COMMUTATIVITY IN LOCALLY COMPACT RINGS

JAMES B. LUCKE
A structure theorem is given for all locally compact rings such that $x$ belongs to the closure of $\{x^n : n \geq 2\}$, in particular, all such rings are commutative, a result which extends a well-known theorem of Jacobson. Similarly we show the commutativity of semisimple locally compact rings satisfying topological analogues of properties studied by Herstein.

Jacobson has shown that a ring is commutative if for every $x$ there is some $n(x) \geq 2$ such that $a^{n(x)} = x$ [5, Th. 1, p. 212]. Herstein has generalized this result, and certain of his and other generalizations are of interest here. A ring is commutative if (and only if) for all $x$ and $y$ there is some $n(x, y) \geq 2$ such that $(a^{n(x,y)} - x)y = y(a^{n(x,y)} - x)$ [4, Th. 2]; a ring is commutative if (and only if) for all $x$ and $y$ there is some $n(x, y) \geq 2$ such that $xy - yx = (xy - yx)^{n(x,y)}$ [3, Th. 6]; a semisimple ring is commutative if (and only if) for all $x$ and $y$ there is some $n(x, y) \geq 1$ such that $x^{n(x,y)}y = yx^{n(x,y)}$ [4, Th. 1] or if for all $x$ and $y$ there are $n, m \geq 1$ such that $x^ny = yx^n$ [1, Lemma 1]. The investigation of analogous conditions for topological rings is the major concern of this paper.

1. A topological analogue of Jacobson's condition. If $a^n = x$ for some $n \geq 2$, then an inductive argument shows that $a^{k(n-1)+1} = x$ for all $k \geq 1$. A possible topological analogue of Jacobson's condition would thus be that for every $x$ there is some $n(x) \geq 2$ such that $\lim_k x^{k(n(x)-1)+1} = x$. But this implies that $a^{n(x)} = x$, since

$$x^{n(x)} = a^{n(x)-1}x = a^{n(x)-1} \lim_k x^{k(n(x)-1)+1} = \lim_k a^{(k+1)(n(x)-1)+1} = x.$$ 

Thus all topological rings having this property have Jacobson's property and hence are commutative.

A less trivial analogue of Jacobson's condition is that for every $x$ in the topological ring $A$, $x$ belongs to the closure of $\{x^n : n \geq 2\}$. In our investigation of these rings, rings with no nonzero topological nilpotents play an important role. Recall that an element $x$ of a topological ring is a topological nilpotent if $\lim_n x^n = 0$. We shall prove that a locally compact ring has no nonzero topological nilpotents if and only if it is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring $B$ that is the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields. From this it is easy to derive a structure theorem for locally compact rings
Lemma 1. If $A$ is a locally compact ring with no nonzero topological nilpotents, then $A$ is totally disconnected.

Proof. The connected component $C$ of zero in $A$ is a closed ideal of $A$ and so is itself a connected locally compact ring with no nonzero topological nilpotents. By hypothesis, $C$ is not annihilated by any of its nonzero elements, for if $xC = (0)$, then $x^2 = 0$, so $x = 0$. Thus $C$ is a finite-dimensional algebra over the real numbers (cf. [6, Th. III]). As the radical of a finite-dimensional algebra is nilpotent, $C$ is a semisimple algebra. If $C \neq (0)$, then by Wedderburn's Theorem, $C$ has an identity $e$, and clearly $(1/2)e$ would then be a nonzero topological nilpotent contrary to our hypothesis. Thus $C = (0)$, and so $A$ is totally disconnected.

Lemma 2. A compact ring $A$ has no nonzero topological nilpotents if and only if $A$ is the Cartesian product of finite fields.

Proof. Necessity: By Lemma 1, $A$ is totally disconnected. Thus the radical $J(A)$ of $A$ is topologically nilpotent [11, Th. 14], and hence is the zero ideal. Thus $A$ is a compact semisimple ring, and so $A$ is topologically isomorphic to the Cartesian product of a family of finite simple rings [11, Th. 16]. A finite simple ring is a matrix ring over a finite field, and unless the matrix ring is just the finite field itself, it will have nonzero nilpotent elements. Thus as $A$ has no nonzero nilpotents, $A$ is topologically isomorphic to the Cartesian product of a family of finite fields. Sufficiency: Clearly zero is the only topological nilpotent in the Cartesian product of a family of finite fields.

