THE ADJOINT GROUP OF LIE GROUPS

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Let $G$ be a Lie group and let $\text{Aut}(G)$ denote the group of automorphisms of $G$. If the subgroup $\text{Int}(G)$ of inner-automorphisms of $G$ is closed in $\text{Aut}(G)$, then we call $G$ a (CA) group (after Van Est.). In this note, we investigate (CA) property of certain classes of Lie groups. The main results are as follows:

**Theorem A.** Let $G$ be an analytic group and suppose that there is no compact semisimple normal subgroup of $G$. If $G$ contains a closed uniform (CA) subgroup $H$, then $G$ is (CA).

**Theorem B.** If $G$ is an analytic group whose exponential map is surjective, then $G$ is (CA).

In [3], Garland and Goto proved that if an analytic group $G$ contains a lattice, then $G$ is (CA). Since a lattice in a solvable group is a uniform lattice, it is finitely generated and so the automorphism group of this uniform lattice is discrete, and thus this lattice is trivially a (CA) subgroup. Thus Theorem A generalizes the above theorem of Garland and Goto for solvable groups. Theorem B is an improvement of the well known theorem that every nilpotent analytic group is (CA) (see [2]). In §1, we introduce some notation and preliminary materials. §2 and §3 are devoted for the proofs of the main theorems together with their immediate corollaries.

1. Preliminaries and notations. The group $\text{Aut}(G)$ of automorphisms of locally compact a topological group $G$ may be regarded as a topological group, the topology being the (generalized) compact open topology defined as in [5]. Thus, if we denote by $N(C, V)$ the set of all $\theta \in \text{Aut}(G)$ for which $\theta(x)x^{-1} \in V$ and $\theta^{-1}(x)x^{-1} \in V$ whenever $x \in C$, then the sets $N(C, V)$ form a fundamental system of neighborhoods of the identity element of $\text{Aut}(G)$ as $C$ ranges over the compact subsets of $G$ and $V$ over the set of neighborhoods of the identity element of $G$.

If $G$ is an analytic group and $\mathfrak{g}$ its Lie algebra, then $\text{Aut}(G)$ may be identified with a closed subgroup of the linear group $\text{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$. Under this identification, $\text{Int}(G)$ coincides with the adjoint group $\text{Int}(\mathfrak{g})$, which is generated by $e^{ad X}$, $X \in \mathfrak{g}$ where $ad$ denotes the adjoint representation of $\mathfrak{g}$. Thus the (CA) property of analytic groups are entirely determined by their Lie algebras. In particular, if $\tilde{G}$ is a covering group of $G$ and if $G$ is (CA), then so is $\tilde{G}$. This fact is used in the proofs of the main theorems.
Throughout this paper the following notation is used: If \( A \) is a subgroup of \( G \), then \( \text{Int}_G(A) \) denotes the subgroup of \( \text{Int}(G) \) which consists of inner automorphisms induced by elements of \( A \). Thus \( \text{Int}_G(G) \) is merely equal to \( \text{Int}(G) \). The center of \( G \) is denoted with \( Z(G) \). Also if \( x \in G \), then \( I_x \) means the inner automorphism induced by \( x \).

2. Proof of Theorem A. Let \( H \) be a closed uniform subgroup of an analytic group \( G \), and \( H_0 \) its identity component. Then \( H/H_0 \) is finitely generated. In order to see this, let \( \tilde{G} \) be the simply connected covering group of \( G \), \( \tilde{H} \) the complete inverse image of \( H \) under the covering projection and \( \tilde{H}_0 \) the identity component of \( \tilde{H} \). Then since the covering projection induces an epimorphism \( \tilde{H}/\tilde{H}_0 \to H/H_0 \), it suffices to show that \( \tilde{H}/\tilde{H}_0 \) is finitely generated. Nothing that \( \tilde{G}/\tilde{H}_0 \) is simply connected (see, for example, Mostow [7], Corollary 1, p. 617), we can identify the discrete group \( \tilde{H}/\tilde{H}_0 \) with the fundamental group of the compact manifold \( \tilde{G}/\tilde{H} \). As the fundamental group of a compact manifold is finitely presented, it follows that \( \tilde{H}/\tilde{H}_0 \) is, in particular, finitely generated.

