ON VISUAL HULLS

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The concept of visual hull has been introduced by G. H. Meisters and S. Ulam. In the following article we study a few of the problems arising from this notion and, in particular, establish (Theorem 3) a conjecture of W. A. Beyer and S. Ulam.

Let $C$ be a set in $\mathbb{R}^n$ and $1 \leq j \leq n - 1$. Then the $j^{th}$ visual hull $H_j(C)$ of $C$ is defined to be the largest set whose $j^{th}$ projections are contained in those of $C$. Alternatively, $H_j(C)$ is the set of points $x$ in $\mathbb{R}^n$ such that each $(n - j)$-flat through $x$ contains a point of $C$.

Let $G_j^n$ denote the Grassmannian of $j$-subspaces in $\mathbb{R}^n$ with $\mu_j(G_j^n) = 1$ for the usual measure $\mu_j$ associated with $G_j^n$ regarded as a metric $0_n$-factorspace. (For further information about $\mu_j$ compare, for example, [3]). The $j^{th}$ virtual hull $V_j(C)$ of $C$ is defined to be the set of points $x$ in $\mathbb{R}^n$ such that almost all (with respect to $\mu_{n-j}$) $(n - j)$-flats through $x$ contain a point of $C$. Thus, if $n = 3, j = 2$, $H_2(C)(V_2(C))$ corresponds to those points in $\mathbb{R}^3$ which are photographically indistinguishable (with probability one) from $C$. A $j^{th}$ minimal hull of $C$ in $\mathbb{R}^n$ is a minimal set in $\mathbb{R}^n$ whose $j^{th}$ projections coincide with those of $C$. In [2] the announced purpose of the paper was to disprove the conjecture that $H_j(C) - C$ is connected to $C$, i.e., $\exists$ disjoint open sets $U, V$ such that $U \supset H_j(C) - C \neq \emptyset$ and $V \supset C \neq \emptyset$. To this we remark that a simple counterexample can be obtained by considering the closed set $C$ formed by removing the relative interiors of alternate sides of a regular hexagon inscribed in a plane circle with centre $a$. The first visual hull $H_1(C)$ is then $C \cup \{a\}$.


**Theorem 1.** Let $A_1, \ldots, A_{j+1}$ be spherically convex, closed subsets (not necessarily nonempty) of the sphere $S^{n-1}$, such that each $(n - j - 1)$-subsphere of $S^{n-1}$ has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. Then $A_1 \cap \cdots \cap A_{j+1} \neq \emptyset$. (so, that, in particular, each set $A_i$ is nonempty).

**Remark.** $S^{n-1}$ is the unit sphere of $\mathbb{R}^n$ and an $(n - j - 1)$-subsphere of $S^{n-1}$ is the intersection of an $n - j$ subspace with $S^{n-1}$. A set $C \subset S^{n-1}$ is spherically convex if $C$ is contained in an open hemisphere of $S^{n-1}$ and, if $x, y \in C$ then $C$ contains the minor arc on the $1$-subsphere determined by $x, y$ and $0$ (the centre of $S^{n-1}$).

**Proof.** The case $n = 1$ is trivial. We assume inductively that
the result is true for all \( n' < n \) and it remains to prove the result for \( j + 1 \) sets on \( S^{n-1} \). Assume on the contrary that there exist spherically convex closed subsets \( A_1, \ldots, A_{j+1} \subset S^{n-1} \) such that

\[
T \cap (A_1 \cup \cdots \cup A_{j+1}) \neq \emptyset
\]

for each \( (n - j - 1) \)-subsphere \( T \) of \( S^{n-1} \), and \( A_1 \cap \cdots \cap A_{j+1} = \emptyset \). Let \( A = A_1 \cap \cdots \cap A_j \). Then \( A, A_{j+1} \) are disjoint spherically convex closed subsets of \( S^{n-1} \), and there exists an \( (n - 2) \)-subsphere \( S' \) of \( S^{n-1} \) which separates \( A \) and \( A_{j+1} \) and such that \( S' \cap A = \emptyset, S' \cap A_{j+1} = \emptyset \). Set \( A'_i = A_i \cap S' \) \( (1 \leq i \leq j) \). Then each \( A'_i \) is a spherically convex closed subset of \( S' \) and, since \( A_{j+1} \cap S' = \emptyset \), each \( (n - j - 1) \)-subsphere of \( S' \) has a nonempty intersection with \( A'_i \cap \cdots \cap A'_{j+1} \). Hence by the inductive assumption \( A'_i \cap \cdots \cap A'_{j+1} = A \cap S' \neq \emptyset \); contradiction.

**Theorem 2.** In \( \mathbb{R}^n \) let \( C_1, \ldots, C_{j+1} \) be \( j+1 \) compact convex sets. If \( x \in H_j(\bigcup_{i=1}^{j+1} C_i) \) then either \( x \in \bigcup_{i=1}^{j+1} C_i \) or there exists a halfline \( l \) emanating from \( x \) such that \( l \cap C_i \neq \emptyset, 1 \leq i \leq j+1 \).

**Corollary.** In \( \mathbb{R}^n \) let \( C_1, \ldots, C_{j+1} \) be compact convex sets. Then a sufficient condition for \( H_j(\bigcup_{i=1}^{j+1} C_i) = \bigcup_{i=1}^{j+1} C_i \) is that the sets do not have a common transversal.

**Proof.** On \( S^{n-1} \) define \( j+1 \) spherically convex closed subsets \( A_1, \ldots, A_{j+1} \) so that \( u \in A_i \) if \( u \in S^{n-1} \) and the half line \( \{ x + \lambda u \mid \lambda \geq 0 \} \) meets \( C_i \). Then, as \( x \in H_j(\bigcup_{i=1}^{j+1} C_i) \) each \( (n - j - 1) \)-subsphere of \( S^{n-1} \) has a nonempty intersection with \( \bigcup_{i=1}^{j+1} A_i \). And so, by Theorem 1, there exists \( u \in \bigcap_{i=1}^{j+1} A_i \), i.e., the halfline \( \{ x + \lambda u \mid \lambda \geq 0 \} \) meets each of \( C_1, \ldots, C_{j+1} \).

**Theorem 3.** In \( \mathbb{R}^n \) let \( C_1, \ldots, C_{j+1} \) be nonempty compact convex sets. Then the number of components of \( H_j(\bigcup_{i=1}^{j+1} C_i) \) is at most \( j+1 \) with equality if and only if \( C_1, \ldots, C_{j+1} \) are pairwise disjoint.

**Proof.** By Theorem 2, if \( x \in H_j(\bigcup_{i=1}^{j+1} C_i) - \bigcup_{i=1}^{j+1} C_i \), then there exists a halfline \( l = \{ x + \lambda u \mid \lambda \geq 0 \} \) such that \( l \) meets each of \( C_1, \ldots, C_{j+1} \).