Lemma 3. If $A$ is a ring with no nonzero nilpotents, then every idempotent is in the center of $A$.

Proof. If $e$ is an idempotent and if $a \in A$, an easy calculation shows that $(ae - eae)^2 = 0$, hence $ae - eae = 0$. Similarly, $ea = eae$ and thus $ae = ea$.

We recall that the local direct sum of a family $(A_{\gamma})_{\gamma \in \Gamma}$ of topological rings with respect to open subrings $(B_{\gamma})_{\gamma \in \Gamma}$ is the subring of the Cartesian product $\prod_{\gamma} A_{\gamma}$ consisting of all $(a_{\gamma})$ such that $a_{\gamma} \in B_{\gamma}$ for all but finitely many $\gamma$, topologized by declaring all neighborhoods of zero in the topological ring $\prod_{\gamma} B_{\gamma}$ to be a fundamental system of neighborhoods of zero in the local direct sum. It is easy to see that the local direct sum equipped with this topology is indeed a topological ring.
Theorem 1. A locally compact ring $A$ has no nonzero topological nilpotents if and only if $A$ is the topological direct sum of a discrete ring having no nonzero nilpotents and a ring $B$ (possibly the zero ring) that is topologically isomorphic to the local direct sum of a family of discrete rings having no nonzero nilpotents with respect to finite subfields.

Proof. Necessity: As $A$ is totally disconnected by Lemma 1, $A$ contains a compact open subring $F$ [7, Lemma 4]. By Lemma 2, $F$ is topologically isomorphic to the product of finite fields. Consequently there exists in $F$ a summable orthogonal family $(e_\gamma)_{\gamma \in \Gamma}$ of idempotents such that $Fe_\gamma$ is a finite field and $\sum_{\gamma \in \Gamma} e_\gamma = e$, the identity of $F$.

By Lemma 3, $e$ is in the center of $A$, so $Ae$ and $A(1-e) = \{a - ae : a \in A\}$ are ideals. The continuous mappings $a \mapsto ae$ and $a \mapsto (a - ae)$ are the projections from $A$ onto $Ae$ and $A(1-e)$. Thus $A$ is the topological direct sum of $Ae$ and $A(1-e)$. As $e$ is the identity of $F$, $Fe_\gamma = F$ for all but finitely many $\gamma \in \Gamma$. Thus as $F$ is open, $A(1-e)$ is discrete and hence has no nonzero nilpotents.

As $F$ is open and as $Ae \cap F = Fe_\gamma$, a finite field, $Ae_\gamma$ is discrete and is an ideal as $e_\gamma$ is in the center of $A$. Consequently $Ae_\gamma$ has no nonzero nilpotents. It will therefore suffice to show that $B = Ae_\gamma$ is topologically isomorphic to the local direct sum of the discrete rings $Ae_\gamma$, with respect to the finite subfields $Fe_\gamma$.

Let $B'$ be the local direct sum of the $Ae_\gamma$'s with respect to the $Fe_\gamma$'s. Let $K : b \mapsto (be_\gamma) \in \prod_{\gamma} Ae_\gamma$. Clearly $b \mapsto be_\gamma$ is a continuous homomorphism for each $\gamma$, hence $K$ is a continuous homomorphism from $B$ into $\prod_{\gamma} Ae_\gamma$. If $b \in B$, then $(be_\gamma)$ is summable and $\sum_\gamma be_\gamma = b(\sum_\gamma e_\gamma) = be = b$. Therefore as $F$ is open in $B$, $be_\gamma \in F \cap Ae_\gamma = Fe_\gamma$ for all but finitely many $\gamma \in \Gamma$. Thus $K(B) \subseteq B'$.

