ is contained in the halfspace $\{x \mid \langle x, (0, 1, 1) \rangle \geq 0\}$ whose bounding hyperplane P passes through the points (0, 0, 0), (1, -1, 1) and $(-1, -1, 1); P \cap \text{aff } W_1$ is a line in direction (0, -1, 1). If $y \in m$, then $y = \mu(1, -1, -1)$ for some $\mu, 0 < \mu < 1$. Consider the line $l = y + \{\lambda(0, -1, 1) \mid \lambda \text{ real}\}$. If $z = (z_1, z_2, z_3) \in l$ then $z_1 = \mu < 1$, i.e., $z \notin C_1$. Also $\langle z, (0, 1, 1) \rangle = -2\mu < 0$ which means that $z \notin C_1 \cup C_2$. Therefore *l* does not meet $C_1 \cup C_2 \cup C_3$, *m* does not belong to B_1 , and B_1 is not convex.

In [6] V. L. Klee proved that if all j^{th} projections of a compact convex body C in \mathbb{R}^n (j fixed ≥ 2) are polytopes, then C is a polytope. As a partial analogue to this for unions of two convex bodies we prove

THEOREM 5. Let C_1 , C_2 be two disjoint compact convex bodies in \mathbb{R}^n such that each j^{th} projection of $C_1 \cup C_2$ (j fixed ≥ 2) is the union of two polytopes. Then (i) ext (C_i) = exp (C_i) and ext (C_i) is countable (i = 1, 2) but (ii) ext (C_i) is not necessarily finite.

Proof. Let a be an extreme point of C_1 and we suppose, without loss of generality, that a = 0, the origin of R^n . Then, to prove (i) it is enough to prove that the convex cone K of outward normals to C_1 at 0 is *n*-dimensional. We assume that dim $K \leq n - 1$ so that K is contained in an (n-1)-subspace P_1 , and seek a contradiction. Let P_2 be an (n-1)-subspace which supports C_1 at 0. Of course $P_1 \neq P_2$. We can choose an (n-1)-subspace P_3 so that there exists a translate of P_3 which strictly separates C_1 and C_2 and such that the normal to P_3 at 0 intersects P_1 only at 0. Then $P_2 \cap P_3$ is a subspace of dimension at least n-2 and we choose an n-j subspace Q in $P_2 \cap P_3$. The orthogonal complement S of Q in R^n is a j-dimensional subspace which meets P_1 in a (j-1)-subspace. The projection of $C_1 \cup C_2$ onto S is the union of two polytopes. Further, as $P_3 \cap C_2 = \emptyset$, 0 is at positive distance from proj C_2 . As 0 is an extreme point of proj C_1 , it follows that 0 is a locally polyhedral extreme point for $\operatorname{proj} C_1$. Hence, in S, the cone of outward normals to $\operatorname{proj} C_1$ at 0 is *j*-dimensional. Further, any (j-1)-plane H of support in S to proj C_1 at 0 can be extended to an (n-1)-plane of support H+Q in R^n to C_1 at 0. Also, the outward normals to these planes form a j-dimensional convex cone lying in S. Hence $j = \dim (K \cap S) = \dim (P_1 \cap S) = j - 1$; contradiction. And so (i) is proved.

To prove (ii) we construct an example in R^3 of two convex bodies C_1, C_2 , both of which have a countable infinity of extreme points but, nevertheless, each 2-projection of $C_1 \cup C_2$ is the union of two convex polygons. Let $l = \{x \mid x_1 = x_2 = 0, -1 \leq x_2 \leq 1\}$ be a line segment and $S = \{x \mid (x_1 - 1)^2 + x_2^2 = 1, x_3 = 0\}$ a plane circle. By T we denote the set of those points on S with x_2 -coordinate $\pm(1/n)$ for $n = 1, 2, \cdots$. We take $C_1 = \operatorname{conv} \{l \cup T\}$, which is a compact convex body in R^3 with extreme points $T \cup \{(0, 0, -1), (0, 0, 1)\}$. It is easily seen that there is precisely one 2-projection of C_1 which is not a convex polygon, and that is in the direction (0, 0, 1). Further the only limit point of extreme points of this projection is (0, 0, 0). Define C_2 as a disjoint copy of

 C_1 formed by placing C_2 above C_1 in such a way that their respective major lines pierce the centres of their respective circles. From above, every 2-projection of $C_1 \cup C_2$ is the union of two convex polygons and and both C_1 and C_2 are compact bodies with a countable infinity of extreme points.