Now we can apply a theorem of Hochschild ([5], Th. 2, p. 212) to see that if \( H \) is a closed subgroup of an analytic group, then \( \text{Aut}(H) \) is a Lie group.

The following lemma enables us to assume that \( G \) is simply connected.

**Lemma.** Let \( \tilde{H} \) be a compactly generated Lie group and \( A \) a closed discrete central subgroup of \( \tilde{H} \). Let \( H = \tilde{H}/A \). If \( H \) is a (CA) group, then so is \( \tilde{H} \). In fact, \( \text{Int}(\tilde{H}) \) is a topological extension of a discrete group by \( \text{Int}(H) \).

**Proof.** Let \( \pi: \tilde{H} \to \tilde{H}/A = H \) be the natural map and define \( \chi: \text{Int}(\tilde{H}) \to \text{Int}(H) \) by \( \chi(\tilde{I}_b) = I_{\pi(\tilde{h})} \), for \( \tilde{h} \in \tilde{H} \).

(i) \( \chi \) is continuous. To see this, note first that we can find a compact neighborhood \( \tilde{D} \) of 1 in \( \tilde{H} \) which generates \( \tilde{H} \). Now let \( C \) be a compact subset of \( H \) and \( U \) a neighborhood of 1 in \( H \). Then we have to find a compact subset \( \tilde{C} \) of \( \tilde{H} \) and a neighborhood \( \tilde{U} \) of 1 in \( \tilde{H} \) so that \( \chi(N(\tilde{C}, \tilde{U}) \cap \text{Int}(\tilde{H})) \subseteq N(C, U) \cap \text{Int}(H) \). Since \( \pi(\tilde{D}) = D \) is also a compact neighborhood of 1 which generates \( H \), we can find a positive integer \( k \) such that \( C \subseteq \tilde{D}^k \) by using the compactness of \( C \). Now letting \( \tilde{C} = \tilde{D}^k \) and \( \tilde{U} = \pi^{-1}(U) \), it is easy to see that \( (\tilde{C}, \tilde{U}) \) is a desired pair. Hence \( \chi \) is continuous.

(ii) \( \chi \) is open. In fact, since \( H \) is (CA), the canonical map \( H/Z(H) \to \text{Int}(H) \) is an isomorphism of topological groups. Hence (ii) follows from the following commutative diagram
where the left vertical map is always continuous and the top one is open.

(iii) The kernel $\mathcal{R}$ of $\chi$ is a discrete subgroup of $\text{Aut}(\tilde{H})$ and hence is closed in $\text{Aut}(\tilde{H})$. To see this let $\mathcal{R}_0$ be the closure of $\mathcal{R}$ in $\text{Aut}(\tilde{H})$, and $\mathcal{R}_0^0$ the identity component of $\mathcal{R}$. Since $\text{Aut}(\tilde{H})$ is a Lie group $\mathcal{R}_0^0/\mathcal{R}_0$ is discrete.

Since $\mathcal{R} = \text{Ker}(\chi)$ and since $A$ is central in $\tilde{H}$ every element of $\mathcal{R}$ induces the identity map on $H = \tilde{H}/A$ and on $A$. Hence $\theta \in \mathcal{R}$ implies that $\theta = 1$ on $A$ and $\theta = 1$ on $H = \tilde{H}/A$, which implies that $\theta(\tilde{h})\tilde{h}^{-1} \in A$ for $\tilde{h} \in \tilde{H}$.