The mapping $K$ is an isomorphism onto $K(B)$, since if $x \in B$ and if $xe_\gamma = 0$ for all $\gamma \in \Gamma$, then $x = xe = x(\sum_\gamma e_\gamma) = \sum_\gamma xe_\gamma = 0$. Let $y_\beta \in Fe_\beta$, and let $x_\gamma = 0$ for all $\gamma \neq \beta$, $x_\beta = y_\beta$; then $(x_\gamma) = K(y_\beta) \in K(F)$ since $(e_\gamma)\gamma$ is an orthogonal family. Thus $K(F)$ contains a dense subring of $\prod_\gamma Fe_\gamma$, and hence $K(F) = \prod_\gamma Fe_\gamma$ as $K(F)$ is compact. As the restriction of $K$ to $F$ is thus a continuous isomorphism from compact $F$ onto $\prod_\gamma Fe_\gamma$, $F$ is topologically isomorphic to $\prod_\gamma Fe_\gamma$ under $K$.

Thus it suffices to show that $K(B) \cong B'$, for $K$ is then, by the definition of the local direct sum, a topological isomorphism from $B$ onto $B'$. If $(b, e_\gamma) \in B'$, then $b, e_\gamma \in Fe_\gamma$ for all but finitely many $\gamma$, say $\gamma_1, \ldots, \gamma_n$. Call this set $\Gamma'$ and let $\Gamma' - \Gamma' = \Gamma''$. Thus $\sum_{\gamma \in \Gamma'} b_\gamma e_\gamma \in B$ and $b, e_\gamma \in F$ for all $\gamma \in \Gamma''$. Hence as $F$ is topologically isomorphic to $\prod_\gamma Fe_\gamma$, $b' = \sum_{\gamma \in \Gamma''} b_\gamma e_\gamma \in B$. Thus $b = b' + \sum_{\gamma \in \Gamma'} b_\gamma e_\gamma$, and $b_\gamma = b_\gamma e_\gamma$, so $K(b) = (b, e_\gamma)$. The sufficiency is clear.
We will call a ring \( A \) a Jacobson ring if given any \( x \in A \) there is an \( n(x) \geq 2 \) such that \( x^{n(x)} = x \). All Jacobson rings are commutative \([5, \text{Th. 1, p. 212}]\), and in extending this result to topological rings we give the following definition, noting that it reduces to Jacobson's condition in the discrete case.

Definition. A topological ring \( A \) is a \( J \)-ring if for each \( x \in A \), \( x \) belongs to the closure of \( \{x^n : n \geq 2\} \).

Lemma 4. If \( A \) is a \( J \)-ring, then \( A \) has no nonzero topological nilpotents.

Proof. If \( \lim_n x^n = 0 \), then since \( x \) belongs to the closure of \( \{x^n : n \geq 2\} \), we conclude that \( x = 0 \).

Theorem 2. A locally compact ring \( A \) is a \( J \)-ring if and only if \( A \) is the topological direct sum of a discrete Jacobson ring and a ring \( B \) which is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields.

Proof. Necessity: By Theorem 1 and Lemma 4, \( A \) is the topological direct sum of a discrete ring \( C \) and a ring \( B \) which is topologically isomorphic to the local direct sum of a family of discrete rings with respect to finite subfields. As each of these rings is an ideal of \( A \), each is a discrete \( J \)-ring and so is a Jacobson ring.

Sufficiency: Let \( B \) be the local direct sum of a family of discrete Jacobson rings \( B_\gamma, \gamma \in \Gamma \) with respect to finite subfields \( F_\gamma, \gamma \in \Gamma \). Let \( (x_\gamma) \in \mathcal{B} \) and let \( U \) be a neighborhood of zero in \( B \). Then we may assume that there is a finite subset \( \mathcal{A} \) of \( \Gamma \) such that \( x_\gamma \in F_\gamma \) for all \( \gamma \in \mathcal{A} \) and \( U = \bigcap_{\gamma \in \mathcal{A}} G_\gamma \), where \( G_\gamma = F_\gamma \) for all \( \gamma \in \mathcal{A} \). For each \( \gamma \in \mathcal{A} \), let \( n(\gamma) > 1 \) be such that \( x_\gamma^{n(\gamma)} = x_\gamma \). Let \( n = 1 + \prod_{\gamma \in \mathcal{A}} (n(\gamma) - 1) \). An inductive argument shows that \( x_\gamma^n = x_\gamma \) for all \( \gamma \in \mathcal{A} \). Hence \( (x_\gamma)^n - (x_\gamma) \in U \). Thus \( B \) is a \( J \)-ring, and consequently \( A \) is also a \( J \)-ring.