3. Visual hulls of more general sets. The following problem can be formulated.

Is the visual (virtual) (minimal) hull of a borel (analytic) set in \mathbb{R}^n necessarily borel (analytic)?

The answer is affirmative (Theorem 6) for virtual hulls and negative (Theorem 7) for minimal hulls. Whilst it is not true (Theorem 8) that the j^{th} visual hull of a borel set is necessarily borel, we have been unable to decide whether or not the j^{th} visual hull of a borel or of an analytic set is always analytic, except in the cases covered by Theorem 9. It is possible also that the j^{th} visual hull of a convex borel (analytic) set is a borel (analytic) set, and we include some partial results (Theorem 9) in this direction. As before we denote by G_j^n the Grassmannian of j-subspaces of \mathbb{R}^n and by μ_j the invariant (with respect to O_n acting in the usual way on G_j^n) measure normalised so that $\mu_j(G_j^n) = 1$.

LEMMA 2. Let A be an analytic set in \mathbb{R}^n and denote by A^* the set of those j-subspaces in \mathbb{G}_j^n which meet A. Then

(i) A^* is an analytic set in G_j^n and hence A^* is μ_j measurable. (ii) If $\mu_j(A^*) > a$ then there exists a compact subset A' of A such that $\mu_j(A'^*) > a$.

(iii) If $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of analytic sets in \mathbb{R}^n then $\mu_j(\bigcup_{i=1}^{\infty} A_i)^* = \lim_{i \to \infty} \mu_j(A_i^*)$.

(iv) If $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of analytic sets in \mathbb{R}^n then $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* = \lim_{i \to \infty} \mu_j(A_i^*)$.

Proof. (i) Let *I* be the set of irrational numbers in [0, 1] and, if $i = (i_1, \dots, i_n, \dots)$ is a typical member of *I* expressed as a continued fraction, set $i \mid n = (i_1, \dots, i_n)$. Then, as *A* is analytic, it can be represented as $A = \sum_{i \in I} \bigcap_{n=1}^{\infty} A(i \mid n)$ where the sets $A(i \mid n)$ form, for each fixed *i*, a decreasing sequence of compact subsets of \mathbb{R}^n . Then $A^* = \sum_{i \in I} \bigcap_{n=1}^{\infty} A^*(i \mid n)$. As each $A^*(i \mid n)$ is a compact subset of G_j^n , we conclude that A^* is an analytic set.

(ii) If $\mu_j(A^*) > a + \delta$ with $\delta > 0$, then we can choose $m_1, 1 \leq m_1 < \infty$, such that if I_1 denotes the set of irrational numbers

$$i = (i_1 \cdots i_n \cdots)$$

with $1 \leq i_1 \leq m_1$ and $A_1^* = \sum_{i \in I_1} \bigcap_{n=1}^{\infty} A^*(i \mid n)$ then $\mu_j(A_1^*) > a + \delta$.

Proceeding by induction we may define natural numbers m_p , $1 \leq p < \infty$, such that if I_q denotes the subset of those irrationals i with $1 \leq i_p \leq m_p$ for $p = 1, \dots, q$, and $A_q^* = \sum_{i \in I_q} \bigcap_{n=1}^{\infty} A^*(i \mid n)$ then $\mu_j(A_q^*) > a + \delta$. Let I' be the compact subset of [0, 1] defined as the set of those irrational numbers i for which $1 \leq i_p \leq m_p$ for $p = 1, 2, \dots$, and

$$A^{\prime *} = \sum_{i \in I^{\prime}} \bigcap_{n=1}^{\infty} A^{*}(i \mid n)$$
 .

Then $\bigcap_{q=1}^{\infty} A_q^* = A'^*$ and so $\mu_j(A'^*) \ge a + \delta > a$. Also

$$A' = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A(i \mid n)$$

is a compact subset of A, as I' is a compact subset of I.

(iii) $\mu_j(\bigcup_{i=1}^{\infty} A_i)^* = \mu_j(\bigcup_{i=1}^{\infty} A_i^*) = \lim_{i \to \infty} \mu_j(A_i^*).$

(iv) Clearly $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* \leq \lim_{i \to \infty} \mu_j(A_i^*)$. Now set $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* = a_i$ and suppose $\lim_{i \to \infty} \mu_j(A_i^*) > a + \varepsilon$, for some positive number ε . By (ii) we find a compact set $B_1 \subset A_1$ such that $\mu_j(B_1^*) \geq \mu_j(A_1^*) - \varepsilon/2$. Now we have $A_2^* = (B_1 \cap A_2)^* \cup (A_2^* - B_1^*)$, where

$$A_2^*-B_1^*=\{F\in G_j^n\,|\,F\cap A_2
eq arnothing$$
 , but $F\cap B_1=arnothing\}$.