Then \( x + \alpha_k u \in C_k \) for some \( \alpha_k > 0 \). We set \( \alpha = \min \{ \alpha_k \mid 1 \leq k \leq j+1 \} \) and want to show that \( x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i) \) for all \( \lambda \) with \( 0 \leq \lambda \leq \alpha \). Set \( y = x + \lambda u \) and let \( P \) be an \( (n-j) \)-subspace. As \( x \in H_j(\bigcup_{i=1}^{j+1} C_i) \) there exists \( i \) such that the \( (n-j) \)-flat \( x + P \) meets \( C_i \) at \( v \), say. Set \( z = x + \alpha_i u \in C_i \). Then, as \( y \) lies between \( x \) and \( z \) on \( l \), there exists \( \mu, 0 \leq \mu \leq 1 \), such that \( y = \mu x + (1 - \mu) z \). Then the \( (n-j) \)-\flat \( y + P \) through \( y \) contains the point \( \mu v + (1 - \mu) z \) of \( C_i \). As \( P \)
was arbitrary we conclude that $y \in H_j(\bigcup_{i=1}^{j+1} C_i)$ and hence that $x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for $0 \leq \lambda \leq \alpha$. Hence, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then $x$ is connected, via a line segment in $H_j(\bigcup_{i=1}^{j+1} C_i)$, to at least one of the sets $C_i$. Hence $H_j(\bigcup_{i=1}^{j+1} C_i)$ has at most $j + 1$ components with equality only if the $C_i$'s are disjoint. If the sets $C_1, \ldots, C_{j+1}$ are pairwise disjoint then in order to show that $H_j(\bigcup_{i=1}^{j+1} C_i)$ has exactly $j + 1$ components it is enough to show that for each $k$, $1 \leq k \leq j + 1$, there exist disjoint open sets $U_k, V_k$ such that $U_k \cup V_k \supset H_j(\bigcup_{i=1}^{j+1} C_i)$ and $U_k \supset C_k$, $V_k \supset \{C_1 \cup \cdots \cup C_{k-1} \cup C_{k+1} \cup \cdots \cup C_{j+1}\}$. We suppose, without loss of generality, that $k = 1$. For $i = 2, \ldots, j + 1$ let $H_i$ denote a hyperplane which strictly separates $C_i$ from $C_i$, and let $H_i$ be the open halfspace bounded by $H_i$ and containing $C_i$. We can assume that the $H_i$'s are in general position. Set $U_1 = \bigcap_{i=2}^{j+1} H_i$, $V_1 = R^n - \bar{U}_1$. Then $U_1$ and $V_1$ are disjoint open sets, $C_1 \subset U_1$, $\bigcup_{i=2}^{j+1} C_i \subset V_1$. It remains to show that $H_j(\bigcup_{i=1}^{j+1} C_i) \subset U_1 \cup V_1$, and it is enough to show that $(\bar{U}_1 \cap \bar{V}_1) \cap H_j(\bigcup_{i=1}^{j+1} C_i) = \emptyset$. Since the $H_i$'s are in general position, their intersection $\bigcap_{i=2}^{j+1} H_i$ is an $(n - j)$-dimensional flat $L$. Let $I$ be the $j$-dimensional subspace orthogonal to $L$. If $M$ is any subset of $R^n$ we denote by $\text{proj}_I M$ the set of all points $x \in I$ for which the flat $L_x$, which is parallel to $L$ and contains $x$, has a nonempty intersection with $M$. $\text{proj}_I U_1$ and $\text{proj}_I V_1$ are two open sets in $I$ with common boundary $\text{proj}_I (\bar{U}_1 \cap \bar{V}_1)$. As $\text{proj}_I C_1 \subset \text{proj}_I U_1$, $\text{proj}_I \bigcup_{i=2}^{j+1} C_i \subset \text{proj}_I V_1$, it follows that $(\text{proj}_I (\bar{U}_1 \cap \bar{V}_1)) \cap (\text{proj}_I \bigcup_{i=2}^{j+1} C_i) = \emptyset$. Now, if $z$ is an arbitrary point in $\bar{U}_1 \cap \bar{V}_1$ it follows that $L_z \cap (\bigcup_{i=2}^{j+1} C_i) = \emptyset$, and since $\dim L_z = n - j$, we find, by the definition of $H_j$, that $z$ does not belong to $H_j(\bigcup_{i=1}^{j+1} C_i)$. Therefore $(\bar{U}_1 \cap \bar{V}_1) \cap H_j(\bigcup_{i=1}^{j+1} C_i) = \emptyset$.

**REMARKS.** The proof of Theorem 3 also shows that any component of $H_j(\bigcup_{i=1}^{j+1} C_i)$ has the property that any two points of it can be joined by a broken line in it, consisting of at most 3 segments. Hence it is natural to ask: When are these components convex? (supposing now that the $C_i$'s are disjoint). In [1] W. A. Beyer has shown an example of three (nondisjoint) polytopes $C_i$ in $R^3$ such that $H_3(C_1 \cup C_2 \cup C_3)$ is not a polyhedron. We don't know whether a similar construction would be possible with disjoint polytopes. Let us mention here a few more technical terms. If $M$ is any subset of $R^n$, we denote by $\text{aff} M$ the affine hull of $M$ and by $\text{conv} M$ the convex hull of $M$. $\text{relint} M$ means the interior of $M$ with respect to the natural topology in $\text{aff} M$. By the dimension $\dim M$ of $M$ we understand the algebraic dimension of the flat $\text{aff} M$. A polytope is the convex hull of some finite set. If $P \subset E^n$ is a convex set we denote by $\text{ext} P$ the set of extreme points of $P$ and by $\text{exp} P$ the set of its exposed points. For an exact definition of these terms the reader may compare, for example, the introductory chapters of [4].
Theorem 4. (i) In $\mathbb{R}^n$ let $C_1, C_2$ be compact convex sets. Then $H_x(d \cup C_2)$ is the union of at most two convex components which are polytopes whenever $C_1$ and $C_2$ are polytopes.

(ii) There exist in $\mathbb{R}^3$ three disjoint polytopes such that one of the components of the second visual hull of their union is not convex.

Lemma 1. Let $C_1, C_2$ be $n$-dimensional polytopes in $\mathbb{R}^n$. If $a \in H_{i}(C_1 \cup C_2)$ there exists a hyperplane $H$ such that

1. $a \not\in H$, $H$ separates $a$ from $C_i$
2. $H \cap C_i = \emptyset$ or $H$ supports $C_i$ ($i = 1, 2$)
3. $\text{aff}(H \cap (C_1 \cup C_2)) = H$

Proof of Lemma 1. The case $n = 1$ is trivial, and we assume $n \geq 2$. If there exists a hyperplane $P$ through $a$ which does not meet $C_1 \cup C_2$ and does not separate $C_1$ and $C_2$ then $\text{conv}(C_1 \cup C_2)$ is an $n$-dimensional polytope not containing $a$, and the lemma follows from standard results on polytopes. Hence it can be supposed that there is a hyperplane $H$ for which (1) and also (2') holds. We choose $H$ in the set $\emptyset$ of hyperplanes for which (1) and (2') holds. We assume that $h = \dim \text{aff} T$ is maximal, where $T = H \cap (C_1 \cup C_2)$. Obviously $h \geq 0$. If $h < n - 1$, let $F \subset H$ be an $(n - 2)$-dimensional hyperplane in $H$ containing $T$, and denote by $\pi: \mathbb{R}^n \rightarrow E$ the projection along $F$ onto a 2-dimensional flat $E$ orthogonal to $F$. It is easy to see that there is a line $L$ in $E$ such that: (a): the singleton $\pi(T)$ is contained in $L$. (b): $\pi(a) \not\in L$, $L$ separates $\pi(a)$ from the polygon $\pi(C_1)(\gamma): L$ separates $\pi(C_1)$ and $\pi(C_2)$.

(\delta) $\text{aff}(L \cap (\pi(C_1) \cup \pi(C_2))) = L$.

