ANalytic Interpolation of Certain Multiplier Spaces

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ANALYTIC INTERPOLATION OF CERTAIN MULTIPLIER SPACES

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Let $W_p$ denote the space of all functions on the circle which are the uniform limit of a sequence of trigonometric polynomials which is bounded as a sequence of multipliers for $l_p, 1 \leq p \leq 2$. Let $U_s$ be the interpolation space $[W_2, W_1]$ (see 1.1). Our main result, Theorem 2.4, states that for a compact subset $E$ of the circle, $U_s | E = C(E)$ if and only if $W_1 | E = C(E)$. A major step in the proof is a maximum principle for interpolation, Theorem 1.7. We also give a direct proof that $U_s \subseteq W_p$ (see Theorem 2.7) for corresponding $s$ and $p$.

1. Some properties of analytic interpolation.

1.1. Let $B^0$ and $B^1$ be two Banach spaces continuously embedded in a topological vector space $V$ such that $B^0 \cap B^1$ is dense in both $B^0$ and $B^1$. For $0 < s < 1$, let $\mathcal{G}, [B^0, B^1], B^0 + B^1$ denote the spaces as defined in [1, §1]. For two Banach spaces $X$ and $Y$ we let $O(X, Y)$ denote the Banach space of bounded linear operators from $X$ into $Y$ where the norm is the usual operator norm. Let $O(X)$ denote $O(X, X)$.

1.2. Assume the notation and conditions of paragraph 1.1 and for convenience let $B_j$ denote the space $[B^0, B^1], 0 < s < 1$. Let $V'$ denote the Banach space

$$O(B^0 \cap B^1, B^0 + B^1).$$

Let $A_j$ be a closed subspace of $O(B^j), j = 0, 1$. By restricting the elements in $A_j$ to $B^0 \cap B^1$ in the obvious way we may regard $A_j$ as continuously embedded in the topological vector space $V'$, and it is with respect to this embedding that we understand $[A_0, A_1]$; in particular, $[A_0, A_1]$ is a subspace of $V'$. We will assume that $A_0 \cap A_1$ is dense in $A_j$ with respect to the norm of $A_j, j = 0, 1$, when these spaces are embedded in $V'$ as described. Since $B^0 \cap B^1$ is dense in $B^0$ and $B^1$, we know from [1, §9.3] that $B^0 \cap B^1$ is dense in $B_j$; thus, since $B_j \subseteq B^0 + B^1$, the restriction of elements of $O(B_j)$ to $B^0 \cap B^1$ gives a continuous embedding of $O(B_j)$ in $V'$ in the obvious manner. Note that each element of $A_0 \cap A_1$ is bounded with respect to the norm $\| \cdot \|_{s_j}$ restricted to $B^0 \cap B^1$ and is, therefore, contained in the embedded $O(B_j)$. Let $A_j$ denote the closure of $A_0 \cap A_1$ in $O(B_j)$ where $O(B_j)$ is regarded as embedded in $V'$ in the manner just described. Finally, we let $M_s$ and $N_s$ denote the norms of the spaces $A_s$ and $[A_0, A_1]$, respectively.
LEMMA 1.3. Assuming 1.2, \([A_0, A_1], \subset A_s\) and \(M_s \leq N_s, 0 < s < 1\).

This lemma is an immediate consequence of [1, § 11.1].

1.4. Assume the notation and conditions of 1.1. Let \(J\) be a closed subspace of \(B^0 + B^1\). We will assume that

\[(1.4.1) \quad I^j = J \cap B_j,\]

is closed in \(B^j, j = 0, 1\). Clearly the map \(\alpha\) defined by

\[
\alpha(x + I^j) = x + J \quad j = 0, 1
\]
is a continuous one to one linear map from \(B^j/I^j\) into \(V/J\). Let

\[
D_s = [\alpha(B^0/I^0), \alpha(B^1/I^1)].
\]

LEMMA 1.5. Assuming 1.4, if \(x \in B_s, 0 < s < 1\), then \(x + J \in D_s\) and

\[
(1.5.1) \quad \| x + J \|_{D_s} \leq \| x + (J \cap B_s) \|_{s_1}/(J \cap B_s).
\]

Proof. Let \(x \in B_s, h \in J \cap B_s\) and \(\varepsilon > 0\). Choose \(f \in \mathcal{G} = \mathcal{G}(B^0, B^1)\) such that \(f(s) = x + h\) and

\[
(1.5.2) \quad \| f \|_{\mathcal{G}} \leq \varepsilon + \| x + h \|_{s_1}.
\]