Since $A_2^* \subset A_1^*$ we derive further $A_2^* \subset (B_1 \cap A_2)^* \cup (A_1^* - B_1^*)$, or $\mu_j(A_2^*) \leq \mu_j(B_1 \cap A_2)^* + \varepsilon/2$. Since $B_1 \cap A_2$ is analytic there exists, again by (ii), a compact set $B_2 \subset (B_1 \cap A_2)$ such that

$$\mu_j(B_2)^* \geq \mu_j(B_1 \cap A_2)^* - arepsilon/4$$

and consequently $\mu_j(B_2)^* \ge \mu_j(A_2)^* - (\varepsilon/2 + \varepsilon/4)$. Continuing this process we obtain a decreasing sequence $\{B_i\}_{i=1}^{\infty}$ of compact subsets of R^n such that $B_i \subset A_i$, $i = 1, 2, \cdots$, and $\mu_j(B_i^*) \ge \mu_j(A_i^*) - \sum_{p=1}^i \varepsilon/(2^p)$. Then $\bigcap_{i=1}^{\infty} B_i^* = (\bigcap_{i=1}^{\infty} B_i)^* \subset (\bigcap_{i=1}^{\infty} A_i)^*$, and $\mu_j(\bigcap_{i=1}^{\infty} B_i^*) = \lim_{i\to\infty} \mu_j(B_i^*) \le a$; but also $\lim_{i\to\infty} \mu_j(B_i^*) \ge \lim_{i\to\infty} \mu_j(A_i^*) - \varepsilon$. Combining the last two inequalities we find $\lim_{i\to\infty} \mu_j(A_i) \le a + \varepsilon$, a contradiction.

THEOREM 6. Let C be a borel (analytic) set in \mathbb{R}^n . Then the j^{th} virtual hull $V_j(C)$ is a borel (analytic) set.

Proof. Suppose first that C is a borel set in \mathbb{R}^n , and we need to show that $V_j(C)$ is a borel set. If D is a subset of \mathbb{R}^n and $x \in \mathbb{R}^n$, let D[x, n-j] denote the set of those n-j subspaces F in \mathbb{G}_{n-j}^n such that $(x + F) \cap D \neq \emptyset$. If $0 < \lambda < 1$ let $D(n - j, \lambda)$ be the set of all x in \mathbb{R}^n such that $\mu_{n-j}(D[x, n-j]) > \lambda$. Let B denote the largest family of subsets of \mathbb{R}^n such that $D \in B$ if (i) D is a borel set in \mathbb{R}^n . (ii) $D(n - j, \lambda)$ is a borel set for all $\lambda, 0 < \lambda < 1$. We shall prove that B coincides with the family of borel subsets of \mathbb{R}^n , and it is enough.

to show that B contains the open sets and is closed under the operations of increasing union and decreasing intersection. If D is an open subset of \mathbb{R}^n , then it is easy to see that $D(n-j,\lambda)$ is open for all $\lambda, 0 < \lambda < 1$, and so B contains all the open sets. Now suppose that $\{E_i\}_{i=1}^{\infty}$ is an increasing sequence of sets in B and set $E = \bigcup_{i=1}^{\infty} E_i$. We want to show that for each λ , $0 < \lambda < 1$, the equality $E(n-j,\lambda) = \bigcup_{i=1}^{\infty} E_i(n-j,\lambda)$ In order to do this we observe the following equivalences: holds. $x \in E(n-j,\lambda) \leftrightarrow \mu_{n-j}(E[x,n-j]) > \lambda \leftrightarrow \lim_{i \to \infty} \lambda_{n-j}(E_i[x,n-j]) > \lambda \leftrightarrow$ $x \in \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$. Here the first equivalence holds by definition, the second one follows directly from Lemma 2, (iii), if we observe that this lemma remains true if M^* denotes, for each $M \subset \mathbb{R}^n$, the set M[x, n-j] ($x \in \mathbb{R}^n$ fixed). (The lemma itself is stated for the special case where x is the origin of R^n .) The last equivalence again follows immediately from the definitions, we only have to observe that the sequence $\{E_i\}_{i=1}^{\infty}$ is increasing. Now suppose that $\{H_i\}_{i=1}^{\infty}$ is a decreasing sequence of subsets of B and set $H = \bigcap_{i=1}^{\infty} H_i$. Suppose λ fixed, $0 < \lambda < 1$, and let *m* be a natural number such that $\lambda + 1/m < 1$. Then, using (iv) of Lemma 2, we find by an argument analogous to the one above, $H(n-j,\lambda) = \bigcup_{p=m}^{\infty} \bigcap_{i=1}^{\infty} H_i(n-j,\lambda+1/p)$. Hence $H(n-j,\lambda)$ is a borel set, and $H \in B$. Therefore, B is the family of borel subsets of \mathbb{R}^n and so, in particular, $C \in B$. Further $V_i(C) =$ $\bigcap_{p=2}^{\infty} C(n-j, 1-(1/p))$ and so $V_j(C)$ is a borel set.