(Notice that the conditions (a) - (\gamma) are fulfilled by $\pi(H)$). The hyperplane $\pi^{-1}(L)$ of $E^*$ intersects $C_1 \cup C_2$ in a set $S$ with $\dim \text{aff} S = h + 1$. Since $S \in \emptyset$ this contradicts the maximality of $h$. Hence the lemma is established.

Proof of Theorem 4. (i) We first prove the result when $C_1, C_2$ are $n$-dimensional polytopes. If $C_1 \cap C_2 \neq \emptyset$ then

$$H_x(C_1 \cup C_2) = \text{conv}(C_1 \cup C_2),$$

which is a polytope. We suppose therefore that $C_1 \cap C_2 = \emptyset$. Let \{\{H_i\}_{i=1}^m\} be the finite set of those hyperplanes which do not contain an interior of $C_j$ ($j = 1, 2$) and for which $\dim (H_i \cap (C_1 \cup C_2)) = n - 1$. By $C^*_j$ we denote the (finite) intersection of those closed half spaces which contain $C_j$ and whose bounding hyperplane is amongst \{\{H_i\}_{i=1}^m, j = 1, 2\}. Then $C^*_j$ is polyhedral and, since $C_1, C_2$ are compact, $C^*_j$ is a polytope,
\[ j = 1, 2. \] We show that \( H_i(C_1 \cup C_2) = C_1^* \cup C_2^*. \) Suppose that \( x^* \in C_1^* \cup C_2^* \). Then there exist closed halfspaces \( H_i^*, H_2^* \) with bounding hyperplanes \( H_i, H_2 \) amongst \( \{ H_i \}_{i=1}^m \) such that \( x^* \in H_i^* \supset C_1, x^* \in H_2^* \supset C_2. \) If

\[ x^* \in H_i(C_1 \cup C_2), H_i \quad \text{and} \quad H_2 \]

must separate \( C_1 \) and \( C_2 \). Consider \( H_i \) and the two disjoint compact sets \( H_1 \cap C_i, H_1 \cap C_2 \) in \( H_1 \). There exists a \( n - 2 \) dimensional flat \( L \) in \( H_1 \) which strictly separates \( H_1 \cap C_i \) and \( H_1 \cap C_2 \). By slightly rotating \( H_1 \) about \( L \) in the appropriate direction we obtain a hyperplane \( H_i' \) which strictly separates \( C_1 \) and \( C_2 \) as well as \( x^* \) and \( C_1 \). Similarly we can obtain a hyperplane \( H_i' \) which strictly separates \( C_1 \) and \( C_2 \), and \( x^* \) and \( C_1 \). We may suppose that \( H_i', H_2 \) are not parallel and so \( H_i' \cap H_2 \) is an \( n - 2 \) flat. Suppose, without loss of generality, that \( H_1' = \{ x \mid \langle x, \xi \rangle = \alpha > 0 \}, H_2 = \{ x \mid \langle x, \eta \rangle = \beta > 0 \}. \) Then

\[ C_1 \subset \{ x \mid \langle x, \xi \rangle > \alpha \} \cap \{ x \mid \langle x, \eta \rangle > \beta \} \]
\[ C_2 \subset \{ x \mid \langle x, \xi \rangle < \alpha \} \cap \{ x \mid \langle x, \eta \rangle < \beta \}. \]

Consider the hyperplane \( H: \{ x \mid \langle x, \lambda \xi + (1 - \lambda) \eta \rangle = 0 \}, \) where \( \lambda \alpha + (1 - \lambda) \beta = 0 \) and \( 0 < \lambda < 1 \). Then \( x^* \in H \) and, using the above inequalities, \( C_i \cap H = \emptyset, i = 1, 2. \) Hence \( x^* \) is not in \( H_i(C_1 \cup C_2) \), and we have \( H_i(C_1 \cup C_2) \subset C_1^* \cup C_2^*. \) Conversely, if \( x^* \in C_1^* \cup C_2^* - H_i(C_1 \cup C_2) \), suppose without loss of generality that \( x^* \in C_1^* \). Then, by Lemma 1, there exists a hyperplane \( H \) amongst \( \{ H_i \}_{i=1}^m \) which does not contain \( x^* \) and which separates \( x^* \) from \( C_1 \). Then, if \( H^* \) donotes the closed halfspace containing \( C_1 \) whose bounding hyperplane is \( H, x^* \in H^* \) and so \( x^* \in C_1^* \); contradiction. And so \( H_i(C_1 \cup C_2) = C_1^* \cup C_2^* \), which is the union of two polytopes. If \( C_i, C_j \) are compact convex sets we choose decreasing sequences \( \{ P_i \}_{i=1}^\infty, \{ P_j \}_{j=1}^\infty \) of polytopes such that \( C_i = \bigcap_{n=1}^\infty P_i \), \( i = 1, 2. \) Then, using the above notation,

\[ H_i(C_1 \cup C_2) = \bigcap_{n=1}^\infty P_i \cap \bigcap_{n=1}^\infty P_j. \]

(ii) Let \( W \) be the cube \( \{ x = (x_1, x_2, x_3) \mid -1 \leq x_i \leq 1, i = 1, 2, 3 \} \) in \( R^3 \), and denote by \( W_i \) the facet of \( W \) defined by \( x_i = 1. \) Set \( C_1 = W_i, C_2 = 2W_i, C_3 = 3W_i. \) Let \( B_i(1 \leq i \leq 3) \) be the components of \( H_i(\bigcup_{i=1}^3 C_i), \) where the indices are chosen such that, for all \( i, C_i \subset B_i. \) Clearly \( (0,0,0) \in B_i \) as does, of course, the point \( (1, -1, -1) \in B_i \cap C_i. \) However we show that the line segment \( m: \{ x = \lambda(1, -1, -1) \mid 0 < \lambda < 1 \} \) is not in \( B_i. \) Now \( C_i \cup C_i \) is contained in the halfspace \( \{ x \mid \langle x, (0,1,1) \rangle \geq 0 \} \) whose bounding hyperplane \( P \) passes through the points \( (0, 0, 0), (1, -1, 1) \) and \( (-1, -1, 1); P \cap \text{aff} W_i \) is a line in direction \( (0, -1, 1). \) If \( y \in m, \) then \( y = \mu(1, -1, -1) \) for some \( \mu, 0 < \mu < 1. \) Consider the line \( l = y + [\lambda(0, -1, 1) \mid \lambda \text{ real}]. \) If \( z = (z_1, z_2, z_3) \in l \) then \( z_1 = \mu < 1, \)
i.e., \( z \notin C_1 \). Also \( \langle s, (0, 1, 1) \rangle = -2\mu < 0 \), which means that \( z \in C_1 \cup C_2 \).

Therefore \( l \) does not meet \( C_1 \cup C_2 \cup C_3 \), \( m \) does not belong to \( B_1 \), and \( B_1 \) is not convex.

In [6] V. L. Klee proved that if all \( j \text{th} \) projections of a compact convex body \( C \) in \( \mathbb{R}^n \) (\( j \) fixed \( \geq 2 \)) are polytopes, then \( C \) is a polytope. As a partial analogue to this for unions of two convex bodies we prove

**Theorem 5.** Let \( C_1, C_2 \) be two disjoint compact convex bodies in \( \mathbb{R}^n \) such that each \( j \text{th} \) projection of \( C_1 \cup C_2 \) (\( j \) fixed \( \geq 2 \)) is the union of two polytopes. Then (i) \( \text{ext}(C_i) = \text{exp}(C_i) \) and \( \text{ext}(C_i) \) is countable (\( i = 1, 2 \)) but (ii) \( \text{ext}(C_i) \) is not necessarily finite.

