SEMI-SIMPLE RADICAL CLASSES

P A T R I C K  N O B L E  S T E W A R T
The purpose of this paper is to characterize all semi-simple radical classes (those classes of rings which are semi-simple classes and at the same time radical classes).

Andrunakievic has shown that the class of Boolean rings is a semi-simple radical class. More recently, Armendariz has considered such classes.

For "I is an ideal of the ring R" we shall write "I <| R".

Following Divinsky [6], but substituting classes of rings for ring properties, we define:

(i) A nonempty class of rings $\mathcal{C}$ is a radical class if and only if $\mathcal{C}$ satisfies the following conditions:

(A) Homomorphic images of rings in $\mathcal{C}$ are in $\mathcal{C}$.
(B) Every ring $R$ has an ideal $\mathcal{C}(R) \in \mathcal{C}$ such that if $I <| R$ and $I \in \mathcal{C}$ then $I \subseteq \mathcal{C}(R)$.
(C) The only ideal of the factor ring $R/\mathcal{C}(R)$ which is in $\mathcal{C}$ is the zero ideal.

(ii) If $\mathcal{C}$ is a radical class, a ring $R$ is $\mathcal{C}$ semi-simple if and only if $\mathcal{C}(R) = (0)$.

(iii) A nonempty class of rings $\mathcal{C}$ is a semi-simple class if and only if $\mathcal{C}$ satisfies the following conditions:

(E) Every nonzero ideal of a ring in $\mathcal{C}$ can be homomorphically mapped onto a nonzero ring in $\mathcal{C}$.
(F) If every nonzero ideal of a ring $R$ can be homomorphically mapped onto a nonzero ring in $\mathcal{C}$ then $R \in \mathcal{C}$.

2. Rings without nilpotent elements. Our purpose in this section is to establish:

THEOREM 2.1. A ring $R$ without nilpotent elements is isomorphic to a subdirect sum of rings without proper divisors of zero.

It will be convenient to first prove:

LEMMA 2.2. If $R$ has no nilpotent elements and $0 \neq x \in R$ then

(i) $x_r = \{y \in R: xy = 0\} <| R$ and $x_r = x_l = \{y \in R: yx = 0\}$,
(ii) $x \in x_r$.

[1 The author wishes to thank the referee for pointing out that this result has also been obtained by V. Andrunakievic and Ju. M. Rjabuhin, Rings without nilpotent elements, and completely simple ideals, Dokl. Akad. Nauk. SSR. 180, 9-11 (Translation, Soviet Mathematics 9 (1968), 565-568).]
(iii) if \( r \in R \) and \( rx \in x_i \) then \( r \in x_i \),
(iv) the factor ring \( R/x_i \) has no nilpotent elements.

Proof. Let \( R \) be a ring with no nilpotent elements and \( 0 \neq x \in R \). If \( a \in R \) and \( ax = 0 \) then \( (xa)^2 = 0 \) so \( xa = 0 \). Similarly if \( xa = 0 \) then \( ax = 0 \). This establishes (i). Since \( x^2 \neq 0 \), (ii) is clear. If \( a, \ b \in R \) and \( ab^2 = 0 \) then \( (bab)^2 = 0 \) so \( bab = 0 \), but then \( (ab)^2 = 0 \) so \( ab = 0 \). From this (iii) and (iv) follow immediately.

To prove the theorem it is sufficient to find, for each \( 0 \neq x \in R \), an ideal \( I(x) \) of \( R \) for which \( R/\I(x) \) has no proper divisors of zero and \( x \notin I(x) \). Let \( Z(x) = \{ I \triangleleft R : x \notin I, \text{ if } rx \in I \text{ then } r \in I, \text{ and } R/I \text{ has no nilpotent elements} \} \). By 2.2 \( x_i \in Z(x) \) so \( Z(x) \neq \emptyset \) and it is clear that the union of an ascending chain in \( Z(x) \) is also in \( Z(x) \). Thus we may choose, by Zorn’s Lemma, \( I(x) \) maximal in \( Z(x) \).