To show that $V_j(A)$ is analytic whenever A is analytic, we use the well known result that there exists an $F_{\sigma\delta}$ set K in \mathbb{R}^{n+1} such that A is the orthogonal projection proj K of K into \mathbb{R}^n (see, for example, [8]). Call an (n - j + 1)-subspace H of \mathbb{R}^{n+1} upright if H has the form $\{\hat{H} + \lambda(0, \dots, 0, 1) \mid -\infty < \lambda < \infty\}$ where $\hat{H} \in \mathbb{G}_{n-j}^n$. Let U_{j+1} be the set of upright (n - j + 1)-subspaces in \mathbb{R}^{n+1} with the measure μ' induced by μ_{n-j} in the obvious manner. We can define $U_{j+1}(C)$ of a set C in \mathbb{R}^{n+1} as the set of all those points x in \mathbb{R}^{n+1} such that almost all (with respect to μ') upright (n - j + 1)-flats through x meet C. As above, it can been shown that $U_{j+1}(C)$ is a borel set whenever Cis a borel set. Clearly proj $U_{j+1}(K) = V_j(A)$ and, since the projection of a borel set is analytic, we conclude that $V_j(A)$ is an analytic subset of \mathbb{R}^n .

THEOREM 7. Let C be an open convex subset of \mathbb{R}^n . Then assuming the continuum hypothesis, C contains a minimal j^{ih} hull D such that every analytic subset of D is countable.¹

Proof. We assume the continuum hypothesis and let Ω be the

¹ As the referee pointed out, Theorem 7 may be a special case of a much more general theorem on effective constructions.

first uncountable ordinal. Let $\{A_{\xi}\}_{\xi<\varrho}$ be an enumeration of the analytic subsets of \mathbb{R}^n of (n-j)-dimensional measure zero; let $\{H_{\xi}\}_{\xi<\varrho}$ be an enumeration of the (n-j)-flats which meet C. Let F be a fixed (n-j)-subspace of \mathbb{R}^n and denote by α a fixed set, which is not a point of \mathbb{R}^n . We now choose a set $E = \{M_{\xi}\}_{\xi<\varrho}$ and a collection of translates $\{F_{\xi}\}_{\xi<\varrho}$ of F inductively as follows. Take $M_1 \in (H_1 - A_1) \cap C$ and let F_1 be a translate of F through M_1 . Suppose now that $M_{\xi'}, F_{\xi'}$ have been defined for all $\xi' < \xi$, where ξ is some ordinal proceeding Ω . If H_{ξ} is a translate of F we take $F_{\xi} = H_{\xi}$ and consider two possibilities:

(a) If $\exists \xi' < \xi$ such that $M_{\xi'} \in H_{\xi}$ then we take $M_{\xi} = \alpha$.

(b) If $\exists \xi' < \xi$ such that $M_{\xi'} \in H_{\xi}$ we choose M_{ξ} in the set $(H_{\xi} - (\bigcup_{\xi' < \xi} H_{\xi'} \cup \bigcup_{\xi' < \xi} A_{\xi'})) \cap C$. Such a choice is possible as $H_{\xi} \cap C$ has positive (n-j)-dimensional measure whereas $H_{\xi} \cap (\bigcup_{\xi' < \xi} H_{\xi'} \cup \bigcup_{\xi' < \xi} A_{\xi'})$ has zero (n-j)-dimensional measure, being a countable union of sets of measure zero. If H_{ξ} is not a translate of F we find, by similar arguments, that the set $(H_{\xi} - (\bigcup_{\xi' < \xi} H_{\xi'} \cup \bigcup_{\xi' < \xi} A_{\xi'} \cup \bigcup_{\xi' < \xi} F_{\xi'})) \cap C$ is not empty. We choose M_{ξ} in this set and let F_{ξ} be the translate of F through M_{ξ} . We claim that the set $D = E - \alpha$ is a j^{th} minimal hull for C which meets each analytic subset in at most a countable number of points. To show that all j^{th} projections of D coincide with those of C, it is enough to show that the j^{th} visual hull of D contains C. Let x be a point of C and let P be an (n-j)-flat through x. Then P is amongst $\{H_{\xi}\}_{\xi<0}$, say $P = H_{\xi'}$. If $M_{\xi'} \neq \alpha$ then $M_{\xi'} \in D \cap H_{\xi'}$. If $M_{\xi'} = \alpha$ then $\exists M_{\xi''}, \xi'' < \xi'$, such that $M_{\xi''} \in D \cap H_{\xi'}$. In either case P meets D and so $x \in H_j(D)$.