Let $\tilde{h} \in \tilde{H}$ be arbitrary and define $\eta_{\tilde{h}} : \mathcal{R} \to A$ by $\eta_{\tilde{h}}(\theta) = \theta(\tilde{h})\tilde{h}^{-1}$, $\theta \in \mathcal{R}$. Then $\eta_{\tilde{h}}$ is continuous and thus $\eta_{\tilde{h}}(\mathcal{R}_0)$ is connected in the discrete $A$. Since $\eta_{\tilde{h}}(\mathcal{R}_0)$ contains 1, $\eta_{\tilde{h}}(\mathcal{R}_0) = 1$ and this then implies that $\theta(\tilde{h}) = \tilde{h}$ for all $\theta \in \mathcal{R}_0$. Since $\tilde{h}$ is arbitrary, $\mathcal{R}_0 = 1$ and $\mathcal{R}$ is discrete. We have thus shown that $\mathcal{R}$ is a discrete subgroup of $\text{Aut}(H)$ and hence $\mathcal{R}$ is closed in $\text{Aut}(\tilde{H})$.

(iv) Since $(\text{Int}(\tilde{H})/\mathcal{R}) \cong \text{Int}(H)$, $\text{Int}(\tilde{H})$ is closed in $\text{Aut}(\tilde{H})$ as a locally compact subgroup of $\text{Aut}(\tilde{H})$ and the lemma is proved.

Now we are ready to present the proof of Theorem A. Let $\tilde{G}$ denote the simply connected covering group of $G$ and let $\pi$ be the covering homomorphism.

Then, by the lemma $\pi^{-1}(H) = \tilde{H}$ is also uniform and (CA). Hence no generality will be lost in assuming that $G$ is simply connected. By the assumption, $\text{Int}(H)$ is closed in $\text{Aut}(H)$. Thus the canonical map $H/Z(H) \to \text{Int}(H)$ is an isomorphism of topological groups. Define $\varphi : \text{Int}_G(H) \to \text{Int}(H)$ to be the restricting homomorphism and let $\mathcal{R}$ be the closure of the kernel of $\varphi$, the closure being taken in $\text{Aut}(G)$. Then $\text{Int}_G(H)\mathcal{R}$ is a subgroup of $\text{Aut}(G)$. We define $H/Z(H) \to \text{Int}_G(H)\mathcal{R}/\mathcal{R}$ and $\text{Int}_G(H)\mathcal{R}/\mathcal{R} \to \text{Int}(H)$ to be the homomorphisms induced by the canonical maps $H \to \text{Int}_G(H)$ and $\text{Int}_G(H) \to \text{Int}(H)$, respectively. Then the following diagram commutes:

$$
\begin{array}{ccc}
\text{Int}_G(H)\mathcal{R}/\mathcal{R} & \to & \text{Int}(H) \\
\| & \downarrow & \\
H/Z(H) & \longrightarrow & \text{Int}(H)
\end{array}
$$

and all three maps are continuous and algebraically isomorphisms. Since the bottom one is an isomorphism of topological groups, $\text{Int}(H)$
is topologically isomorphic with $\text{Int}_G(H) \cdot \mathcal{N}$ and thus the latter is a locally compact subspace of the quotient space $\text{Aut}(G)/\mathcal{N}$. Hence it is closed in $\text{Aut}(G)/\mathcal{N}$ and, accordingly, $\text{Int}_G(H) \cdot \mathcal{N}$ is closed in $\text{Aut}(G)$.

We next claim that $\mathcal{N} \subseteq \text{Int}(G)$. In fact, if $\theta \in \mathcal{N}$, define $P(g) = \theta(g)g^{-1}$, for $g \in G$. Then $P: G \to G$ is continuous, and $P(H) = \{1\}$ and $G/H$ compact imply that $P(G)$ is compact. Thus we see that $\theta$ is an automorphism of bounded displacement in the sense of Tits [8] and $\theta$ is therefore an inner automorphism induced by a central element of the nilradical of $G$ ([8], Lemma (6), p. 102). Thus $\mathcal{N} \subseteq \text{Int}(G)$.

By what we have shown, it is clear now that the closure $\overline{\text{Int}_G(H)}$ of $\text{Int}_G(H)$ is contained in $\text{Int}(G)$. Since $G/H$ is compact and since $G/H \to \text{Int}(G)/\overline{\text{Int}_G(H)}$ is continuous, $\text{Int}(G)$ is compact, modulo $\overline{\text{Int}_G(H)}$ and hence $\text{Int}(G)$ is closed, proving that $G$ is (CA).

