PRODUCT INTEGRAL REPRESENTATION OF TIME
DEPENDENT NONLINEAR EVOLUTION EQUATIONS IN
BANACH SPACES

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The object of this paper is to use the method of product integration to treat the time dependent evolution equation
\[ u'(t) = A(t)(u(t)), \quad t \geq 0, \]
where \( u \) is a function from \([0, \infty)\) to a Banach space \( S \) and \( A \) is a function from \([0, \infty)\) to the set of mappings (possibly nonlinear) on \( S \). The basic requirements made on \( A \) are that for each \( t \geq 0 \) \( A(t) \) is the infinitesimal generator of a semi-group of nonlinear nonexpansive transformations on \( S \) and a continuity condition on \( A(t) \) as a function of \( t \).

The product integration method has been used by T. Kato in [5] to treat evolution equations in which \( A(t) \) is the infinitesimal generator of a semi-group of linear contraction operators. In [6] Kato treats the nonlinear evolution equation in which \( A(t) \) is \( m \)-monotone and the Banach space \( S \) is uniformly convex. For other investigations of nonlinear evolution equations one should see P. Sobolevski [9], F. Browder [1], J. Neuberger [8], and J. Dorroh [3].

1. Definitions and theorems. In this section definitions and theorems will be stated. For examples satisfying the definitions and theorems below, one should see § 4. Let \( S \) denote a real Banach space.

**Definition 1.1.** The function \( T \) from \([0, \infty)\) to the set of mappings (possibly nonlinear) on \( S \) will be said to be a \( \mathcal{C} \)-semi-groups of mappings on \( S \) provided that the following are true:

1. \( T(x + y) = T(x)T(y) \) for \( x, y \geq 0 \).
2. \( T(x) \) is nonexpansive for \( x \geq 0 \).
3. If \( p \in S \) and \( g_p(x) \) is defined as \( T(x)p \) for \( x \geq 0 \) then \( g_p \) is continuous and \( g_p(0) = p \).
4. The infinitesimal generator \( A \) of \( T \) is defined on a dense subset \( D_A \) of \( S \) (i.e., if \( p \in D_{g_p^+}(0) \) exists and \( Ap = g_p^+(0) \)) and if \( p \in D_{g_p^+}(x) = Ag_p(x) \) for \( x \geq 0 \), \( g_p(x) = p + \int_0^x Ag_p(u)du \) for \( x \geq 0 \), \( g_p^+ \) is continuous from the right on \([0, \infty)\), and \( \| g_p^+ \| \) is nonincreasing on \([0, \infty)\).

**Definition 1.2.** The mapping \( A \) from a subset of \( S \) to \( S \) will be said to be a \( \mathcal{C} \)-mapping on \( S \) provided that the following are true:

1. The domain \( D_A \) of \( A \) is dense in \( S \).
(2) \(A\) is monotone on \(S\), i.e., if \(\varepsilon > 0\) and \(p, q \in D\), \(\|(I - \varepsilon A)p - (I - \varepsilon A)q\| \geq \|p - q\|\).

(3) \(A\) is \(m\)-monotone on \(S\), i.e. \(A\) is monotone on \(S\) and if \(\varepsilon > 0\) then \(\text{Range } (I - \varepsilon A) = S\).

(4) \(A\) is the infinitesimal generator of a \(C\)-semi-group of mappings on \(S\).

**DEFINITION 1.3.** Let each of \(m\) and \(n\) be a nonnegative integer and for each integer \(i\) in \([m, n]\) let \(K_i\) be a mapping from \(S\) to \(S\). If \(m > n\) define \(\prod_{i=m}^{n} K_i = I\). If \(m \leq n\) define \(\prod_{i=m}^{n} K_i = K_m\) and if \(m + 1 \leq j \leq n\) define \(\prod_{i=m}^{j} K_i = K_j \prod_{i=j+1}^{n} K_i\). Define \(\prod_{i=m}^{n} K_i = \prod_{i=m}^{n} K_n^{-1}\) if each of \(a\) and \(b\) is a nonnegative number then a chain \(\{s_i\}_{i=0}^{2m}\) from \(a\) to \(b\) is a nondecreasing or nonincreasing number-sequence such that \(s_0 = a\) and \(s_{2m} = b\). The norm of \(\{s_i\}_{i=0}^{2m}\) is max \(|s_{2i} - s_{2i-2}|\) \(i \in [1, m]\).

**DEFINITION 1.4.** Let \(F\) be a function from \([0, \infty) \times [0, \infty)\) to the set of mappings on \(S\). Suppose that \(p \in S, a, b \geq 0\), and \(u\) is a point in \(S\) such that if \(\varepsilon > 0\) there exists a chain \(\{s_i\}_{i=0}^{2m}\) from \(a\) to \(b\) such that if \(\{t_i\}_{i=0}^{n}\) is a refinement of \(\{s_i\}_{i=0}^{2m}\) then
\[
\left\|u - \prod_{i=1}^{n} F(t_{2i-1}, t_{2i} - t_{2i-2})p\right\| < \varepsilon.
\]
Then \(u\) is said to be the product integral of \(F\) from \(a\) to \(b\) with respect to \(p\) and is denoted by \(\prod_{i=0}^{n} F(I, dI)p\).