If D is not minimal then there exists $M_{\xi}, \xi < \Omega$, such that

$$H_j(D-M_{\xi})=C$$
.

But, projecting C and $D - M_{\varepsilon}$ onto the orthogonal complement of Fwe see that by construction proj $C \cap \operatorname{proj} F_{\varepsilon} \neq \emptyset$, but proj $(D - M_{\varepsilon}) \cap$ proj $F_{\varepsilon} = \emptyset$. Hence D is a j^{in} minimal hull for C. Finally, suppose that B is an uncountable analytic subset of D. If B has positive jdimensional measure then it is possible to find an uncountable analytic subset of B of zero j-dimensional measure. Hence it can be supposed that B has zero j-dimensional measure and so $B = A_{\varepsilon}$ for some $\varepsilon < \Omega$. But $A_{\varepsilon} = A_{\varepsilon} \cap D \subset \bigcup_{\varepsilon' < \varepsilon} M_{\varepsilon'}$, which is countable; contradiction.

Of course, if G is an open or compact set in \mathbb{R}^n then $H_j(G)$ will accordingly be an open or compact set. Apart from these cases it does not seem entirely trivial to determine the nature of $H_j(G)$ for a given subset G of \mathbb{R}^n . Here we prove the following

THEOREM 8. (i) There exists, in the plane R^2 , a borel set C such that $H_1(C)$ is analytic but not borel.

(ii) If D is an F_{σ} -subset of \mathbb{R}^n then $H_j(D)$ is the complement of an analytic set.

REMARKS. We note that by (i) if C is analytic then $H_i(C)$ is not necessarily the complement of an analytic set. To disprove the statement that whenever A is analytic then $H_j(A)$ is analytic, it would be enough, using (ii), to find an F_{σ} -subset D of R^n such that $H_j(D)$ is not borel. (Notice that, a subset, M of R^n is borel if and only if M and $R^n - M$ are both analytic. Compare, for example, [5]).

Proof. (i) As already observed, every analytic set in R^1 can be represented as the projection into R^1 of some $F_{\sigma\delta}$ set in R^2 . Let Abe an analytic subset of R^1 such that A is not a borel set and let Bbe an $F_{\sigma\delta}$ set in R^2 such that proj B = A. Take C to be the union of B and the "y-axis" $(R^1)^{\perp}$. Then it is easily seen that $H_1(C)$ is the union of all lines which are parallel to $(R^1)^{\perp}$ and contain a point of C. However this is not a borel set as $H_1(C) \cap R^1 = A \cup \{(0, 0)\}$ is not a borel set.

(ii) We define a complete separable metric space Ω , whose points are the (n-j)-flats of \mathbb{R}^n , as follows. For each (n-j)-flat F in \mathbb{R}^n let y be the nearest point of F to 0 and set $F \cap (S^{n-1} + y) = \hat{F}$. Then the distance $\rho(F, F')$ of two (n-j)-flats in Ω is defined as the Hausdorff distance of \hat{F}, \hat{F}' in \mathbb{R}^n . Let $D \subset \mathbb{R}^n$ be an F_σ set, say $D = \bigcup_{i=1}^{\infty} D_i$ with $D_i \subset D_{i+1}$, each D_i compact, $i = 1, 2, \cdots$. Let $D_i^*, i =$ $1, 2 \cdots$ denote the closed subsets of Ω such that $F \in D_i^*$ if F meets D_i in \mathbb{R}^n . Similarly defined, relative to D, is D^* . Then $D^* = \bigcup_{i=1}^{\infty} D_i^*$ and so D^* is an F_σ subset of Ω . Hence $\Omega - D^*$ is a G_i set and so, in particular, $\Omega - D^*$ is an analytic subset of Ω . Set

$$arOmega - D^* = \sum\limits_{i \, \epsilon \, I} \, igcap_{p=1}^{igca} A(i \,|\, p)$$
 ,

where the A(i | p), $p = 1, 2, \dots$, form a decreasing sequence of compact subsets of Ω , for each $i \in I$. Set

$$B_m = \{x \mid x \in \mathbb{R}^n, -m \leq x_i \leq m, i = 1, \dots, n\}$$
.