As all Jacobson rings are commutative we have the following analogue of Jacobson's Theorem:

Corollary. A locally compact \( J \)-ring is commutative.

Theorem 3. A locally compact ring \( A \) is a Jacobson ring if and only if there exists \( N \geq 2 \) such that \( A \) is the topological direct sum of a discrete Jacobson ring and a ring \( B \) that is topologically isomorphic to the local direct sum of a family of discrete Jacobson rings with respect to finite subfields of order \( \leq N \).
Proof. Necessity: Let \( |B_r| \) be the order of \( B_r \). By Theorem 2 it suffices to show that \( \sup |B_r| < +\infty \). If \( \sup |B_r| = +\infty \), then there exists \( (x_r) \in \prod_r B_r \) such that the orders of the \( x_r \)'s are unbounded. Consequently for no \( n \) does \( x_r^n = x_r \) for all \( r \), i.e., for no \( n \) does \( (x_r)^n = (x_r) \).

Sufficiency: Let \( (A_r)_{r \in \Gamma} \) be a family of discrete Jacobson rings with finite subfields \( B_r \) such that \( |B_r| \leq N \) for all \( r \). Let \( (x_r) \) be in the local direct sum of the \( A_r \)'s with respect to the \( B_r \)'s. There exists a finite subset \( A \) of \( \Gamma \) such that if \( r \in A \), \( x_r \in B_r \). Since each \( A_r \) is a Jacobson ring, for \( r \in A \) there is \( n(r) \) such that \( x_r^n = x_r \).

If \( x_r^n = x_r \), an inductive argument shows that \( x_r^{k(n(r)-1)+1} = x_r \) for all \( k \). If \( x_r \in B_r \), then \( |B_r| \leq N \), so since \( |B_r| - 1 < N \), \( x_r^{k(n(r)-1)+1} = x_r \) for all \( k \). Let \( n = 1 + [(N!) \prod_{r \in A} (n(r) - 1)] \). Then \( x_r^n = x_r \) for all \( r \), i.e., \( (x_r)^n = (x_r) \).

2. Analogues of four of Herstein's results. An analogue for topological rings of the first of Herstein's conditions that are mentioned above is that for all \( x \) and \( y \), \( xy - yx \) is in the closure of \( \{x^n y - y x^n : n \geq 2\} \), and we say such a topological ring is an \( H \)-ring. An analogue of the second of Herstein's conditions is that for all \( x \) and \( y \), \( xy - yx \) is in the closure of \( \{(xy - yx)^n : n \geq 2\} \), and we say such a topological ring is an \( H_2 \)-ring. (If \( (xy - yx)^m(x, y) = xy - yx \) for all \( m \geq 1 \); hence another topological analogue is the assumption that for each \( x, y \in A \), there exists \( n(x, y) \geq 2 \) that \( \lim_n (xy - yx)^{k(n(x, y)-1)+1} = xy - yx \); however by an argument similar to that of the first paragraph of § 1, this condition implies that \( (xy - yx)^m(x, y) = xy - yx \).) Similarly an analogue of the third of Herstein's conditions is that for all \( x, y \) in \( A \), \( \lim_n x^n y - y x^n = 0 \), and we say such topological rings are \( H_3 \)-rings, just as we will call \( H_i \)-rings those topological rings in which for all \( x, y \) there is an \( m(x, y) \geq 1 \) such that \( \lim_n x^n y - y x^n = 0 \). We shall prove that those \( H_i \)-rings which are semisimple and locally compact are commutative, \( i = 1, 2, 3, 4 \).

Lemma 5. All idempotents in an \( H_i \)-ring, \( i = 1, 2, 3, 4 \), commute.