**Proof.** Let \( a \) be an extreme point of \( C_1 \) and we suppose, without loss of generality, that \( a = 0 \), the origin of \( \mathbb{R}^n \). Then, to prove (i) it is enough to prove that the convex cone \( K \) of outward normals to \( C_1 \) at \( 0 \) is \( n \)-dimensional. We assume that \( \dim K \leq n - 1 \) so that \( K \) is contained in an \((n - 1)\)-subspace \( P_1 \), and seek a contradiction. Let \( P_2 \) be an \((n - 1)\)-subspace which supports \( C_1 \) at \( 0 \). Of course \( P_1 \neq P_2 \).

We can choose an \((n - 1)\)-subspace \( P_3 \) so that there exists a translate of \( P_3 \) which strictly separates \( C_1 \) and \( C_2 \) and such that the normal to \( P_3 \) at \( 0 \) intersects \( P_1 \) only at \( 0 \). Then \( P_2 \cap P_3 \) is a subspace of dimension at least \( n - 2 \) and we choose an \( n - j \)-subspace \( Q \) in \( P_2 \cap P_3 \). The orthogonal complement \( S \) of \( Q \) in \( \mathbb{R}^n \) is a \( j \)-dimensional subspace which meets \( P_1 \) in a \((j - 1)\)-subspace. The projection of \( C_1 \cup C_2 \) onto \( S \) is the union of two polytopes. Further, as \( P_3 \cap C_2 = \varnothing \), \( 0 \) is at positive distance from \( \text{proj} C_2 \). As \( 0 \) is an extreme point of \( \text{proj} C_1 \), it follows that \( 0 \) is a locally polyhedral extreme point for \( \text{proj} C_1 \). Hence, in \( S \), the cone of outward normals to \( \text{proj} C_1 \) at \( 0 \) is \( j \)-dimensional. Further, any \((j - 1)\)-plane \( H \) of support in \( S \) to \( \text{proj} C_1 \) at \( 0 \) can be extended to an \((n - 1)\)-plane of support \( H + Q \) in \( \mathbb{R}^n \) to \( C_1 \) at \( 0 \). Also, the outward normals to these planes form a \( j \)-dimensional convex cone lying in \( S \). Hence \( j = \dim(K \cap S) = \dim(P_1 \cap S) = j - 1 \); contradiction. And so (i) is proved.

To prove (ii) we construct an example in \( \mathbb{R}^3 \) of two convex bodies \( C_1, C_2 \), both of which have a countable infinity of extreme points but, nevertheless, each \( 2 \)-projection of \( C_1 \cup C_2 \) is the union of two convex polygons. Let \( l = \{x \mid x_1 = x_2 = 0, -1 \leq x_3 \leq 1\} \) be a line segment and \( S = \{x \mid (x_1 - 1)^2 + x_2^2 = 1, x_3 = 0\} \) a plane circle. By \( T \) we denote the set of those points on \( S \) with \( x_2 \)-coordinate \( \pm(1/n) \) for \( n = 1, 2, \ldots \). We take \( C_1 = \text{conv}\{l \cup T\} \), which is a compact convex body in \( \mathbb{R}^3 \) with extreme points \( T \cup \{(0, 0, -1), (0, 0, 1)\} \). It is easily seen that there is precisely one \( 2 \)-projection of \( C_1 \) which is not a convex polygon, and that is in the direction \((0, 0, 1)\). Further the only limit point of extreme points of this projection is \((0, 0, 0)\). Define \( C_2 \) as a disjoint copy of
formed by placing $C_2$ above $C_1$ in such a way that their respective major lines pierce the centres of their respective circles. From above, every 2-projection of $C_1 \cup C_2$ is the union of two convex polygons and and both $C_1$ and $C_2$ are compact bodies with a countable infinity of extreme points.

3. Visual hulls of more general sets. The following problem can be formulated.

*Is the visual (virtual) (minimal) hull of a borel (analytic) set in $\mathbb{R}^n$ necessarily borel (analytic)?*

The answer is affirmative (Theorem 6) for virtual hulls and negative (Theorem 7) for minimal hulls. Whilst it is not true (Theorem 8) that the $j^{th}$ visual hull of a borel set is necessarily borel, we have been unable to decide whether or not the $j^{th}$ visual hull of a borel or of an analytic set is always analytic, except in the cases covered by Theorem 9. It is possible also that the $j^{th}$ visual hull of a convex borel (analytic) set is a borel (analytic) set, and we include some partial results (Theorem 9) in this direction. As before we denote by $G_j^n$ the Grassmannian of $j$-subspaces of $\mathbb{R}^n$ and by $\mu_j$ the invariant (with respect to $0_n$ acting in the usual way on $G_j^n$) measure normalised so that $\mu_j(G_j^n) = 1$.

**Lemma 2.** Let $A$ be an analytic set in $\mathbb{R}^n$ and denote by $A^*$ the set of those $j$-subspaces in $G_j^n$ which meet $A$. Then

(i) $A^*$ is an analytic set in $G_j^n$ and hence $A^*$ is $\mu_j$ measurable.

(ii) If $\mu_j(A^*) > a$ then there exists a compact subset $A'$ of $A$ such that $\mu_j(A'^*) > a$.

(iii) If $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of analytic sets in $\mathbb{R}^n$ then $\mu_j(\bigcup_{l=1}^{\infty} A_l)^* = \lim_{l \to \infty} \mu_j(A_l^*)$.

(iv) If $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of analytic sets in $\mathbb{R}^n$ then $\mu_j(\bigcap_{l=1}^{\infty} A_l)^* = \lim_{l \to \infty} \mu_j(A_l^*)$.

**Proof.** (i) Let $I$ be the set of irrational numbers in $[0, 1]$ and, if $i = (i_1, \ldots, i_n, \ldots)$ is a typical member of $I$ expressed as a continued fraction, set $i \mid n = (i_1, \ldots, i_n)$. Then, as $A$ is analytic, it can be represented as $A = \bigcup_{i \in I} A(i \mid n)$ where the sets $A(i \mid n)$ form, for each fixed $i$, a decreasing sequence of compact subsets of $\mathbb{R}^n$. Then $A^* = \bigcup_{i \in I} A^*(i \mid n)$. As each $A^*(i \mid n)$ is a compact subset of $G_j^n$, we conclude that $A^*$ is an analytic set.

(ii) If $\mu_j(A^*) > a + \delta$ with $\delta > 0$, then we can choose $m_1, 1 \leq m_1 < \infty$, such that if $I_i$ denotes the set of irrational numbers

$$i = (i_1, \ldots, i_n, \ldots)$$

with $1 \leq i_1 \leq m_1$ and $A_i^* = \bigcup_{i \in I_i} A^*(i \mid n)$ then $\mu_j(A_i^*) > a + \delta$. 
Proceeding by induction we may define natural numbers \( m_p, 1 \leq p < \infty \), such that if \( I_q \) denotes the subset of those irrationals \( i \) with \( 1 \leq i_p \leq m_p \) for \( p = 1, \ldots, q \), and \( A_q^* = \sum_{i \in I_q} \bigcap_{n=1}^{\infty} A(i \mid n) \) then \( \mu_j(A_q^*) \geq a + \delta \). Let \( I' \) be the compact subset of \([0, 1]\) defined as the set of those irrational numbers \( i \) for which \( 1 \leq i_p \leq m_p \) for \( p = 1, 2, \ldots \), and

\[
A' = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A(i \mid n).
\]

Then \( \bigcap_{q=1}^{\infty} A_q^* = A' \) and so \( \mu_j(A') \geq a + \delta > a \). Also

\[
A' = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A(i \mid n)
\]

is a compact subset of \( A \), as \( I' \) is a compact subset of \( I \).