Let $K_m(i \mid p)$ be the closed subset of B_m such that $x \in K_m(i \mid p)$ if x is contained in an (n - j)-flat F with $F \in A(i \mid p)$. Similarly, we define $K_m \subset B_m$ relative to $\mathcal{Q} - D^*$. Then $K_m = \sum_{i \in I} \bigcap_{p=1}^{\infty} K_m(i \mid p)$ is an analytic subset of R^n and so, therefore, is $K = \bigcup_{m=1}^{\infty} K_m$. We claim that $H_j(D) = R^n - K$. If $x \in K$ then $x \in K_m$ for some m and so x is contained in some (n - j)-flat F which is contained (in Ω) in some set $\bigcap_{p=1}^{\infty} A(i \mid p)$. Hence $F \in \mathcal{Q} - D^*$ which means that F does not meet D; i.e., $x \notin H_j(D)$. Therefore $R^n - K \supset H_j(D)$. Conversely if $x \notin H_j(D)$ then there exists an (n - j)-flat F through x such that F does not meet D. Hence $F \in \mathcal{Q} - D$ D^* and so $F \in \bigcap_{p=1}^{\infty} A(i | p)$ for some $i \in I$. Hence $x \in \bigcap_{p=1}^{\infty} K_m(i | p)$ for some positive integer m, i.e., $x \in K$. Therefore $R^n - K \subset H_j(D)$ and so $H_j(D) = R^n - K$ is the complement of the analytic set K.

DEFINITION. An irregular point x of some closed convex set C in \mathbb{R}^3 is an extreme point x of C such that x lies in two distinct 1-faces l_1, l_2 of C, with neither of l_1, l_2 being contained in a 2-face of C. Let C be a closed subset of a simple closed curve in the plane OXY. We say that a set $B \subset C \times (-\infty, \infty)$ is vertically convex if every line which is perpendicular to OXY meets B in a (possibly empty) line segment. We shall make use of the following immediate corollary to a theorem of K. Kunugui [7].

LEMMA 3. (Kunugui) Let B be a vertically convex borel set in $C \times (-\infty, \infty)$. Then the projection of B into C is a borel set.

As an immediate consequence of Lemma 3, we have

LEMMA 4. Let B be a vertically convex borel subset of some vertically convex closed subset D in $C \times (-\infty, \infty)$. Then the set $D \cap \{(\text{proj. } B) \times (-\infty, \infty)\}$ is a vertically convex borel set.

In [9] the authors have derived properties of visual hulls for the class of convex sets. Our contribution in this direction is

THEOREM 9. (i) If C is a convex borel (analytic) set in \mathbb{R}^3 then $H_2(C)$ is a borel (analytic) set.

(ii) If C is a convex borel (analytic) set in \mathbb{R}^3 and $\overline{\mathbb{C}}$ does not have irregular points then $H_1(\mathbb{C})$ is a borel (analytic) set.

Proof. (i) We first show that if C is a convex borel (analytic) set in \mathbb{R}^2 then $H_1(C)$ is a borel (analytic) set. If dim C = 1 then the result is trivial and so it can be supposed that dim C = 2. Note that $C^0 \subset H_1(C) \subset \overline{C}$. Let the 1-faces of \overline{C} be $\{F_i\}_{i=1}^{\infty}$. Then

$$H_{\scriptscriptstyle 1}(C)\cap (ar{C}-igcup_{i=1}^\infty F_i)=C-igcup_{i=1}^\infty F_i$$
 ,