If \( a \in R \) and \( a \notin I(x) \) let \( J = \{ y \in R : ay \in I(x) \} \supseteq I(x) \). Then \( J/I(x) = (a + I(x))_a \in R/I(x) \) and by 2.2 (i) \( (a + I(x))_a = (a + I(x))_a \triangleleft R/I(x) \). Since \( a \notin I(x) \), \( ax \in I(x) \) so \( x \notin J \). If \( rx \in J \) then \( arx \in I(x) \) so \( ar \in I(x) \), hence \( r \in J \). Finally by 2.2 (iv) \( R/J \cong R/I(x)/J/I(x) \) has no nilpotent elements, so \( J \in Z(x) \). Hence \( J = I(x) \) so \( R/I(x) \) has no proper divisors of zero.

Note 2.3. The generalized nil radical \( N^g \) of Andrunakievic [4] and Thierrin [10] (see also [6]) is the upper radical with respect to the class of rings without proper divisors of zero. A ring \( R \) is \( N^g \) semi-simple if and only if \( R \) is isomorphic to a subdirect sum of rings without proper divisors of zero. In this context, 2.1 can be restated as: A ring \( R \) is \( N^g \) semi-simple if and only if \( R \) has no nilpotent elements.

3. \( \mathcal{B}_1 \)-rings. If \( x \in R \), let \([x] = \) the subring of \( R \) generated by \( x \).

Definition 3.1. \( R \) is a \( \mathcal{B}_1 \)-ring if for all \( x \in R \), \([x] = [x]^2 \).

Let \( R \) be a ring and \( x \in R \). Clearly \([x] = [x]^2 \) if and only if \( x \in [x]^2 \) if and only if there are integers \( a_2, \cdots, a_k \) such that \( x = \sum_{i=2}^{k} a_i x^i \). Using this it is clear that homomorphic images of \( \mathcal{B}_1 \)-rings are \( \mathcal{B}_1 \)-rings and that if \( A/B \) and \( B \) are \( \mathcal{B}_1 \)-rings then \( A \) is a \( \mathcal{B}_1 \)-ring. It then easily follows that the class of \( \mathcal{B}_1 \)-rings (which we shall denote by \( \mathcal{B}_1 \)) is a radical class.

Lemma 3.2. A nonzero \( \mathcal{B}_1 \)-ring without proper divisors of zero is a field of prime characteristic which is algebraic over its prime subfield.

Proof. Let \( R \) be a nonzero \( \mathcal{B}_1 \)-ring without proper divisors of
zero. If \( x \) is a nonzero element of \( R \) there are integers \( a_2, \ldots, a_k \) such that \( x = \sum_{i=2}^{k} a_i x^i \), hence \( e_x = \sum_{i=2}^{k} a_i x^{i-1} \) is an identity for \([x]\). Since \( x \) is not a zero divisor \( e_x \) is an identity for \( R \). If \( w \in R, w \neq 0, e_w \in [w] = [w]^2 \) so \( e_w \in [w] \cdot w \subseteq Rw \) thus \( R = Rw \). Since \( R \) is nonzero, \( R \) is a division ring.

Let \( e \) be the identity of \( R \). Then \([2e] = [2e]^2 = [4e] \) so \( Ne = 0 \) for some positive integer \( N \). Consequently the characteristic of \( R \) is a prime and since \( e = e_w \in [w] \) for all nonzero \( w \in R, R \) is algebraic over its prime subfield. Therefore, by Theorem 2, page 183 of Jacobson [7] \( R \) is a field.

**Corollary 3.3.** If \( R \) is a \( \mathfrak{B}_1 \)-ring then \( R \) is isomorphic to a subdirect sum of algebraic fields of prime characteristic. So, in particular, \( R \) is commutative.

**Proof.** If \( x \in R, x^N = 0 \) and \( R \in \mathfrak{B}_1 \), then \([x] = [x]^2 = \cdots = [x]^N = (0) \) so \( x = 0 \). Hence \( \mathfrak{B}_1 \)-rings do not have nilpotent elements so the corollary follows from 2.1 and 3.2.

