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## **RINGS OF FUNCTIONS WITH CERTAIN LIPSCHITZ PROPERTIES**

CHARLES HARRIS SCANLON

## RINGS OF FUNCTIONS WITH CERTAIN LIPSCHITZ PROPERTIES

C. H. SCANLON

Let  $(X, d)$  denote a metric space,  $L_c(X)$  the ring of real valued functions on  $X$  which are Lipschitz on each compact subset of  $X$ ,  $L_1(X)$  the ring of real valued functions on  $X$  which are locally Lipschitz relative to the completion of  $X$ , and  $L_c^*(X)$ ,  $L_1^*(X)$  the bounded elements of  $L_c(X)$ ,  $L_1(X)$ . The relations between equality of these rings and the topological properties of  $X$  are studied. It is shown that a subspace  $(S, d)$  of  $(X, d)$  is  $L_c$ -embedded (or  $L_c^*$ -embedded) in  $(X, d)$  if and only if  $S$  is closed. Further, every subspace of  $(X, d)$  is  $L_1$ - and  $L_1^*$ -embedded in  $(X, d)$ .

Su [3] investigated algebraic properties of the rings  $L_c(X)$  and  $L_c^*(X)$  similar to those of  $C(X)$  and  $C^*(X)$  by Gillman and Jerrison [2].

2. Equality of rings. Let  $f$  denote a real valued function defined on  $X$ .  $f$  is Lipschitz on  $S \subset X$  if and only if there is a real number  $m$ , called a Lipschitz constant for  $f$  on  $S$ , such that if  $x, y \in S$ , then  $|f(x) - f(y)| \leq md(x, y)$ .  $f$  is locally Lipschitz on  $X$  if and only if for each  $x \in X$ , there is a neighborhood  $N$  of  $x$  such that  $f$  is Lipschitz on  $N$ . If  $\text{comp } X$  denotes the completion of  $X$ , then  $f$  is locally Lipschitz with respect to  $\text{comp } X$  if and only if for each  $x \in \text{comp } X$  there is a neighborhood  $N$  of  $x$  such that  $f$  is Lipschitz on  $N \cap X$ .

**THEOREM 2.1.**  $f \in L_c(X)$  if and only if  $f$  is locally Lipschitz on  $X$ .

*Sufficiency.* Let  $f$  be locally Lipschitz on  $X$  and  $S$  a compact subset of  $X$ . Then there exists a finite collection  $N_1, N_2, \dots, N_m$  of open sets covering  $S$ , on each of which  $f$  is Lipschitz and thus bounded. Assuming  $f$  is not Lipschitz on  $S$  implies that there exists a sequence  $\{x_n\}$  from  $S$  converging to  $x \in S$  and a sequence  $\{y_n\}$  from  $S$  such that  $|f(x_n) - f(y_n)|/d(x_n, y_n) > n$  for each positive integer  $n$ . Since  $f$  is bounded on  $S$ , it follows that  $\{y_n\}$  converges to  $x$ . Since  $x \in N_j$  for some  $j = 1, 2, \dots, m$ ,  $f$  is not Lipschitz on  $N_j$  which contradicts the definition of  $N_j$ .

*Necessity.* Let  $f \in L_c(X)$  and  $x \in X$ . Assuming  $f$  is not locally Lipschitz at  $x$  implies there exists sequences  $\{x_n\}$  and  $\{y_n\}$  such that

$d(x, x_n) < 1/n$ ,  $d(x, y_n) < 1/n$ , and  $|f(x_n) - f(y_n)|/d(x_n, y_n) > n$ . Then  $\{p: p \in \{x_n\}, p \in \{y_n\}, \text{ or } p = x\}$  is a compact subset of  $X$  on which  $f$  is not Lipschitz.

**COROLLARY 2.2.**  $f \in L_c^*(X)$  if and only if  $f$  is locally Lipschitz on  $X$  and bounded.

*Proof.* Follows immediately from the definition of  $L_c^*(X)$ .