which is a borel set. Let $\{F_{i_{\nu}}\}_{\nu=1}^{\infty}$ be the 1-faces of \overline{C} which meet C. Then relint $F_{i_{\nu}} \subset H_1(C) \cap F_{i_{\nu}}, \nu = 1, 2, \cdots$. The two endpoints of $F_{i_{\nu}}$ may, or may not, be in $H_1(C)$. Nevertheless, $H_1(C)$ differs from the borel set $(C - \bigcup_{i=1}^{\infty} F_i) \cup \bigcup_{\nu=1}^{\infty}$ relint $F_{i_{\nu}}$ by at most a countable number of points. And so $H_1(C)$ is a borel set. Similarly, if C is a convex analytic set in R^2 , then $H_1(C)$ is an analytic set. Suppose now that C is a convex borel set in R^3 . If dim $C \leq 2$ then $H_2(C) = C$, and so it can be supposed that dim C = 3. Let $\{F_i\}_{i=1}^{\infty}$ be an enumeration of the 2-faces of \overline{C} . Then each F_i is closed and $H_2(C) \cap (\overline{C} - \bigcup_{i=1}^{\infty} F_i) = C \cap (\overline{C} - \bigcup_{i=1}^{\infty} F_i)$, which is a borel set. As $H_2(C) \subset \overline{C}$, it is now enough to show that $H_2(C) \cap F_i$ is a borel set for $i = 1, 2, \cdots$. Let $H'_1(C \cap F_i)$ denote the first visual hull of $C \cap F_i$ relative to aff F_i . Then, from above, $H'_1(C \cap F_i)$ is a borel set. Let $\{F_{i_j}\}_{j=1}^{\infty}$ be an enumeration of the 1-faces of F_i . Then $H_2(C) \cap (F_i - \bigcup_{j=1}^{\infty} F_{i_j}) = H'_1(C \cap F_i) - \bigcup_{j=1}^{\infty} F_{i_j}$ which is a borel set K_i , say. Let $\{F_{i_j\nu}\}_{\nu=1}^{\infty}$ be the 1-faces of F_i which meet C and have the property that the only plane of support to \overline{C} which contains $F_{i_{j\nu}}$ is aff F_i . Then relint $F_{i_{j\nu}} \subset H_2(C)$ and the end points of $F_{i_{j\nu}}$ may or may not be in $H_2(C)$. Hence $H_2(C) \cap F_i$ differs from the borel set $K_i \cup (\bigcup_{\nu=1}^{\infty} \operatorname{relint} F_{i_{j\nu}}) \cup (\bigcup_{j=1}^{\infty} (F_{i_j} \cap C))$ by at most a countable number of points. Therefore $H_2(C) \cap F_i$ is a borel set, and so, therefore, is $H_2(C)$. Similarly, it can be shown that if C is a convex analytic set in R^3 then $H_2(C)$ is an analytic set.

(ii) Again we shall prove the result for convex borel sets, and indicate at the end the modifications required for convex analytic sets. Let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers and let P_{ik} denote the 2-flat $\{x | x_k = r_i\} k = 1, 2, 3; i = 1, 2, \cdots$ For each i, j, k, let B(i, j, k)denote the closed set formed by the point set union of all maximal line segments in $\overline{C} - C^{\circ}$ which meet both both P_{ik} and P_{jk} . Let $\{G_m\}_{m=1}^{\infty}$ be the 2-faces of \overline{C} . If a 2-face G_m of \overline{C} meets B(i, j, k) then G_m meets $C_i(C_i=(ar{C}-C^{\scriptscriptstyle 0})\cap P_{ik})$ and $C_j(C_j=(ar{C}-C^{\scriptscriptstyle 0})\cap P_{jk})$ in line segments 1_{im} and 1_{jm} respectively. Let $1_m^1, 1_m^2$ denote the (at most) two maximal line segments in G_m such that each segment contains an endpoint of $\mathbf{1}_{im}$ and $\mathbf{1}_{jm}$ but $\mathbf{1}_m^{\scriptscriptstyle 1}$ and $\mathbf{1}_m^{\scriptscriptstyle 2}$ do not intersect except possibly at end points. Set $C^* = (\bar{C} - C^{\scriptscriptstyle 0}) \cap P$, where P is a plane parallel to P_{ik} and lying strictly between P_{ik} and P_{jk} . Then G_m cuts C^* in an interval I_m . Let 1_m denote the subinterval of I_m with endpoints $1_m^1 \cap C^*$, $1_m^2 \cap C^*$, and let 1_m° be the relative interior of 1_m . Then

$$C'=B(i,j,k)\cap \left(C^*-igcup_{m=1}^\infty 1^{\scriptscriptstyle 0}_{m}
ight)$$

is a closed subset of C^* . If $x \in C'$, let \hat{x} denote the unique maximal line segment in B(i, j, k) which passes through x and meets C_1 and C_2 . Let X denote the closed set formed by the point set union of the line segments $\hat{x}, x \in C'$, and set $Q(i, j, k) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C \neq \emptyset, y \in \hat{x}\}$. We now show that Q(i, j, k) is a borel set. Every point y of X can be given a coordinate vector $y = \langle x, h \rangle$, where $y \in \hat{x}$ and h is the height, relative to the j^{th} coordinate, of y above C^* . Because \overline{C} does not have irregular points, the number of points y in X which receive two different coordinate vectors is countable. Let Φ be the mapping $X \to C^* \times (-\infty, \infty)$ defined by taking $\Phi \langle x, h \rangle = (x, h), x \in C'$. Then K is a borel subset of X if and only if $\Phi(K)$ is a borel subset of the