**COROLLARY.** If a solvable analytic group $G$ contains a closed abelian uniform subgroup, then $G$ is a (CA) group.

**COROLLARY.** (See, Garland and Goto [3]). If a solvable analytic group $G$ contains a lattice, then $G$ is a (CA) group.

**REMARK.** In [6], we have shown that any extension of a simply connected (CA) group by a compact connected group is a (CA) group. Thus Theorem A generalizes this for the solvable case.

**REMARK.** We have failed to see whether or not the nonexistence of compact semi-simple normal subgroup in the theorem is necessary. This was needed in order to apply the result of Tits in the proof.

3. **Proof of Theorem B.** In order to prove Theorem B, we first note that an analytic group $G$ is (CA) if and only if its radical is (CA)(See Van Est [2]). Thus we may assume that the group in the theorem is solvable.

Let $\mathcal{G}$ be a finite-dimensional real solvable Lie algebra and let $G$ be an analytic group with its Lie algebra $\mathcal{G}$. If an exponential map $\exp: \mathcal{G} \to G$ is surjective, then the exponential map into its simply connected covering group is a bijection. Thus by the remark in §1, it suffices to prove:

**THEOREM B'.** Let $\mathcal{G}$ be a finite-dimensional real solvable Lie algebra. If the exponential map is a bijection, then $\mathcal{G}$ is a (CA) Lie algebra (that is, the adjoint group $\text{Int}(\mathcal{G})$ is closed in $\text{Aut}(\mathcal{G})$).

In order to prove this, we need the following lemma:
LEMMA. Let \( \mathcal{N} \) be the nilradical of \( \mathcal{G} \). If \( X \in \mathcal{N} \) then the one-parameter subgroup \( \{ e^{ad(tX)} : t \in \mathbb{R} \} \) is closed in \( \text{Aut}(\mathcal{G}) \).

Proof of lemma. Let \( T \) denote the given one-parameter subgroup. We show if \( T \) is not closed, then \( T \) is trivial. In fact, if \( T \) is not closed in \( \text{Aut}(\mathcal{G}) \), then the closure \( \overline{T} \) of \( T \) is compact. Define \( \varphi : \text{Aut}(\mathcal{G}) \to \text{Aut}(\mathcal{N}) \) to be the restricting homomorphism. Since \( \mathcal{N} \) is a characteristic ideal in \( \mathcal{G} \), \( \varphi \) is well defined and is continuous.

Now let \( ad_r \) denote the adjoint representation of the nilpotent Lie algebra \( \mathcal{N} \). Since \( \mathcal{N} \) is nilpotent, \( \text{Int}(\mathcal{N}) \) is closed in \( \text{Aut}(\mathcal{N}) \) ([2], Proposition 1.2.2, p. 322), and thus \( \overline{\varphi(T)} \subset \text{Int}(\mathcal{N}) \). By using the fact that the maximal compact subgroup of any nilpotent analytic group is contained in its center, it follows that \( \text{Int}(\mathcal{N}) \) is always simply connected. Hence the compact subgroup \( \overline{\varphi(T)} \) must be trivial, which means that \( ad_r X = 0 \) and so \( X \) is central in \( \mathcal{N} \).

Next we show that \( X \) is central in \( \mathcal{G} \). In order to see this, note first that \( [X, \mathcal{G}] \subseteq \mathcal{G}' \subseteq \mathcal{N} \), \( \mathcal{G}' \) being the commutator subalgebra of \( \mathcal{G} \). Thus \( X \) being a central element of \( \mathcal{N} \) implies that \( ad(X)^2 = 0 \). Therefore \( e^{ad(tX)} = 1 + ad(tX) \) for \( t \in \mathbb{R} \). Let \( Y \in \mathcal{G} \) be arbitrary. Thus we have

\[
\exp (R[X, Y]) = \exp (ad(RX)(Y)) = \exp (e^{ad(RX)} - 1)(Y)
= \exp ((T - 1)(Y)).
\]

Since \( \overline{T} \) is compact, the closure of \( T - 1 \) is compact in the matrix topology of \( \text{End}(\mathcal{G}) \), the ring of endomorphisms of the vector space \( \mathcal{G} \). Therefore, the continuity of \( \exp \) implies that \( \exp ((T - 1)(Y)) \) is bounded in \( G \). Consequently, the one-parameter subgroup \( \exp (R[X, Y]) \) is relatively compact. But \( G \) is simply connected and thus this one-parameter subgroup must be trivial, which implies that \( ad(X) = 0 \) and we have proved that \( X \) is central in \( \mathcal{G} \). Therefore \( T = 1 \) as desired.