**COROLLARY 2.3.**  $L_1(X) \subset L_c(X)$  and  $L_1^*(X) \subset L_c^*(X)$ .

*Proof.* If  $f$  is locally Lipschitz relative to  $\text{com } X$ , then  $f$  is locally Lipschitz.

**LEMMA 2.4.** If  $K$  is a uniformly bounded set of Lipschitz functions defined on  $S \subset X$  and there is a real number  $m$  which is a Lipschitz constant for each element of  $K$ , then  $f(x) = \sup \{g(x): g \in K\}$  for each  $x \in S$  is Lipschitz on  $S$  and  $m$  is a Lipschitz constant for  $f$  on  $S$ .

*Proof.*  $f$  exists since  $K$  is a uniformly bounded set. Assume  $x \in S, y \in S$ , and

$$(1) \quad f(y) - f(x) - md(x, y) = e > 0.$$

Let  $g \in K$  such that

$$(2) \quad f(y) - g(y) < e,$$

then

$$(3) \quad g(y) - g(x) \leq md(x, y).$$

Combining (2) and (3) yields  $f(y) - g(x) - md(x, y) < e$ , which when combined with (1) gives  $f(x) < g(x)$ . This contradicts the definition of  $f$ .

**LEMMA 2.5.** Suppose each of  $c$  and  $r > 0, p \in X$ , and for

$$\text{each } x \in X, f(x) = \begin{cases} (c/r)\{r - d(x, p)\} & \text{for } d(x, p) \leq r, \\ 0 & \text{otherwise} \end{cases}$$

then  $f$  is Lipschitz on  $X$  and  $(c/r)$  is a Lipschitz constant for  $f$  on  $X$ .

*Proof.* Let  $g(x) = (c/r)\{r - d(x, p)\}$  for each  $x \in X$ . Then for  $x, y \in X$ ,

$$\begin{aligned}g(x) - g(y) &= g(x) - g(p) + g(p) - g(y) , \\g(x) - g(y) &= -(c/r)d(x, p) + (c/r)d(y, p) ,\end{aligned}$$

and  $g(x) - g(y) \leq (c/r)d(x, y)$  by the triangle property. Since  $\sup\{g, 0\}$  is Lipschitz with a Lipschitz constant  $\sup\{(c/r), 0\}$  by Lemma 2.4, the conclusion follows.

**THEOREM 2.6.** *Each of the following is equivalent to each of the others:*

- (1)  $L_1(X) = L_c(X)$  ,
- (2)  $L_1^*(X) = L_c^*(X)$ , and
- (3)  $X$  is complete.

*Proof.* (1)  $\Rightarrow$  (2) obviously. The remaining order is (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1).

Assume (2) and that  $X$  is not complete. Then there exists an  $x \in (\text{comp } X) - X$  and a sequence  $\{x_n\}$  of distinct points in  $X$  such that  $\{x_n\}$  converges to  $x$ . For each odd integer  $n$ , let

$$r_n = \frac{1}{3} \inf \{y: y = d(x_n, x_m) \text{ for } m \neq n \text{ or } y = (1/n)\} ,$$

$$C(x_n, r_n) = \{t \in X: d(t, x_n) \leq r_n\} ,$$

and

$$f_n(t) = \begin{cases} (1/r_n)\{r_n - d(x_n, t)\} & \text{for } t \in C(x_n, r_n) \\ 0 & \text{otherwise} \end{cases}$$

for each  $t \in X$ . Let  $f(t) = \sup \{f_n(t)\}$  for each  $t \in X$ . If  $S$  is a compact subset of  $X$ , then  $S$  can intersect at most a finite number of the elements of  $\{C(x_n, r_n)\}$  and since only a finite number of elements of  $\{f_n\}$  are nonzero on  $S$ , by Lemma 2.4  $f$  is Lipschitz on  $S$  and  $f \in L_c^*(X)$ . For each neighborhood  $N$  in  $\text{comp } X$  of  $x$ , there is a point  $t \in N$  and a point  $y \in N$  such that  $f(t) = 1$  and  $f(y) = 0$ . Thus  $f \notin L_1(X)$  and by contradiction, (2)  $\Rightarrow$  (3).