(iii) \( \mu_j(\bigcup_{i=1}^{\infty} A_i^*) = \mu_j(\bigcup_{i=1}^{\infty} A_i^*) = \lim_{i \to \infty} \mu_j(A_i^*) \).

(iv) Clearly \( \mu_j(\bigcap_{i=1}^{\infty} A_i^*) \leq \lim_{i \to \infty} \mu_j(A_i^*) \). Now set \( \mu_j(\bigcap_{i=1}^{\infty} A_i^*) = a \) and suppose \( \lim_{i \to \infty} \mu_j(A_i^*) > a + \varepsilon \), for some positive number \( \varepsilon \). By (ii) we find a compact set \( B_i \subseteq A_i \) such that \( \mu_j(B_i) \geq \mu_j(A_i^*) - \varepsilon/2 \). Now we have \( A_i^* = (B_i \cap A_i)^* \cup (A_i^* - B_i^*) \), where \( A_i^* - B_i^* = \{ F \in G_j \mid F \cap A_i \neq \emptyset \text{, but } F \cap B_i = \emptyset \} \).

Since \( A_i^* \subseteq A_i^* \) we derive further \( A_i^* \subseteq (B_i \cap A_i)^* \cup (A_i^* - B_i^*) \), or \( \mu_j(A_i^*) \leq \mu_j(B_i \cap A_i)^* + \varepsilon/2 \). Since \( B_i \cap A_i \) is analytic there exists, again by (ii), a compact set \( B_{i+1} \subseteq (B_i \cap A_i) \) such that

\[
\mu_j(B_{i+1}) \geq \mu_j(B_i \cap A_i)^* - \varepsilon/4
\]

and consequently

\[
\mu_j(B_{i+1}) \geq \mu_j(A_i^*) - (\varepsilon/2 + \varepsilon/4). 
\]

Continuing this process we obtain a decreasing sequence \( \{ B_i \}_{i=1}^{\infty} \) of compact subsets of \( R^n \) such that \( B_i \subseteq A_i \), \( i = 1, 2, \ldots \), and \( \mu_j(B_i) \geq \mu_j(A_i^*) - \varepsilon/(2^n) \). Then \( \bigcap_{i=1}^{\infty} B_i = (\bigcap_{i=1}^{\infty} B_i^*) \subseteq (\bigcap_{i=1}^{\infty} A_i)^* \), and \( \mu_j(\bigcap_{i=1}^{\infty} B_i^*) = \lim_{i \to \infty} \mu_j(B_i^*) \leq a \); but also \( \lim_{i \to \infty} \mu_j(B_i) \geq \lim_{i \to \infty} \mu_j(A_i^*) - \varepsilon \). Combining the last two inequalities we find \( \lim_{i \to \infty} \mu_j(A_i) \leq a + \varepsilon \), a contradiction.

**Theorem 6.** Let \( C \) be a borel (analytic) set in \( R^* \). Then the \( j^{th} \) virtual hull \( V_j(C) \) is a borel (analytic) set.

**Proof.** Suppose first that \( C \) is a borel set in \( R^* \), and we need to show that \( V_j(C) \) is a borel set. If \( D \) is a subset of \( R^* \) and \( x \in R^* \), let \( D[x, n - j] \) denote the set of those \( n - j \) subspaces \( F \) in \( G_{n-j}^j \) such that \( (x + F) \cap D \neq \emptyset \). If \( 0 < \lambda < 1 \) let \( D(n - j, \lambda) \) be the set of all \( x \) in \( R^* \) such that \( \mu_{n-j}(D[x, n - j]) > \lambda \). Let \( B \) denote the largest family of subsets of \( R^* \) such that \( D \in B \) if (i) \( D \) is a borel set in \( R^* \). (ii) \( D(n - j, \lambda) \) is a borel set for all \( \lambda, 0 < \lambda < 1 \). We shall prove that \( B \) coincides with the family of borel subsets of \( R^* \), and it is enough
to show that $B$ contains the open sets and is closed under the operations of increasing union and decreasing intersection. If $D$ is an open subset of $R^n$, then it is easy to see that $D(n-j, \lambda)$ is open for all $\lambda$, $0 < \lambda < 1$, and so $B$ contains all the open sets. Now suppose that $\{E_i\}_{i=1}^{\infty}$ is an increasing sequence of sets in $B$ and set $E = \bigcup_{i=1}^{\infty} E_i$. We want to show that for each $\lambda$, $0 < \lambda < 1$, the equality $E(n-j, \lambda) = \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$ holds. In order to do this we observe the following equivalences: $x \in E(n-j, \lambda) \iff \mu_{n-j}(E[x, n-j]) > \beta \iff \lim_{\lambda \to \infty} \mu_{n-j}(E[x, n-j]) > \lambda \iff x \in \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$. Here the first equivalence holds by definition, the second one follows directly from Lemma 2, (iii), if we observe that this lemma remains true if $M^*$ denotes, for each $M \subset R^n$, the set $M[x, n-j]$ (for $x \in R^n$ fixed). (The lemma itself is stated for the special case where $x$ is the origin of $R^n$.) The last equivalence again follows immediately from the definitions, we only have to observe that the sequence $\{E_i\}_{i=1}^{\infty}$ is increasing. Now suppose that $\{H_i\}_{i=1}^{\infty}$ is a decreasing sequence of subsets of $B$ and set $H = \bigcap_{i=1}^{\infty} H_i$. Suppose $\lambda$ fixed, $0 < \lambda < 1$, and let $m$ be a natural number such that $\lambda + 1/m < 1$. Then, using (iv) of Lemma 2, we find by an argument analogous to the one above, $H(n-j, \lambda) = \bigcup_{p=m}^{\infty} \bigcap_{i=1}^{\infty} H_i(n-j, \lambda + 1/p)$. Hence $H(n-j, \lambda)$ is a borel set, and $H \in B$. Therefore, $B$ is the family of borel subsets of $R^n$ and so, in particular, $C \in B$. Further $V_j(C) = \bigcap_{p=2}^{\infty} C(n-j, 1-(1/p))$ and so $V_j(C)$ is a borel set.

To show that $V_j(A)$ is analytic whenever $A$ is analytic, we use the well known result that there exists an $F_{\sigma\delta}$ set $K$ in $R^{n+1}$ such that $A$ is the orthogonal projection $\text{proj} K$ into $R^n$ (see, for example, [8]). Call an $(n-j+1)$-subspace $H$ of $R^{n+1}$ upright if $H$ has the form $\{\hat{H} + \lambda(0, \cdots, 0, 1) \mid -\infty < \lambda < \infty\}$ where $\hat{H} \in G_{n-j}$. Let $U_{j+1}$ be the set of upright $(n-j+1)$-subspaces in $R^{n+1}$ with the measure $\mu'$ induced by $\mu_{n-j}$ in the obvious manner. We can define $U_{j+1}(C)$ of a set $C$ in $R^{n+1}$ as the set of all those points $x$ in $R^{n+1}$ such that almost all (with respect to $\mu'$) upright $(n-j+1)$-flats through $x$ meet $C$. As above, it can been shown that $U_{j+1}(C)$ is a borel set whenever $C$ is a borel set. Clearly $\text{proj} U_{j+1}(K) = V_j(A)$ and, since the projection of a borel set is analytic, we conclude that $V_j(A)$ is an analytic subset of $R^n$.