Proof. Let \( e \) and \( f \) be idempotents in such a ring \( A \). Then \( (efe - ef)^2 = 0 \), so \( \{(efe - ef)^n e - e(efe - ef) : n \geq 2\} = \{0\} \). Therefore, if \( A \) is an \( H_i \)-ring, then \( (efe - ef)e - e(efe - ef) = 0 \), so

\[
0 = (efe - ef)e = e(efe - ef) = efe - ef.
\]

If \( A \) is an \( H_2 \)-ring, then \( (ef)e - e(ef) = efe - ef = 0 \) since \( efe - ef \) is in the closure of \( \{[(ef)e - e(ef)]^n : n \geq 2\} = \{0\} \). Similarly in either case
efe = fe, so ef = fe. As 0 = \lim_n e^f - fe^* = \lim_n e^*f^* - f^*e^* = ef - fe, the assertion also holds for H_i and H_\tau-rings.

Since it is clear that all subrings and quotient rings determined by closed ideals of H_\tau-rings are H_\tau-rings, i = 1, 2, 3, 4, and since all idempotents in such rings commute, we see that the following is applicable.

**Lemma 6.** Let P be a property of Hausdorff topological rings such that:

1. if A is a Hausdorff topological ring with property P, then every subring of A has property P and A/B has property P where B is any closed ideal of A,

2. if A has property P, then all idempotents in A commute.

If A is a locally compact primitive ring with property P, then A is a division ring.

**Proof.** Since A is a semisimple ring, A is the topological direct sum of a connected ring B and a totally disconnected ring C, where B is a semisimple algebra over R of finite dimension [7, Th. 2]. As A is primitive, either A = B or A = C. In the former case A is a matrix ring since it is primitive, and so has idempotents which do not commute unless it is a division ring.

It suffices, therefore, to consider the case in which A is totally disconnected. We shall first prove the assertion under the additional assumption that A is a Q-ring (i.e., the set of quasi-invertible elements is a neighborhood of zero). We may consider A to be a dense ring of linear operators on a vector space E over a division ring D. If E is not one-dimensional, then E has a two-dimensional subspace M with basis \{z_1, z_2\}. Let B = \{a \in A: a(M) \subseteq M\}, and let

\[ N = \{a \in A: a(M) = 0\} = K_1 \cap K_2 \]

where \( K_i = \{a \in A: a(z_i) = 0\}, i = 1, 2 \).

There exists \( u \in A \) such that \( u(z_i) = z_i \), and hence \( x - xu \in K_i \), for all \( x \in A \). If \( v \in K_i \), then there exists \( w \in A \) such that \( vw(z_i) = z_i \), so as \( u = vw + (u - vw) \) and \( u - vw \in K_i \), A = Au + K_i = Av + K_i \). Therefore \( K_i \), and similarly \( K_2 \), is a regular maximal left ideal, an observation of the referee that simplifies the proof. Hence \( K_1 \) and \( K_2 \) are closed (cf. [11, Th. 2]), so \( N \) is a closed ideal of \( B \). By hypothesis \( B/N \) is therefore a Hausdorff topological ring having property P. Thus all idempotents in \( B/N \) commute; but \( B/N \) is isomorphic to the ring of all linear operators on \( M \), a ring containing idempotents which do not commute. Hence E is one-dimensional and A is a division ring.

Next we shall show that A is necessarily a Q-ring, from which
the result follows by preceding. As $A$ is totally disconnected $A$ has a compact open subring $D$ [7, Lemma 4]. If $D = J(D)$, the radical of $D$, then $D$ and hence $A$ are $Q$-rings. Assume therefore that $J(D) \subset D$. We shall show that $D/J(D)$ is a finite ring and hence is discrete.

The radical, $J(D)$, of $D$ is closed [8, Th. 1], $D/(J(D))$ is compact semisimple ring and thus $D/J(D)$ is topologically isomorphic to the Cartesian product of a family $(F_r)_{r \in \Gamma}$ of finite simple rings with identities $(f_r)_{r \in \Gamma}$ [11, Th. 16]. As $J(D)$ is topologically nilpotent [11, Th. 14], $D$ is suitable for building idempotents [12, Lemma 4] (cf. [11, Lemma 12]).