closed set $\Phi(X)$. Hence $\Phi(C \cap X)$ is a vertically convex borel subset of $C' \times (-\infty, \infty)$. Hence the set $D = X \cap \{ \operatorname{proj} \Phi(C \cap X) \times (-\infty, \infty) \}$ is a convex borel set and so $Q(i, j, k) = \Phi^{-1}(D)$ is a borel set. Hence the set $R(i, j, k) = Q(i, j, k) - \bigcup_{m=1}^{\infty} G_m$ is a borel set. Consider now the set $S = \bigcup_{i,j,k} R(i,j,k)$ and consider the borel set T defined as the point set union of all 1-faces of \overline{C} which are not contained in some 2-face of \overline{C} . We assert that the set $H_1(C) = H_1(C) \cap (T - \bigcup_{m=1}^{\infty} G_m)$ equals S. For if $y \in H^1_1(C)$ then, because \overline{C} does not have any irregular points, there exists a unique 1-face l, not contained in $\bigcup_{m=1}^{\infty} G_m$, such that $y \in l$. Then $y \in H_1(C)$ if and only if $l \cap C = \emptyset$, which happens if and only if $l \subset Q(i, j, k)$ or in other words $y \in R(i, j, k)$ for some i, j, k. Hence $H_1^1(C) = S$. Let V denote the borel set of exposed points of \overline{C} and $H^2_{\scriptscriptstyle 1}(C)=V\cap H_{\scriptscriptstyle 1}(C),\, H^3_{\scriptscriptstyle 1}(C)=igcup_{m=1}^\infty \left(H_{\scriptscriptstyle 1}(C)\cap (G_m-V)\right).$ Now $H_1(C) = H_1^1(C) \cup H_1^2(C) \cup H_1^3(C)$. $H_1^1(C) = S$ is a borel set and, since $H_1^2(C) = V \cap C$, $H_1^2(C)$ is a borel set. Hence it is enough to show that $H_1(C) \cap (G_m - V)$ is a borel set for all m. Now let $\{G_m\}_{\nu=1}^{\infty}$ be those 2-faces of \overline{C} which meet C. Then relint $G_{m_{\nu}} \subset H^{3}_{1}(C)$ for all ν . Let $\{G_{m_{\nu}n}\}_{n=1}^{\infty}$ be the 1-faces of $G_{m_{\nu}}$. Then either relint $G_{m_{\nu}n} \subset H^{3}_{1}(C)$ or relint $G_{m,n}\cap H^s_{\scriptscriptstyle 1}(C)= \oslash$. Then the endpoints of $G_{m,n}$ may or may not be in $H_1^3(C)$. Let H_m be the countable set of those endpoints of $\{G_{m_{\nu}n}\}_{\nu=1}^{\infty}$ which lie in $H_1^3(C)$ and let $\{G_{m_{\nu}n_{\mu}}\}_{\nu=1}^{\infty}$ be the 1-faces of $G_{m_{\nu}}$ whose relative interiors are contained in $H^3_1(C)$. We have $G_{m_u} \cap H^3_1(C) =$ relint $G_{m_{\nu}} \cup (\bigcup_{\mu=1}^{\infty} \text{ relint } G_{m_{\nu},n_{\mu}}) \cup H_{m_{\nu}}$, which is a borel set. If, on the other hand, a 2-face of C does not meet C, its intersection with $H_1^{\mathfrak{l}}(C)$ is empty. Therefore $H^{\scriptscriptstyle 3}_{\scriptscriptstyle \rm I}(C)\cap G_m$ is a borel set for all m, and $H_{\scriptscriptstyle 1}(C)$ is a borel set.

For the case when C is an analytic set, say $C = \sum_{i \in I} \bigcap_{n=1}^{\infty} C(i \mid n)$ in the usual representation, the only modification required to the above proof is to show that the set Q(i, j, k) is an analytic set. With the previous notation, $Q(i \mid n) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C(i \mid n) \neq \emptyset, y \in \hat{x}\}$. Then $Q(i \mid n)$ is a closed set and $Q(i, j, k) = \sum_{i \in I} \bigcap_{n=1}^{\infty} Q(i \mid n)$. Therefore Q(i, j, k) is an analytic set.

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