**Theorem 3.4.** A ring \( R \) is a \( \mathfrak{B}_1 \)-ring if and only if every finitely generated subring of \( R \) is isomorphic to a finite direct sum of finite fields.

**Proof.** Let \( R \in \mathfrak{B}_1 \) and \( R' \) be a finitely generated subring of \( R \). Then \( R' \in \mathfrak{B}_1 \) and hence is commutative, so by the Hilbert Basis Theorem \( R' \) has maximum condition on ideals. If \( P' \neq R' \) and \( P' \) is a prime ideal of \( R' \) then \( P' \) is a maximal ideal of \( R' \) since by 3.2 \( R'/P' \) is a field. Since \( R' \) is finitely generated, commutative, and \([g]\) has an identity for each generator \( g \) of \( R', R' \) has an identity. Then by Theorem 2, page 203 of [11] \( R' \) has minimum condition on ideals. But then \( R' \) is a commutative Wedderburn ring so \( R' \) is isomorphic to a finite direct sum of fields each of which must be finite since they are finitely generated, algebraic and of prime characteristic.

The converse is obvious; in fact, if \( x \in R' \) and \( R' \) is isomorphic to a finite direct sum of finite fields then there is an integer \( n(x) \geq 2 \) such that \( x^{n(x)} = x \). Thus we have:

**Corollary 3.5.** \( R \) is a \( \mathfrak{B}_1 \)-ring if and only if for each \( x \in R \) there exists an integer \( n(x) \geq 2 \) such that \( x^{n(x)} = x \).

A class of rings \( \mathfrak{C} \) is said to be hereditary if \( I \triangleleft R \in \mathfrak{C} \) implies that \( I \in \mathfrak{C} \). Analogously we say:

**Definition 3.6.** A class of rings \( \mathfrak{C} \) is strongly hereditary if \( S \) is a subring of \( R \in \mathfrak{C} \) then \( S \in \mathfrak{C} \).
PROPOSITION 3.7. If $\mathcal{F}$ is a strongly hereditary finite set of finite fields then a ring $R$ is isomorphic to a subdirect sum of fields in $\mathcal{F}$ if and only if every finitely generated subring of $R$ is isomorphic to a finite direct sum of fields in $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is a finite set of finite fields there exists an integer $N \geq 2$ such that $x^N = x$ for all $x \in F \in \mathcal{F}$.

Let $R$ have ideals $I_{\alpha}: \alpha \in A$ such that $R/I_{\alpha} \cong F_{\alpha} \in \mathcal{F}$ and $\cap \{I_{\alpha}: \alpha \in A\} = (0)$. Let $R'$ be a finitely generated subring of $R$. Then $R' \in \mathcal{B}_1$ since $x^N = x$ for all $x \in R \supseteq R'$, so by 3.4 $R' \cong A_1 \oplus \cdots \oplus A_k$ and the ideals are finite fields. Choose $a_i \in R'$ such that $[a_i] \cong A_i$. Then $a_i \neq 0$ for some $\beta_i \in A_i$ but $I_{\beta_i} \cap [a_i] \not\subset [a_i]$ so $I_{\beta_i} \cap [a_i] = (0)$. Therefore $A_i \cong [a_i] \cong [a_i] + I_{\beta_i}/I_{\beta_i}$ is isomorphic to a subring of $F_{\beta_i}$. Since $\mathcal{F}$ is strongly hereditary $R'$ is isomorphic to a finite direct sum of fields in $\mathcal{F}$.