**Theorem 7.** Let $C$ be an open convex subset of $R^n$. Then assuming the continuum hypothesis, $C$ contains a minimal $j$th hull $D$ such that every analytic subset of $D$ is countable.\(^1\)

**Proof.** We assume the continuum hypothesis and let $\Omega$ be the

\(^1\) As the referee pointed out, Theorem 7 may be a special case of a much more general theorem on effective constructions.
first uncountable ordinal. Let \( \{ A_\xi \}_{\xi \in \Omega} \) be an enumeration of the analytic subsets of \( R^n \) of \( (n-j) \)-dimensional measure zero; let \( \{ H_\xi \}_{\xi \in \Omega} \) be an enumeration of the \( (n-j) \)-flats which meet \( C \). Let \( F \) be a fixed \( (n-j) \)-subspace of \( R^n \) and denote by \( \alpha \) a fixed set, which is not a point of \( R^n \). We now choose a set \( E = \{ M_\xi \}_{\xi \in \Omega} \) and a collection of translates \( \{ F_\xi \}_{\xi \in \Omega} \) of \( F \) inductively as follows. Take \( M_\xi \in (H_\xi - A_\xi) \cap C \) and let \( F_\xi \) be a translate of \( F \) through \( M_\xi \). Suppose now that \( M_{\xi'}, F_{\xi'} \) have been defined for all \( \xi' < \xi \), where \( \xi \) is some ordinal proceeding \( \Omega \). If \( H_\xi \) is a translate of \( F \) we take \( F_\xi = H_\xi \) and consider two possibilities:

(a) If \( \exists \xi' < \xi \) such that \( M_{\xi'} \in H_\xi \) then we take \( M_\xi = \alpha \).

(b) If \( \exists \xi' < \xi \) such that \( M_{\xi'} \in H_\xi \), we choose \( M_\xi \) in the set \( (H_\xi - (\bigcup_{\xi' < \xi} H_{\xi'} \cup \bigcup_{\xi' < \xi} A_{\xi'})) \cap C \). Such a choice is possible as \( H_\xi \cap C \) has positive \( (n-j) \)-dimensional measure whereas \( H_\xi \cap (\bigcup_{\xi' < \xi} H_{\xi'} \cup \bigcup_{\xi' < \xi} A_{\xi'}) \) has zero \( (n-j) \)-dimensional measure, being a countable union of sets of measure zero. If \( H_\xi \) is not a translate of \( F \) we find, by similar arguments, that the set \( (H_\xi - (\bigcup_{\xi' < \xi} H_{\xi'} \cup \bigcup_{\xi' < \xi} A_{\xi'} \cup \bigcup_{\xi' < \xi} F_{\xi'})) \cap C \) is not empty. We choose \( M_\xi \) in this set and let \( F_\xi \) be the translate of \( F \) through \( M_\xi \). We claim that the set \( D = E - \alpha \) is a \( j^\text{th} \) minimal hull for \( C \) which meets each analytic subset in at most a countable number of points. To show that all \( j^\text{th} \) projections of \( D \) coincide with those of \( C \), it is enough to show that the \( j^\text{th} \) visual hull of \( D \) contains \( C \). Let \( x \) be a point of \( C \) and let \( P \) be an \( (n-j) \)-flat through \( x \). Then \( P \) is amongst \( \{ H_\xi \}_{\xi \in \Omega} \), say \( P = H_\xi \). If \( M_{\xi'} \neq \alpha \) then \( M_{\xi'} \in D \cap H_\xi \). If \( M_{\xi'} = \alpha \) then \( \exists M_{\xi''}, \xi'' < \xi' \), such that \( M_{\xi''} \in D \cap H_\xi \). In either case \( P \) meets \( D \) and so \( x \in H_\xi(D) \).

If \( D \) is not minimal then there exists \( M_\xi, \xi < \Omega \), such that \( H_\xi(D - M_\xi) = C \).

But, projecting \( C \) and \( D - M_\xi \) onto the orthogonal complement of \( F \) we see that by construction \( \text{proj} C \cap \text{proj} F_\xi \neq \emptyset \), but \( \text{proj} (D - M_\xi) \cap \text{proj} F_\xi = \emptyset \). Hence \( D \) is a \( j^\text{th} \) minimal hull for \( C \). Finally, suppose that \( B \) is an uncountable analytic subset of \( D \). If \( B \) has positive \( j \)-dimensional measure then it is possible to find an uncountable analytic subset of \( B \) of zero \( j \)-dimensional measure. Hence it can be supposed that \( B \) has zero \( j \)-dimensional measure and so \( B = A_\xi \) for some \( \xi < \Omega \). But \( A_\xi = A_\xi \cap D \subset \bigcup_{\xi' < \xi} M_{\xi'} \), which is countable; contradiction.

Of course, if \( G \) is an open or compact set in \( R^n \) then \( H_j(G) \) will accordingly be an open or compact set. Apart from these cases it does not seem entirely trivial to determine the nature of \( H_j(G) \) for a given subset \( G \) of \( R^n \). Here we prove the following

**Theorem 8.** (i) There exists, in the plane \( R^2 \), a borel set \( C \) such that \( H_j(C) \) is analytic but not borel.
(ii) If $D$ is an $F_\sigma$-subset of $\mathbb{R}^n$ then $H_j(D)$ is the complement of an analytic set.

REMARKS. We note that by (i) if $C$ is analytic then $H^C_\sigma(D)$ is not necessarily the complement of an analytic set. To disprove the statement that whenever $A$ is analytic then $H_j(A)$ is analytic, it would be enough, using (ii), to find an $F_\sigma$-subset $D$ of $\mathbb{R}^n$ such that $H_j(D)$ is not borel. (Notice that, a subset, $M$ of $\mathbb{R}^n$ is borel if and only if $M$ and $\mathbb{R}^n - M$ are both analytic. Compare, for example, [5]).

Proof. (i) As already observed, every analytic set in $\mathbb{R}^1$ can be represented as the projection into $\mathbb{R}^1$ of some $F_\sigma\delta$ set in $\mathbb{R}^n$. Let $A$ be an analytic subset of $\mathbb{R}^n$ such that $A$ is not a borel set and let $B$ be an $F_\sigma\delta$ set in $\mathbb{R}^2$ such that $\text{proj} B = A$. Take $C$ to be the union of $B$ and the "$y$-axis" ($\mathbb{R}^1$). Then it is easily seen that $H_j(C)$ is the union of all lines which are parallel to ($\mathbb{R}^1$) and contain a point of $C$. However this is not a borel set as $H_j(C) = \mathbb{R}^1 \cup \{(0, 0)\}$ is not a borel set.