Let \(g(\xi) = f(\xi) + J\) for \(1 \leq |\xi| \leq \varepsilon\). Then it is clear that \(g \in \mathcal{G}_1\) where

\[
\mathcal{G}_1 = \mathcal{G}(\alpha(B^0/I^0), \alpha(B^1/I^1))
\]

and that

\[
(1.5.3) \quad g(s) = x + J.
\]

Hence, \(x + J \in D_s\). Furthermore, since it is clear that

\[
(1.5.4) \quad \| g \|_{\mathcal{G}_1} \leq \| f \|_{\mathcal{G}},
\]

(1.5.1) follows from (1.5.2), (1.5.3), (1.5.4) and the fact that \(h\) and \(\varepsilon\) were chosen arbitrarily.

The following lemma can be proved by the usual method of successive approximations.

LEMMA 1.6. Suppose that \(D_1\) is a Banach space that is continuously embedded in a Banach space \(D_0\) such that \(D_1\) is dense in \(D_0\) with respect to the norm of \(D_0\). Suppose that there exist constants \(c, c_0, c < 1\), with the property that for each \(x \in D_1\) there is a corresponding element \(z \in D_0\) such that
We will now establish a "maximum principle" for analytic interpolation.

**Theorem 1.7.** If, in addition to the assumptions of paragraph 1.1, \( B^0 = [B^0, B^1] \), for some \( s \) \((0 < s < 1)\), then \( B^0 = B^1 \).

**Proof.** From the fact that \( B^0 \) and \( B^1 \) are continuously embedded in \( V \) and the closed graph theorem we conclude that the norms \( \| \cdot \| \) and \( \| \cdot \|_s \) on \( B^0 \) and \([B^0, B^1]_s \), respectively, are equivalent. In particular, there is a constant \( c \) such that

\[
|x|_s \leq c|x|_0 \quad \text{for all } x \text{ in } B^0.
\]

From [1, 9.4. (ii)] we conclude that

\[
|x|_s \leq |x|_0^{1-s} |x|_s \quad \text{for all } x \text{ in } B^0 \cap B^1.
\]

We conclude from (1.7.1) and (1.7.2) that

\[
|x|_0 \leq c^{1/s} |x|_s \quad \text{for all } x \text{ in } B^0 \cap B^1.
\]

Thus, \( B_1 \) is continuously embedded in \( B^1 \). We shall now prove that (1.7.3) there is a constant \( c_i \) with the property that for each \( x \) in \( B^1 \) there is a corresponding \( y \) in \( B^1 \) such that

\[
|y|_1 \leq c_i |x|_0 \quad \text{and} \quad |y - x|_0 \leq (1/2)|x|_0.
\]

Let \( x \in B^1 \). In particular, \( x \in [B^0, B^1] \), and, therefore, there exists an \( f \in \mathbb{R}(B^0, B^1) \) such that \( f(s) = x \) and \( |f|_{\mathbb{R}(B^0, B^1)} \leq 2|\cdot|_s \). Since the norms \( \| \cdot \|_0 \) and \( \| \cdot \|_s \) are equivalent we can choose a real number \( \lambda \) so that \( 2|u|_s e^{i\xi} \leq (1/2)|u|_0 \) for every \( u \) in \( B^0 \). Let \( g(\xi) = f(\xi)e^{-\lambda(1-\xi)} \)

where \( \lambda(0, 1) \) and \( \lambda(1) \) denote the first and second integrals, respectively, appearing in (1.7.4). Since

\[
|g(it)|_0 \leq 2|x|_s e^{i\xi} \leq (1/2)|x|_0 \quad \text{(all real } t) \quad \text{and}
\]

\[
|g(1 + it)|_1 \leq 2|x|_s e^{-\lambda(1-\xi)} \leq (1/2)e^{-\lambda} |x|_0
\]
(all real t), it follows that |x - z|₀ ≤ (1/2) |x|₀ and |z|₁ ≤ (1/2)e⁻² |x|₀. This proves (1.7.3). Since B¹ is continuously embedded as a dense subspace in B° and (1.7.3) holds, the conclusion of Theorem 1.7 follows from Lemma 1.6.