Conversely, if every finitely generated subring of $R$ is isomorphic to a finite direct sum of fields in $\mathcal{F}$, $R$ must be a $\mathcal{B}_1$-ring since again $x^N = x$ for all $x \in R$. Thus by 3.3 there are ideals $I_{\alpha}: \alpha \in A$ of $R$ such that $\cap \{I_{\alpha}: \alpha \in A\} = (0)$ and $R/I_{\alpha}$ is a field of prime characteristic; moreover, $R/I_{\alpha}$ must be a finite field since $x^N - x = 0 \in I_{\alpha}$ for all $x \in R$. Therefore, for each $\alpha \in A$, there exists $x_{\alpha} \in R$ such that $[x_{\alpha}] + I_{\alpha}/I_{\alpha} = R/I_{\alpha}$. But then $R/I_{\alpha}$ is a homomorphic image of $[x_{\alpha}]$ so $R/I_{\alpha}$ is isomorphic to a field in $\mathcal{F}$.

4. Semi-simple radical classes.

LEMMA 4.1. If $\mathcal{C}$ is a class of rings such that subdirect sums of rings in $\mathcal{C}$ are in $\mathcal{C}$ and $\mathcal{C}$ satisfies (A) then $\mathcal{C}$ is strongly hereditary.

Proof. Let $R \in \mathcal{C}$ and $S$ be a subring of $R$.

Set $R_i = R$ for all $i \in \mathbb{Z}^+$ = the set of positive integers. Now the (discrete) direct sum $\sum \{R_i: i \in \mathbb{Z}^+\}$ is an ideal of the direct product (complete direct sum) $\prod \{R_i: i \in \mathbb{Z}^+\}$. If $s \in S$ let $\hat{s}(i) = s$ for all $i \in \mathbb{Z}^+$. Then $S \rightarrow \mathcal{J}(S) = \{\hat{s}: s \in S\}$ is an embedding of $S$ into $\prod \{R_i: i \in \mathbb{Z}^+\}$. $\mathcal{J}(S) + \sum \{R_i: i \in \mathbb{Z}^+\}$ is a subdirect sum of copies of $R$ and hence is in $\mathcal{C}$, so

$$S \cong \mathcal{J}(S) \cong \frac{\mathcal{J}(S) + \sum \{R_i: i \in \mathbb{Z}^+\}}{\sum \{R_i: i \in \mathbb{Z}^+\}} \in \mathcal{C}.$$ 

Using a theorem of Amitsur [1] which states that every ring is a homomorphic image of a subdirect sum of total matrix rings of finite order over the ring of all integers, Armendariz in [5] proves
that if a hypernilpotent radical class \( C \) is a semi-simple class, then \( C \) contains all rings. A hypernilpotent radical class is a hereditary radical class which contains all nilpotent rings.

**Theorem 4.2.** If \( C \) is a semi-simple radical class and \( C \not\subseteq \mathcal{B}_1 \), then \( C \) consists of all rings.

*Proof.* Let \( C \) be a semi-simple radical class. If \( C \not\subseteq \mathcal{B}_1 \), then there is a \( R \in C \) and \( x \in R \) such that \( [x] \neq [x]^2 \). In [8] Kurosh shows that for any semi-simple class \( \mathcal{S} \), subdirect sums of rings in \( \mathcal{S} \) are in \( \mathcal{S} \). Thus, by 4.1, \( [x] \in C \) and since \( [x]^2 \not\subseteq [x] \), \( [x]/[x]^2 \in C \). Now \( [x]/[x]^2 \) is a zero ring on a cyclic group and since \( C \) satisfies \((F)\), \( C^\infty \) = the zero ring on the infinite cyclic group is in \( C \). This implies (see [3] and [6]) that \( C \) contains all nilpotent rings. Since \( C \) is a semi-simple class (see [2] and [6]) \( C \) is hereditary, hence \( C \) is hypernilpotent. Therefore, by [5], \( C \) is the class of all rings.

**Theorem 4.3.** If \( C \) is not the class of all rings then the following are equivalent:

1. \( C \) is a semi-simple radical class,
2. there is a strongly hereditary finite set \( C(F) \) of finite fields such that: \( R \in C \) if and only if \( R \) is isomorphic to a subdirect sum of fields in \( C(F) \),
3. there is a strongly hereditary finite set \( C(F) \) of finite fields such that: \( R \in C \) if and only if every finitely generated subring of \( R \) is isomorphic to a finite direct sum of fields in \( C(F) \).