Suppose that $\Gamma$ has more than one element, say $\{\alpha, \beta\} \subseteq \Gamma$. Then there are nonzero orthogonal idempotents $e_\alpha$, $e_\beta$ in $D$ such that $e_\alpha + J(D)$, $e_\beta + J(D)$ correspond, respectively, under the isomorphism to $(/)_{\alpha}$, $(/)_\beta$ if $\gamma \neq \alpha$, $\beta$, and $(/)_\gamma = /$; $.. Let $^\phi$ be the canonical mapping $x \mapsto x + J(D)$ from $D$ onto $D/J(D)$. As $(/_{\alpha}) + (/_{\beta})$ annihilates the open neighborhood $\{y \in G_r \mid y \neq 0\}$ for $\gamma \neq \alpha, \beta$, we conclude that $^\phi(e_\alpha + e_\beta)$ annihilates a neighborhood $V$ of zero in $D/J(D)$. Consequently $U = ^\phi^{-1}(V)$ is a neighborhood of zero in $D$, and $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) \subseteq J(D)$ (cf. [7, proof of Th. 11]). Therefore as $(e_\alpha + e_\beta)U(e_\alpha + e_\beta) = U \cap (e_\alpha + e_\beta)A(e_\alpha + e_\beta)$, $(e_\alpha + e_\beta)U(e_\alpha + e_\beta)$ is a neighborhood of zero in $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ consisting of quasi-invertable elements, so $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a $Q$-ring. As $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is primitive [6, Proposition 1, p. 48] and is clearly closed, $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a locally compact, primitive $Q$-ring with property $P$, so $(e_\alpha + e_\beta)A(e_\alpha + e_\beta)$ is a division ring. But it contains nonzero $e_\alpha, e_\beta$ satisfying $e_\alpha e_\beta = 0$, a contradiction. Thus $\Gamma$ can contain only one element, so $D/J(D)$ is isomorphic to a finite ring. Hence $J(D)$, being closed in $D$, is open in $D$ and thus in $A$, so $A$ is a $Q$-ring.

**Lemma 7.** If $A$ is an $H_i$-ring, $i = 1, 2, 3, 4$ and if $A$ is a locally compact division ring, then $A$ is a field.

**Proof.** If $A$ is discrete and is an $H_i$-ring ($i = 1, 2, 3, 4$) then $A$ is commutative [3, Th. 2; 4, Th. 1; 3, Th. 1; 1, Lemma 1].

If $A$ is not discrete, then $A$ has a nontrivial absolute value giving its topology, and $A$ is a finite-dimensional algebra over its center, on which the absolute value is nontrivial [10, Th. 8].

If $A$ is an $H_i$-ring and $x$ is nonzero in $A$, then there exists some nonzero $z$ in the center of $A$ such that $|z| < 1/|x|$. Thus $|xz| < 1$, so $\lim_{n \to \infty} (xz)^n = 0$. Hence for any $y \in A$, $\lim_{n \to \infty} (xz)^n y - y(xz)^n = 0$, so as $(xz)y - y(xz)$ is in the closure of $\{(xz)^n y - y(xz)^n : n \geq 2\}$, $0 = (xz)y - y(xz) = z(xy - yx)$. Hence $xy = yx$, as $z \neq 0$. Thus $A$ is commutative.

If $A$ is an $H_i$-ring and if $x, y \in A$ satisfy $xy - yx \neq 0$, then there exists some nonzero $z$ in the center such that $|z| < 1/|xy - yx|$. Thus
\[(xz)y - y(xz) \leq 1, \text{ so } \lim_n [(xz)y - y(xz)]^n = 0. \] Hence \(0 = (xz)y - y(xz) = (xy - yx)z,\) so \(xy - yx = 0\) as \(z \neq 0,\) a contradiction. Thus \(A\) is commutative.

Assume that \(A\) is an \(H_3\)-ring. As \(A\) is a division ring, \(A\) is either totally disconnected or connected \([7, \text{ Th. 2}]\).