2. The spaces Wₚ and Uₚ. Let lₚ, 1 ≤ p < ∞, denote the Banach space of complex valued functions x on the integers such that

\[ ||x||ₚ = \left( \sum |x(n)|^p \right)^{1/p} < \infty \]

where the sum is over all integers n. Each function α on the integers which vanishes outside some finite set determines a linear transformation Tα on lₚ defined by

\[ Tαx(n) = \sum_{-\infty < k < \infty} x(n-k)α(k). \]

Let Wₚ denote the closure of the operators Tα in O(lₚ). Since l₁ is a dense subspace of each space lₚ, 1 ≤ p ≤ 2, the restriction of elements in O(lₚ), 1 ≤ p ≤ 2, to the subspace l₁ gives a one-to-one continuous linear embedding of O(lₚ), 1 ≤ p ≤ 2, into the space

\[ R = O(l₁, l₁). \]

Throughout this section we will identify O(lₚ) with its image under this embedding without further comment. Let Uₚ denote the space [W₂⁻, W₁⁺], where V in 1.1 is, in this case, R.

Our immediate purpose is to define a “Fourier transform” on Wₚ and to prove Lemmas 2.2 and 2.3.

If x is a complex valued function on the integers Z, let τₙx(k) = x(k - n). Let δₙ denote the function on Z such that δₙ(n) = 1 and δₙ(k) = 0, k ≠ n. If x and y are two complex valued function on Z let

\[ x*y(m) = \sum_{n \in Z} x(m-n)y(n) \]

define the function x*y provided the sum converges absolutely for each m ∈ Z. For each H in W₁⁺ let H⁻ denote the function H(δₙ) in lₚ. The following lemma states the needed properties of the map H → H⁻. Note that τₙx = δₙ*x for each n ∈ Z and for each complex valued function x on z.

**Lemma 2.1.**

(2.1.1) H → H⁻ is a one-to-one linear transformation from W₂⁻ into lₚ.
(2.1.2) Hx = H⁻*x, H ∈ W₂⁻, x ∈ lₚ.
(2.1.3) (HK)⁻ = H⁻*K⁻, H, K ∈ W₂⁻.

**Proof.** The map H → H⁻ is clearly linear. Evidently, each H in
$W_p$ commutes with all operators $\tau_m$, $m \in \mathbb{Z}$, since the operators of the form $T_\alpha$ commute with the operators $\tau_m$, $m \in \mathbb{Z}$. Thus for $H \in W_p$ and $m \in \mathbb{Z}$, we see that

$$H(\delta_m) = H(\tau_m \delta_0) = \tau_m H(\delta_0) = \tau_m H^\sim = H^\sim \delta_m.$$  

From this we see that since the linear span of the elements $\delta_m$ is dense in $l_p$, the map $H \to H^\sim$ is one-to-one. Obviously, $H^\sim$ is in $l_p$. To establish (2.1.2) we first note that since $H^\sim$ is in $l_q(q^{-1} + p^{-1} = 1)$ the map $x \to H^\sim x$ is a continuous linear map from $l_p$ into $c_0$, the space of complex valued functions on $\mathbb{Z}$ which tend to 0 at $\pm \infty$. The map $x \to Hx$ is also a continuous linear map from $l_p$ into $c_0$. These observations together with (2.1.4) and the density property of the $\delta_m$'s noted above complete the proof of (2.1.2). To prove (2.1.3) we note that for $H$ and $K$ in $W_p$, $K \in l_p$, so by (2.1.2) we have

$$H^\sim K^\sim = H(K^\sim) = H(K\delta_0) = (HK)\delta_0 = (HK)^\sim.$$  

This completes the proof of the lemma.

Let $L_p(1 \leq p < \infty)$ denote the Banach space of measurable functions $g(\theta)$ on the circle (reals mod $2\pi$) whose norm $\|g\|_{L_p}$,

$$\|g\|_{L_p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |g(\theta)|^{1/p} d\theta \right)^{1/p},$$

is finite. Let $L_\infty$ denote the space of essentially bounded measurable functions $g$ with $\|g\|_{L_\infty}$ denoting the essential supremum of $g$.

Since each function $H^\sim$, $H \in W_p$, is in $l_p$, which is contained in $l_2$, there is a unique function $H^\sim$ in $L_2$ such that $\sum H^\sim(n)e^{in\theta}$ is the Fourier series of $H^\sim$.

**Lemma 2.2.** For $1 \leq p \leq 2$ the map $H \to H^\sim$ is a norm decreasing algebraic isomorphism from $W_p$ into $L_\infty$.