*Proof.* By 3.7 we have that (2) and (3) are equivalent.

Assume that \( C \) satisfies condition (3). Clearly \( C \) satisfies (A) and (E).

If \( B \not\subseteq A \) and both \( A/B \) and \( B \) are in \( C \) and \( A' \) is a finitely generated subring of \( A \) then \( A' + B/B \equiv A'/A' \cap B \) is isomorphic to a finite direct sum of fields in \( C(F) \). A slight modification of the proof given for Proposition 1 on page 241 of Jacobson [7] shows that \( A' \cap B \) is finitely generated as a ring. Thus \( A' \cap B \) is also isomorphic to a finite direct sum of fields in \( C(F) \) and so \( A' \equiv A'/A' \cap B \oplus A' \cap B \). Therefore \( A \in C \). From this it is easy to show that if \( C(R) = \) the sum of all ideals of \( R \) which are in \( C \) then \( C(R) \in C \) and \( C(R/C(R)) = (0) \). Thus, \( C \) satisfies (B) and (C).

If every nonzero ideal of a ring \( R \) can be homomorphically mapped onto a nonzero ring in \( C \) then by 3.7, every nonzero ideal of \( R \) can be homomorphically mapped onto a ring in \( C(F) \). Sulinski [9] (see also [6], Theorem 46) shows that this implies that \( R \) is isomorphic to a subdirect sum of rings in \( C(F) \) and hence by 3.7 again, \( R \in C \). So
\( \mathscr{C} \) satisfies (F) and hence \( \mathscr{C} \) is a semi-simple radical class.

Conversely, suppose \( \mathscr{C} \) satisfies condition (1). Let \( \mathscr{C}(F) = \) the class of all fields which are in \( \mathscr{C} \) and define \( A = \prod \{ R : R \in \mathscr{C}(F) \} \).

Since \( \mathscr{C} \) is a semi-simple class subdirect sums of rings in \( \mathscr{C} \) are in \( \mathscr{C} \); thus \( A \in \mathscr{C} \). By hypothesis, \( \mathscr{C} \subseteq \mathcal{B} \), so by 3.4 all elements of \( A \) must be torsion. From this it follows that there is a finite number of primes \( p_1, \ldots, p_N \) such that every field in \( \mathscr{C}(F) \) is of characteristic \( p_i \) for some \( 1 \leq i \leq N \). For each finite field \( R \in \mathscr{C}(F) \) choose \( a(R) \) such that \( [a(R)] = R \) and for each infinite field \( R \in \mathscr{C}(F) \) set \( a(R) = 0 \). Then \( a = \{a(R)\}_{R \in \mathscr{C}(F)} \) is in \( A \) and by 3.5 \( a^K = a \) for some integer \( K \geq 2 \). Thus, for all finite fields \( R \) in \( \mathscr{C}(F) \), the dimension of \( R \) over its prime subfield is \( \leq K - 1 \). Hence there is only a finite number of finite fields in \( \mathscr{C}(F) \). Suppose there is an infinite field \( R \in \mathscr{C}(F) \).

By 3.2 \( R \) is of prime characteristic and is algebraic over its prime subfield so \( R \) has an infinite number of non-isomorphic finite subfields. All these subfields are in \( \mathscr{C}(F) \) since \( \mathscr{C} \) is strongly hereditary by 4.1. This is impossible since there is only a finite number of finite fields in \( \mathscr{C}(F) \). Therefore \( \mathscr{C}(F) \) is a strongly hereditary finite set of finite fields. If \( R \in \mathscr{C} \) then \( R \in \mathcal{B} \), so by 3.3 \( R \) is isomorphic to a subdirect sum of fields all of which are in \( \mathscr{C}(F) \) since \( \mathscr{C} \) satisfies (A). Conversely, any ring isomorphic to a subdirect sum of rings in \( \mathscr{C}(F) \) is in \( \mathscr{C} \) since \( \mathscr{C} \) is semi-simple class. Thus \( \mathscr{C} \) satisfies (2).

**References**


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