If (3) is true,  $f \in L_1(X)$  if and only if  $f$  is locally Lipschitz. Thus by Theorem 2.1,  $L_1(X) = L_c(X)$  and (3)  $\Rightarrow$  (1).

**THEOREM 2.7.**  $L_c(X) = L_c^*(X)$  if and only if  $X$  is compact.

*Proof.* If  $X$  is compact, then each element of  $L_c(X)$  is bounded.

Assume  $L_c(X) = L_c^*(X)$  and  $X$  is not compact. Then there exists a sequence  $\{x_n\}$  of distinct points in  $X$  which has no convergent subsequence. Let

$$r_n = \frac{1}{3} \inf \left\{ y: y = d(x_n, x_m) \text{ for } n \neq m \text{ or } y = \frac{1}{n} \right\} ,$$

and

$$f(x) = \begin{cases} (n/r_n)\{r_n - d(x_n, x)\} & \text{for } d(x_n, x) \leq r_n \\ 0 & \text{otherwise} \end{cases}$$

for each  $x \in X$ . By an argument similar to the one for Theorem 2.6,  $f \in L_c(X)$ . Since  $f(x_n) = n$  for each  $n$ ,  $f \in L_c(X) - L_c^*(X)$  which contradicts the assumption.

**THEOREM 2.8.**  $L_1(X) = L_1^*(X)$  if and only if  $\text{comp } X$  is compact.

*Proof.* Each element of  $L_1(X)$ ,  $L_1^*(X)$  can be uniquely extended to an element of  $L_1(\text{comp } X) = L_c(\text{comp } X)$ ,  $L_1^*(\text{comp } X) = L_c^*(\text{comp } X)$ . Since  $L_c(\text{comp } X) = L_c^*(\text{comp } X)$  if and only if  $\text{comp } X$  is compact by Theorem 2.7, the conclusion follows.

3. If  $A$  denotes one of  $L_1, L_1^*, L_c, L_c^*$  and  $S \subset X$ , then the statement that  $S$  is  $A$ -embedded in  $X$  means that if  $f \in A(S)$ , there is a  $g \in A(X)$  such that  $g|S = f$  where  $g|S = \{(x, y) \in g : x \in S\}$ .

**THEOREM 3.1.** If  $S$  is a subset of  $X$ , then each of the following is equivalent to each of the others:

- (1)  $S$  is  $L_c$ -embedded in  $X$ ,
- (2)  $S$  is  $L_c^*$ -embedded in  $X$ , and
- (3)  $S$  is closed.

*Proof.* Czipser and Geher [1] proved that if  $S$  is a closed subset of  $X$  and  $f$  is a real valued locally Lipschitz function with domain  $S$ , then there is a real valued locally Lipschitz function  $g$  with domain  $X$  such that  $g|S = f$ . Furthermore, they proved that if  $f$  is bounded, then there exists a bounded such  $g$ . Consequently, by Theorem 2.1, (3)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (2).

Assume (2) and  $S$  is not closed. Then there exists a sequence  $\{x_n\}$  of distinct points in  $S$  and a point  $x \in X - S$  such that  $\{x_n\}$  converges to  $x$ . Construct  $f$  as in Theorem 2.6. Then  $f \in L_c^*(S)$  which has no extension to  $X$  in  $L_c(X)$ . Thus (2)  $\Rightarrow$  (3). Note that this also shows (1)  $\Rightarrow$  (3).

**COROLLARY 3.2.** Every subset of  $X$  is  $L_1$ -embedded and  $L_1^*$ -embedded in  $X$ .

*Proof.* If  $S \subset X$ , then every element of  $L_1(S)$  has a unique extension to the closure of  $S$  in  $\text{comp } X$  and by Theorems 2.6 and 3.1

an extension in  $L_1(\text{comp } X)$  which when restricted to  $X$  is an element of  $L_1(X)$ .

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