**Proof.** The fact that $H \to H^\sim$ is a one-to-one linear map from $W_p$ into $L_2$ is clear from (2.1.1) and the fact that each function in $L_2$ is uniquely determined by its Fourier coefficients. For each $f \in L_2$, let $\lambda(f)$ denote the function on $\mathbb{Z}$ defined by:

$$\lambda(f)(n) = \left(\frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta} d\theta \right).$$

It is clear from the Schwarz inequality that the map $(f, g) \to \lambda(f \cdot g)(n)$ is a continuous bilinear functional on $L_2 \oplus L_2$ for each integer $n$. On the other hand, the map

$$(f, g) \to (\lambda(f) \ast \lambda(g))(n)$$
is also a continuous bilinear functional on $L_2 \oplus L_2$. Since these functionals (for each $n$) clearly agree when $f$ and $g$ are trigonometric polynomials, they must agree on $L_2 \oplus L_2$. Since $\lambda$ is a one-to-one map, the multiplicative property of $H \to H^\wedge$ now follows from (2.1.3).

To prove that the map is norm decreasing we first note the following inequalities:

$$\|H^n\|_{L_2^p} \geq \|H^n\|_{L_2^q} \geq \|(H^n)^\wedge\|_{L_2} = \|(H^n)^\wedge\|_{L_2} = \|(H^n)^\wedge\|_{L_2}.$$ 

It is well known that $(\|H^n\|_{L_2^p})^{1/n}$ converges to the spectral radius of $H$, which is dominated by $\|H\|_{L_2^p}$, and that $(\|(H^n)^\wedge\|_{L_2})^{1/n}$ converges to $\|H^\wedge\|_{L_2}$ as $n \to \infty$. This proves the lemma.

Let $W_p$ and $U_s$ denote the functions on the circle of the form $H^\wedge$ where $H \in W_p$, $U_s$, respectively. The following lemma is an immediate consequence of Lemma 2.2.

**Lemma 2.3.** $W_p$ consists precisely of the functions on the circle which are the uniform limits of sequences $H^n$ of trigonometric polynomials such that $H^n$ is a Cauchy sequence in $W_p$.

For any subset $E$ of the circle group $U_s \mid E$ denotes the functions on $E$ obtained by restricting the functions of $U_s$ to $E$ and $C(E)$ denotes the continuous complex valued functions on $E$.

**Theorem 2.4.** Suppose that $E$ is a compact subset of the circle group and $0 < s < 1$. Then $U_s \mid E = C(E)$ if and only if $W_1 \mid E = C(E)$.

**Proof.** First assume that $W_1 \mid E = C(E)$. By Lemma 1.3, $U'_s \subset W_1'$; consequently, $U_s \subset W_p$. We conclude from Lemma 2.3 that $W_p \subset C(T)$.

Thus, $U_s \mid E \subset C(E)$. Since $W'_1 \supset W'_s$, it is clear from the definition of interpolation that $U'_s \supset W'_1$. Thus, $U_s \mid E \supset C(E)$.

Consider the converse and assume that $U_s \mid E = C(E)$. In 1.4 we let $B' = W'_s$, $B = W'_s$, $V = R$ and

$$J = \{a \in W'_s : \hat{a}(\theta) = 0, \theta \in E\}.$$ 

The assumptions on $J$ in 1.4 are clearly satisfied since by Lemma 2.2, the maps $x \to \hat{a}$ are continuous on $W'_s$ and $W'_s$. By Theorem 1.5, if $x \in U'_s$, then $x + J$ is in the space

$$(2.4.1) \quad \{\alpha(W'_s/J), \alpha(W'_s/(J \cap W'_s))\}.$$ 

However, by hypothesis, the cosets in $V$ of the form $x + J$, $x \in U'_s$, are the same as the cosets $y + J$, $y \in W'_s$. Therefore, the space in $(2.4.1)$ is $\alpha(W'_s/J)$. Since $W'_s \supset W'_s$,

$$\alpha(W'_s/J) \supset \alpha(W'_s/(J \cap W'_s));$$

where $\alpha(W'_s/J)$, $\alpha(W'_s/(J \cap W'_s))$.
therefore, we conclude from 1.7 that 
\[ \alpha(W_i/J) = \alpha(W_i/(J \cap W_i)) \]
or, what is the same thing, that \( W_1 | E = C(E) \). This completes the proof.