(ii) We define a complete separable metric space $\Omega$ whose points are the $(n-j)$-flats of $\mathbb{R}^n$, as follows. For each $(n-j)$-flat $F$ in $\mathbb{R}^n$ let $y$ be the nearest point of $F$ to 0 and set $F' = (S^{n-1} + y) = \hat{F}$. Then the distance $\rho(F, F')$ of two $(n-j)$-flats in $\Omega$ is defined as the Hausdorff distance of $\hat{F}, \hat{F}'$ in $\mathbb{R}^n$. Let $D \subset \mathbb{R}^n$ be an $F_\sigma$ set, say $D = \bigcup_{i=1}^\infty D_i$ with $D_i \subset D_{i+1}$, each $D_i$ compact, $i = 1, 2, \cdots$. Let $D_i^*$, $i = 1, 2, \cdots$ denote the closed subsets of $\Omega$ such that $F \in D_i^*$ if $F$ meets $D_i$ in $\mathbb{R}^n$. Similarly defined, relative to $D$, is $D^*$. Then $D^* = \bigcup_{i=1}^\infty D_i^*$ and so $D^*$ is an $F_\sigma$ subset of $\Omega$. Hence $\Omega - D^*$ is an analytic subset of $\Omega$. Set

$$\Omega - D^* = \bigcap_{i \in I, p \in \mathbb{Z}} A(i \mid p),$$

where the $A(i \mid p)$, $p = 1, 2, \cdots$, form a decreasing sequence of compact subsets of $\Omega$, for each $i \in I$. Set

$$B_m = \{x \mid x \in \mathbb{R}^n, -m \leq x_i \leq m, i = 1, \cdots, n\}.$$

Let $K_m(i \mid p)$ be the closed subset of $B_m$ such that $x \in K_m(i \mid p)$ if $x$ is contained in an $(n-j)$-flat $F$ with $F \in A(i \mid p)$. Similarly, we define $K_m \subset B_m$ relative to $\Omega - D^*$. Then $K_m = \bigcap_{i \in I} \bigcap_{p=1}^\infty K_m(i \mid p)$ is an analytic subset of $\mathbb{R}^n$ and so, therefore, is $K = \bigcup_{m=1}^\infty K_m$. We claim that $H_j(D) = \mathbb{R}^n - K$. If $x \in K$ then $x \in K_m$ for some $m$ and so $x$ is contained in some $(n-j)$-flat $F$ which is contained (in $\Omega$) in some set $\bigcap_{p=1}^\infty A(i \mid p)$. Hence $F \in \Omega - D^*$ which means that $F$ does not meet $D$; i.e., $x \in H_j(D)$. Therefore $\mathbb{R}^n - K \supset H_j(D)$. Conversely if $x \in H_j(D)$ then there exists an $(n-j)$-flat $F$ through $x$ such that $F$ does not meet $D$. Hence $F \in \Omega -$
$D^*$ and so $F \in \bigcap_{p=1}^{\infty} A(i\mid p)$ for some $i \in I$. Hence $x \in \bigcap_{p=1}^{\infty} K_m(i\mid p)$ for some positive integer $m$, i.e., $x \in K$. Therefore $R^n - K \subseteq H_j(D)$ and so $H_j(D) = R^n - K$ is the complement of the analytic set $K$.

**Definition.** An irregular point $x$ of some closed convex set $C$ in $R^n$ is an extreme point $x$ of $C$ such that $x$ lies in two distinct 1-faces $l_1$, $l_2$ of $C$, with neither of $l_1$, $l_2$ being contained in a 2-face of $C$. Let $C$ be a closed subset of a simple closed curve in the plane $OXY$. We say that a set $B \subset C \times (-\infty, \infty)$ is vertically convex if every line which is perpendicular to $OXY$ meets $B$ in a (possibly empty) line segment. We shall make use of the following immediate corollary to a theorem of K. Kunugui [7].

**Lemma 3.** (Kunugui) Let $B$ be a vertically convex borel set in $C \times (-\infty, \infty)$. Then the projection of $B$ into $C$ is a borel set.

As an immediate consequence of Lemma 3, we have

**Lemma 4.** Let $B$ be a vertically convex borel subset of some vertically convex closed subset $D$ in $C \times (-\infty, \infty)$. Then the set $D \cap \{ (\text{proj. } B) \times (-\infty, \infty) \}$ is a vertically convex borel set.

In [9] the authors have derived properties of visual hulls for the class of convex sets. Our contribution in this direction is

**Theorem 9.** (i) If $C$ is a convex borel (analytic) set in $R^3$ then $H_2(C)$ is a borel (analytic) set.

(ii) If $C$ is a convex borel (analytic) set in $R^3$ and $C$ does not have irregular points then $H_2(C)$ is a borel (analytic) set.

**Proof.** (i) We first show that if $C$ is a convex borel (analytic) set in $R^3$ then $H_2(C)$ is a borel (analytic) set. If $\dim C = 1$ then the result is trivial and so it can be supposed that $\dim C = 2$. Note that $C^v \subset H_1(C) \subset \bar{C}$. Let the 1-faces of $\bar{C}$ be $\{ F_i \}_{i=1}^{\infty}$. Then

$$H_1(C) \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) = C - \bigcup_{i=1}^{\infty} F_i,$$

which is a borel set. Let $\{ F_{i,v} \}_{v=1}^{\infty}$ be the 1-faces of $\bar{C}$ which meet $C$. Then relint $F_{i,v} \subset H_1(C) \cap F_{i,v}$, $v = 1, 2, \ldots$. The two endpoints of $F_{i,v}$ may, or may not, be in $H_1(C)$. Nevertheless, $H_1(C)$ differs from the borel set $(C - \bigcup_{i=1}^{\infty} F_i) \cup \bigcup_{i=1}^{\infty}$ relint $F_{i,v}$ by at most a countable number of points. And so $H_1(C)$ is a borel set. Similarly, if $C$ is a convex analytic set in $R^3$, then $H_1(C)$ is an analytic set. Suppose now that $C$ is a convex borel set in $R^3$. If $\dim C \leq 2$ then $H_2(C) = C$, and so
it can be supposed that \( \dim C = 3 \). Let \( \{F_i\}_{i=1}^{\infty} \) be an enumeration of the 2-faces of \( C \). Then each \( F_i \) is closed and \( H_2(C) \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) = C \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) \), which is a borel set. As \( H_2(C) \subset \bar{C} \), it is now enough to show that \( H_2(C) \cap F_i \) is a borel set for \( i = 1, 2, \ldots \). Let \( H'_i(C \cap F_i) \) denote the first visual hull of \( C \cap F_i \) relative to aff \( F_i \). Then, from above, \( H'_i(C \cap F_i) \) is a borel set. Let \( \{F_{ij}\}_{j=1}^{\infty} \) be an enumeration of the 1-faces of \( F_i \). Then \( H_2(C) \cap \bigcup_{j=1}^{\infty} F_{ij} = H'_i(C \cap F_i) - \bigcup_{j=1}^{\infty} F_{ij} \) which is a borel set. As \( H_2(C) \subset C \), it is now enough to show that \( H_2(C) \cap \bigcup_{j=1}^{\infty} F_{ij} \) is a borel set for \( i = 1, 2, \ldots \). Let \( \{F_{ijv}\}_{j,v=1}^{\infty} \) be an enumeration of the 1-faces of \( F_{ij} \) which meet \( C \) and have the property that the only plane of support to \( \bar{C} \) which contains \( F_{ijv} \) is aff \( F_{ij} \). Then relint \( F_{ijv} \subset H_2(C) \) and the end points of \( F_{ijv} \) may or may not be in \( H_2(C) \). Hence \( H_2(C) \cap F_i \) differs from the borel set \( K_i \cup \bigcup_{j=1}^{\infty} \text{relint } F_{ijv} \cup \bigcup_{j=1}^{\infty} (F_{ijv} \cap C) \) by at most a countable number of points. Therefore \( H_2(C) \cap F_i \) is a borel set, and so, therefore, is \( H_2(C) \). Similarly, it can be shown that if \( C \) is a convex analytic set in \( R^3 \) then \( H_2(C) \) is an analytic set.