Proof of Theorem B'. By a theorem of Goto ([4], Theorem III, p. 165), it suffices to show that every one-parameter subgroup of \( \text{Int}(\mathcal{G}) \) is closed in \( \text{Aut}(\mathcal{G}) \). Noting that every one-parameter subgroup of \( \text{Int}(\mathcal{G}) \) is of the form \( e^{ad(RX)} \) for some \( X \in \mathcal{G} \), assume that there is a nonzero \( X \) such that \( T = e^{ad(RX)} \) is not closed in \( \text{Aut}(\mathcal{G}) \). We see from the lemma that \( X \) is not in \( \mathcal{N} \).

Next we select a decreasing sequence of ideals of \( \mathcal{G} \):

\[
\mathcal{G}_0 = \mathcal{G} > \mathcal{G}_1 > \mathcal{G}_2 > \cdots > \mathcal{G}_{n+1} = (0)
\]
such that \( \dim_{\mathbb{C}}((\mathcal{G}_i : \mathcal{G}_{i+1})) \leq 2 \). Let \( A_i \) denote the endomorphism on \( \mathcal{G}_i / \mathcal{G}_{i+1} \) which is induced by \( ad(X) \), \( i = 0, 1, \cdots, n \). Then there exists
such that $A_p \neq 0$. For, if $A_i = 0$ for all $i$, then $ad(X)$ would be a nilpotent transformation and hence $X \in \mathcal{N}$, which is impossible. Since $T$ is relatively compact in $\text{Aut}(\mathcal{G})$, so is $S = e^{R_A} \in \text{Aut}(\mathcal{G}/\mathcal{G}_{p+1})$. Since $A_p$ is nonzero, $S$ is nontrivial and thus $\dim_S(\mathcal{G}/\mathcal{G}_{p+1}) = 2$. Since a maximal compact subgroup of $\text{Aut}(\mathcal{G}/\mathcal{G}_{p+1})$ is a circle group, it follows that $S$ is a circle group in $\text{Aut}(\mathcal{G}/\mathcal{G}_{p+1})$. Now let $\pi: \mathcal{G} \to \mathcal{G}/\mathcal{G}_{p+1}$ be the natural homomorphism and let $\mathcal{H}$ be the sub-algebra of $\mathcal{G}/\mathcal{G}_{p+1}$ which is generated by $\pi(X)$ and $\mathcal{G}/\mathcal{G}_{p+1}$. Then from what we have seen above, it is easy to see that $\mathcal{H}$ is the Lie algebra of the group of the rigid motions on the plane. Thus $\text{exp}$ is not a bijection by the well known theorem of Dixmier ([1], Th. 3, p. 120). Hence every one-parameter subgroup of $\text{Int}(\mathcal{G})$ is closed in $\text{Aut}(\mathcal{G})$, which proves the Theorem B'.

In the proof of Theorem B', we have actually shown that $\text{Int}(\mathcal{G})$ contains no compact subgroups. Hence we have:

**COROLLARY.** Let $G$ be a solvable analytic group such that the exponential map is surjective. Then $\text{Int}(G)$ is simply connected.

**COROLLARY.** Let $G$ be as above. Then $Z(G)$ is connected.

**Proof.** By Theorem B, $G/Z(G) = \text{Int}(G)$ is an isomorphism of topological groups. Since $\text{Int}(G)$ is simply connected, it follows that $Z(G)$ is connected.

**REMARK.** The converse of the Theorem B is false. The group of rigid motions on the plane is perhaps the simplest example.

**Bibliography**


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