**COMMENT 2.5.** It is natural to compare \( U_s \) and \( W_p \) where \([l_2, l_1] = l_p\), i.e., \((1-s)/2 + s = 1/p\). In [3] we showed that Theorem 2.4 is not valid for \( W_p \). To be exact, there is a compact subset \( E \) of the circle such that \( W_p | E \not= C(E) = W_{4/3} | E, 1 \leq p < 4/3 \). We had originally used this result to show that \( W_p \not= U_s \); however, the referee has suggested a direct proof which we will now give.

**LEMMA 2.6.** Let \( h_n \) be a sequence in \( U_s \), \( 0 < s < 1 \), such that \( ||h_n||_s \leq M \) (here \( || ||_s \) is the norm in \( U_s \)) and \( h_n \to h \) almost everywhere. Then \( h \) agrees with some continuous function almost everywhere.

**Proof.** Since \( ||h_n||_s \leq M \) there exist functions \( f_n(\theta, \xi) \), analytic in \( \xi \) for \( 0 < B(\xi) < 1 \) and continuous in \( 0 \leq B(\xi) \leq 1 \), such that for any real number \( t \), \( ||f_n(\theta, it)||_s \leq 2M, ||f_n(\theta, 1 + it)||_s \leq 2M \) and \( f_n(\theta, s) = h_n(\theta) \). Let \( g_n(\theta, \xi) = f_n(\theta, \xi)e^{\frac{\xi}{1+\xi}} \). Then

\[
\begin{align*}
    h_n(\theta) &= f_n(\theta, s) = g_n(\theta, s) = \int_{-\infty}^{\infty} g_n(\theta, it)\mu_0(s, t)dt \\
    &\quad + \int_{-\infty}^{\infty} g_n(\theta, 1 + it)\mu_1(s, t)dt \\
    &= u_n(\theta) + v_n(\theta)
\end{align*}
\]

where \( \mu_0 \) and \( \mu_1 \) are the Poisson Kernels for the strip (see [1, 9.4]). Evidently \( ||u_n||_s \leq 2e^{-is}M, ||v_n||_s \leq 2e^{-is-1}M \). Since the \( v_n \) are uniformly bounded, by taking a subsequence if necessary, we may assume that \( v_n \) converges weakly to a bounded function \( v(\theta) \), that is

\[
\lim_{n \to \infty} \int v_n(\theta)\varphi(\theta)d\theta = \int v(\theta)\varphi(\theta)d\theta
\]

for every integrable \( \varphi \). Furthermore, as is readily seen, \( v(\theta) \) belongs to \( U_i \) and therefore is continuous. Since \( h_n \) is uniformly bounded and converges almost everywhere, \( h_n \) converges almost everywhere. Since \( h_n \) and \( v_n \) converge weakly, \( u_n \) converges weakly to some function \( u \). From the fact that \( ||u_n(\theta)||_s \leq ||u_n||_0 \leq 2e^{-is}M \), it follows that \( ||u(\theta)||_s \leq 2e^{-is} \) almost everywhere. Since \( h = u + v \) almost everywhere and \( \lambda \) can be taken arbitrarily large, \( h \) agrees almost everywhere with the uniform limit of continuous functions. This completes the proof of the lemma.
THEOREM 2.7. $U_s$ is properly contained in $W_p$ for $1 < p < 2$.

Proof. To prove the theorem it suffices to exhibit a sequence of functions in $U_s$ whose norms in $U_s$ tend to infinity and whose norms in $W_p$ remain bounded. Let $h(e^{it}) = 1$ for $0 \leq t \leq \pi$ and $h(e^{it}) = 0$ for $\pi < t < 2\pi$. Then $h$ is a multiplier for $L_p$ (see [2]), which does not agree almost everywhere with any continuous function. Let $\varphi_n$ be defined by: $\varphi_n(e^{it}) = n$ for $|t| \leq 1/2n$, $\varphi_n(e^{it}) = 0$ otherwise, $n = 1, 2, \ldots$. Let $h_n = h*\varphi_n$, $n = 1, 2, \ldots$. Since $\int_0^{2\pi} |h_n(e^{it})| dt = 1$, it follows that the $W_p$ norm of $h_n$ is the same as the $W_s$ norm of $h$; thus, $h_n$ is bounded in $W_p$. Since both $h$ and $\varphi_n$ belong to $L_2(0, 2\pi)$, $h_n \in W_1 \subset U_s$. Obviously, $h_n$ converges to $h$ almost everywhere. Since $h$ does not agree almost everywhere with any continuous function, it follows from Lemma 2.6 that $h_n$ is not bounded in $U_s$.

Bibliography


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