(ii) Again we shall prove the result for convex borel sets, and indicate at the end the modifications required for convex analytic sets. Let \( \{r_i\}_{i=1}^{\infty} \) be an enumeration of the rational numbers and let \( P_{ik} \) denote the 2-flat \( \{x \mid x_k = r_i\} \) \( k = 1, 2, 3; i = 1, 2, \ldots \). For each \( i, j, k \), let \( B(i, j, k) \) denote the closed set formed by the point set union of all maximal line segments in \( \bar{C} - C^0 \) which meet both both \( P_{ik} \) and \( P_{jk} \). Let \( \{G_{m}\}_{m=1}^{\infty} \) be the 2-faces of \( \bar{C} \). If a 2-face \( G_m \) of \( \bar{C} \) meets \( B(i, j, k) \) then \( G_m \) meets \( C_i(C_i = (\bar{C} - C^0) \cap P_{ik}) \) and \( C_j(C_j = (\bar{C} - C^0) \cap P_{jk}) \) in line segments \( 1_{im} \) and \( 1_{jm} \) respectively. Let \( 1_{im}, 1_{jm} \) denote the (at most) two maximal line segments in \( G_m \) such that each segment contains an endpoint of \( 1_{im} \) and \( 1_{jm} \) but \( 1_{im} \) and \( 1_{jm} \) do not intersect except possibly at end points. Set \( C^* = (\bar{C} - C^0) \cap P \), where \( P \) is a plane parallel to \( P_{ik} \) and lying strictly between \( P_{ik} \) and \( P_{jk} \). Then \( G_m \) cuts \( C^* \) in an interval \( I_m \). Let \( 1_m \) denote the subinterval of \( I_m \) with endpoints \( 1_m \cap C^*, 1_m \cap C^* \), and let \( 1^0_m \) be the relative interior of \( 1_m \). Then

\[
C' = B(i, j, k) \cap (C^* - \bigcup_{m=1}^{\infty} 1_m^0)
\]

is a closed subset of \( C^* \). If \( x \in C' \), let \( \hat{x} \) denote the unique maximal line segment in \( B(i, j, k) \) which passes through \( x \) and meets \( C_i \) and \( C_j \). Let \( X \) denote the closed set formed by the point set union of the line segments \( \hat{x}, x \in C' \), and set \( Q(i, j, k) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C \neq \emptyset \}, y \in \hat{x} \). We now show that \( Q(i, j, k) \) is a borel set. Every point \( y \) of \( X \) can be given a coordinate vector \( y = \langle x, h \rangle \), where \( y \in \hat{x} \) and \( h \) is the height, relative to the \( j^{th} \) coordinate, of \( y \) above \( C^* \). Because \( \bar{C} \) does not have irregular points, the number of points \( y \) in \( X \) which receive two different coordinate vectors is countable. Let \( \Phi \) be the mapping \( X \rightarrow C^* \times ( - \infty, \infty ) \) defined by taking \( \Phi \langle x, h \rangle = (x, h), x \in C' \). Then \( K \) is a borel subset of \( X \) if and only if \( \Phi(K) \) is a borel subset of the
closed set $\Phi(X)$. Hence $\Phi(C \cap X)$ is a vertically convex borel subset of $C' \times (-\infty, \infty)$. Hence the set $D = X \cap \{\text{proj } \Phi(C \cap X) \times (-\infty, \infty)\}$ is a convex borel set and so $Q(i, j, k) = \Phi^{-1}(D)$ is a borel set. Hence the set $R(i, j, k) = Q(i, j, k) \setminus \bigcup_{m=1}^{\infty} G_m$ is a borel set. Consider now the set $S = \bigcup_{i,j,k} R(i, j, k)$ and consider the borel set $T$ defined as the point set union of all 1-faces of $C$ which are not contained in some 2-face of $\bar{C}$. We assert that the set $H_1(C) = H_1(C) \cap (T \setminus \bigcup_{m=1}^{\infty} G_m)$ equals $S$. For if $y \in H_1(C)$ then, because $C$ does not have any irregular points, there exists a unique 1-face $I$, not contained in $\bigcup_{m=1}^{\infty} G_m$, such that $y \in I$. Then $y \in H_1(C)$ and only if $I \cap C = \emptyset$, which happens if and only if $I \subseteq Q(i, j, k)$ or in other words $y \in R(i, j, k)$ for some $i, j, k$. Hence $H_1(C) = S$. Let $V$ denote the borel set of exposed points of $C$ and $H_1^v(C) = V \cap H_1(C)$, $H_1(C) = \bigcup_{m=1}^{\infty} (H_1(C) \cap (G_m - V))$. Now $H_1(C) = H_1^v(C) \cup H_1^c(C) \cup H_1^i(C)$. $H_1^v(C) = S$ is a borel set and, since $H_1^c(C) = V \cap C$, $H_1^c(C)$ is a borel set. Hence it is enough to show that $H_1^i(C) \cap (G_m - V)$ is a borel set for all $m$. Now let $\{G_{m,v}\}_{v=1}^{\infty}$ be the 1-faces of $G_m$. Then either $\text{relint } G_{m,v} \subset H_1^i(C)$ or $\text{relint } G_{m,v} \cap H_1^i(C) = \emptyset$. Then the endpoints of $G_{m,v}$ may or may not be in $H_1^i(C)$. Let $H_m$ be the countable set of those endpoints of $\{G_{m,v}\}_{v=1}^{\infty}$ which lie in $H_1^i(C)$ and let $\{G_{m,v,\mu}\}_{\mu=1}^{\infty}$ be the 1-faces of $G_{m,v}$ whose relative interiors are contained in $H_1^i(C)$. We have $G_{m,v} \cap H_1^i(C) = \text{relint } G_{m,v} \cup (\bigcup_{\mu=1}^{\infty} \text{relint } G_{m,v,\mu}) \cup H_m$, which is a borel set. If, on the other hand, a 2-face of $C$ does not meet $C$, its intersection with $H_1^i(C)$ is empty. Therefore $H_1^i(C) \cap G_m$ is a borel set for all $m$, and $H_1(C)$ is a borel set.

For the case when $C$ is an analytic set, say $C = \sum_{i \in I} \bigcap_{n=1}^{\infty} C(i \mid n)$ in the usual representation, the only modification required to the above proof is to show that the set $Q(i, j, k)$ is an analytic set. With the previous notation, $Q(i \mid n) = \{y \mid y \in X, \exists x \in C', \exists \cap C(i \mid n) \neq \emptyset, y \in \hat{x}\}$. Then $Q(i \mid n)$ is a closed set and $Q(i, j, k) = \sum_{i \in I} \bigcap_{n=1}^{\infty} Q(i \mid n)$. Therefore $Q(i, j, k)$ is an analytic set.

References

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