**Case 1.** \(A\) is totally disconnected. Then the topology of \(A\) is given by a nonarchimedean absolute value. Suppose \(A\) is not commutative. Then as \(A\) is a finite-dimensional and hence an algebraic extension of its center \(C,\) there exists some \(x \in C\) having minimal degree \(m > 1\) over \(C.\) Let \(y\) be arbitrary in \(A,\) and assume that for no \(1 \leq i \leq m - 1,\) does \(x^i y = yx^i.\) Hence \(x^i y - yx^i \neq 0, 1 \leq i \leq m - 1,\) and we claim \(\{x^i y - yx^i: 1 \leq i \leq m - 1\}\) is a linearly independent set over \(C.\)

Suppose \(\sum_{i=1}^{m-1} \beta_i (x^i y - yx^i) = 0,\) where \(\beta_i \in C,\) and let \(z = \sum_{i=1}^{m-1} \beta_i x^i.\) Then \(zy = yz.\) By the definition of \(m,\) either \(z \in C\) on \(z\) has degree \(\geq m\) over \(C.\) Suppose \(z \notin C.\) Then \(C[x]\) has dimension \(m\) over \(C,\) so is the degree of \(z\) as \(z \in C[x].\) Therefore \(C[x] = C[z],\) so as \(zy = yz,\) every element of \(C[x]\) commutes with \(y,\) contrary to our assumption. Thus \(z \in C;\) let \(-\beta_0 = z.\) Then \(\sum_{i=0}^{m-1} \beta_i x^i = 0,\) so \(\beta_i = 0, 0 \leq i \leq m - 1\) since \(\{1, x, \cdots, x^{m-1}\}\) is linearly independent over \(C.\)

Since \(x\) is algebraic of degree \(m\) over the center \(C\) of \(A,\) there exist \(\alpha_i \in C, 0 \leq i \leq m - 1,\) such that \(x^m = \sum_{i=0}^{m-1} \alpha_i x^i;\) thus for all \(n \geq m,\) there exist \(\alpha_{i,n} \in C, 0 \leq i \leq m - 1,\) such that \(x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i.\) We may also assume that \(|x| > 1,\) since all our assumption on \(x\) are true for any \(\lambda x, \lambda \in C^*.\) We note that there is therefore some \(r\) such that \(|x|^r \geq |\alpha_i|, 0 \leq i \leq m - 1.\)

Since \(x^n = \sum_{i=0}^{m-1} \alpha_{i,n} x^i,\)

\[
x^ny - yx^n = \sum_{i=0}^{m-1} \alpha_{i,n} (x^iy - yx^i);
\]

so \(\lim_n x^ny - yx^n = 0\) if and only if \(\lim_n \alpha_{i,n} = 0, 1 \leq i \leq m - 1.\)

Since \(|x^n| \leq \max \{|\alpha_{i,n}| |x|^i: 0 \leq i \leq m - 1\},\) if \(|\alpha_{i,n}| < 1, 1 \leq i \leq m - 1,\) then \(|x|^n \leq |\alpha_{0,n}|.\) Let \(r_0\) be such that \(|x|^{r_0} > |x| + 1.\) Since \(\lim_n \alpha_{i,n} = 0, 1 \leq i \leq m - 1,\) there exists \(n_0 > r + r_0\) such that \(|\alpha_{i,n}| < 1,\) for all \(n \geq n_0\) and all \(i\) such that \(1 \leq i \leq m - 1.\) But for any \(n > n_0,\)

\[
x^{n+1} = \sum_{i=0}^{m-2} \alpha_{i,i+1} x^{i+1} + \alpha_{m-1,n} \left(\sum_{i=0}^{m-1} \alpha_i x^i\right)
= \alpha_{m-1,n} \alpha_0 + \sum_{i=1}^{m-1} [\alpha_{i-1,n} + (\alpha_{m-1,n}) \alpha_i] x^i,
\]

so

\[
|\alpha_{i,n+1}| = |\alpha_{0,n} + \alpha_{m-1,n} \alpha_i| \geq |\alpha_{0,n}| - |\alpha_{m-1,n}| |\alpha_i|
\geq |x|^r - |\alpha_i| \geq |x|^{r+r_0} - |x|^r = |x|^r (|x|^{r_0} - 1) > 1.
\]

a contradiction. Hence \(A\) is commutative.
Case 2. A is connected. Then the center $C$ of $A$ contains the real number field $R$, $A$ is finite-dimensional over $R$, so the degree of each element of $A$ over $R$ is less than or equal to 2, and the topology is given by an absolute value. Suppose $x \in C$. Then $\deg x = 2$; let $x^2 = \alpha_1 + \alpha_2x$, and for each $n \geq 2$, let $x^n = \alpha_{i,n} + \alpha_{z,n}x$, where $\alpha_{i,n}$, $\alpha_{z,n} \in R$. As before we may assume that $|x| > 1$. Let $r$ be such that $|x|^r > \max \{|\alpha_1|, |\alpha_2|\}$. Let $y \in A$ be such that $xy \neq yx$. Then $0 = \lim_n (x^ny - yxn) = \lim_n \alpha_{z,n}(xy - yx)$, so $\lim_n \alpha_{z,n} = 0$. Let $n_0 > r$ be such that $|\alpha_{z,n}| < 1$ for all $n \geq n_0$. But if $n \geq n_0$ is such that $|x|^n > 3$, then
\[ |x|^n = |\alpha_{i,n} + \alpha_{z,n}x| \leq |\alpha_{i,n}| + |\alpha_{z,n}| |x| < |\alpha_{i,n}| + |x|, \]
so $|x^n| < |\alpha_{i,n}|$. As
\[ x^{n+1} = \alpha_{i,n}x + \alpha_{z,n}(\alpha_1 + \alpha_2x) = \alpha_{z,n}(\alpha_1 + \alpha_2x) |x|, \]
\[ |\alpha_{z,n+1}| \leq |\alpha_{i,n} + (\alpha_{i,n})\alpha_2| \geq |\alpha_{i,n}| - |\alpha_{z,n}| |\alpha_2|. \]
Hence $|\alpha_{z,n+1}| \geq (|x|^n - |x|) - |x|^r \geq 3 |x|^r - |x|^r - |x|^r = |x|^r > 1$, a contradiction. Hence $A$ is commutative.

Finally let $A$ be an $H_i$-ring. If for all $x$ and $y$, $\lim_n x^ny - yxn = 0$, then $A$ is an $H_i$-ring and so a field; so assume there are $x$ and $y$ in $A$ such that $\lim_n x^ny - yxn \neq 0$. Let $W = \{w \in A : \lim_n x^nw - wxn = 0\}$. Clearly $W$ is a division subring of $A$, and since $y \in W$, $W$ is a proper division subring. By hypothesis, for all $a \in A$ there is an $r \geq 1$ such that $a^r \in W$; thus $A$ is a field [2, Th. B].

**Theorem 4.** All $H_i$-rings that are locally compact and semisimple are commutative, $i = 1, 2, 3, 4$.

**Proof.** $P$ is a primitive ideal of such a ring $A$ if and only if $P = (B: A)$ (by definition $(B: A) = \{x \in A : Ax \subseteq B\}$) where $B$ is a regular maximal to left ideal [5, Corollary to Proposition 2, p. 7]. Let $e \in A$ be such that $x - ex \in B$ for all $x \in A$. If $x \in (B: A)$, then $ex \in B$, so $x \in B$. Hence $(B: A) \subseteq B$.

If $B$ is closed, then $(B: A)$ is closed for if $(x_n)$ is a directed set of elements of $(B: A)$ converging to $x$, then for all $a \in A$, $ax_n \in B$, whence $ax = \lim ax_n \in B$.

As $A$ is semisimple, $(0) = \bigcap \{B : B$ is a closed regular maximal left ideal\} $\supseteq \bigcap \{P : P$ is a closed primitive ideal\} [8, Th. 1]. By Lemma 6 and 7, $A/P$ is a field if $P$ is a closed primitive ideal. Thus for all $x, y \in A$, $xy - yx \in P$, so $xy - yx \in \bigcap \{P : P$ is a closed primitive ideal\} = $(0)$.
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