**REMARK 1.1.** Let \(A\) be a \(C\)-mapping on \(S\) and define the function \(F\) from \([0, \infty) \times [0, \infty)\) to the set of mappings on \(S\) by \(F(u, v) = (I - vA)^{-1}\) for \(u, v \geq 0\) (Note that \((I - vA)^{-1}\) exists and has domain \(S\) by virtue of the \(m\)-monotonicity of \(A\)). The following result in [10] will be used in the theorems below:

If \(A\) is a \(C\)-mapping on \(S\), \(T\) is the \(C\)-semi-group generated by \(A\), and \(F\) is defined as above, then for \(p \in S\) and \(x \geq 0\) \(T(x)p = \prod_{i=0}^{n} F(I, dI)p\).

In this case let \(T(x)\) be denoted by \(\exp(xA)\) for \(x \geq 0\).

Let \(A\) be a function from \([0, \infty)\) to the set of mappings on \(S\) such that the following are true:

(1) For each \(t \geq 0\) \(A(t)\) is a \(C\)-mapping on \(S\)

(II) There is a dense subset \(D\) of \(S\) such that if \(t \geq 0\) the domain of \(A(t)\) is \(D\)

(III) \(A\) is continuous in the following sense: If \(a, b \geq 0, M\) is a bounded subset of \(D\), and \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(u, v \in [a, b]\) and \(|u - v| < \delta\) then \(\|A(u)z - A(v)z\| < \varepsilon\) for each \(z \in M\).
THEOREM 1. Let $A$ satisfy conditions (I), (II) and (III). If $p \in S$ and $a, b \geq 0$ the following are true:

1. If $T(u, v) = \exp(vA(u))$ for $u, v \geq 0$, then $\prod_{a}^{b} T(I, dI)p$ exists.
2. If $L(u, v) = (I - vA(u))^{-1}$ for $u, v \geq 0$, then $\prod_{a}^{b} L(I, dI)p$ exists and $\prod_{a}^{b} L(I, dI)p = \prod_{a}^{b} T(L, dI)p$.

THEOREM 2. Let $A$ satisfy conditions (I), (II) and (III) and define $U(b, a)p = \prod_{a}^{b} T(I, dI)p$ for $p \in S$ and $a, b \geq 0$. The following are true:

1. $U(b, a)$ is nonexpansive for $a, b \geq 0$.
2. $U(b, c)U(c, a) = U(b, a)$ for $a, b \geq 0$ and $c \in [a, b]$ and $U(a, a) = I$ for $a \geq 0$.
3. If $p \in S$ and $a \geq 0$ then $U(a, t)p$ is continuous in $t$.
4. If $p \in S$, $0 \leq a \leq t$, and $U(t, a)p \in D$, then $\frac{\partial}{\partial t} U(t, a)p|_{t=a} = A(t)U(t, a)p$ and if $p \in S$, $0 < s \leq b$, and $U(s, b)p \in D$, then $\frac{\partial}{\partial s} U(s, b)p|_{s=b} = -A(s)U(s, b)p$.

2. Product integral representations. In this section, Theorems 1 and 2 will be proved. Before proving part (1) of Theorem 1 three lemmas will be proved each under the hypothesis of Theorem 1.

**Lemma 1.1.** If $p \in D$, $a, b \geq 0$, and $\{s_{i}\}_{i=0}^{m}$ is a chain from $a$ to $b$ then

$$\left\| \prod_{i=1}^{m} T(s_{2i} - s_{2i-2})p - p \right\| \leq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \left\| A(s_{2i-2})p \right\|.$$

**Proof.**

$$\left\| \prod_{i=1}^{m} T(s_{2i} - s_{2i-2})p - p \right\|$$

$$\leq \sum_{i=1}^{m} \left\| \prod_{j=1}^{m} T(s_{2j} - s_{2j-2})p - \prod_{j=1}^{m} T(s_{2j-2} - s_{2j-4})p \right\|$$

$$\leq \sum_{i=1}^{m} \left\| T(s_{2i-2}, s_{2i} - s_{2i-2})p - p \right\|$$

$$= \sum_{s_{i-1}}^{m} \int_{0}^{s_{2i-2} - s_{2i-4}} A(s_{2i-2})T(s_{2i-2}, t)pdt$$

$$\leq \sum_{s_{i-1}}^{m} |s_{2i} - s_{2i-2}| \left\| A(s_{2i-4})p \right\|.$$

**Lemma 1.2.** If $p \in D$, $a, b \geq 0$, $\{s_{i}\}_{i=0}^{m}$ is a chain from $a$ to $b$, and $\{s'_{i}\}_{i=1}^{m}$ is a sequence in $[a, b]$, then

$$\left\| \prod_{i=1}^{m} L(s_{i}, s_{2i} - s_{2i-2})p - p \right\| \leq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \left\| A(s'_{i})p \right\|.$$

**Proof.**
\[
\left\| \prod_{i=1}^{m} L(s'_i, |s_{2i} - s_{2i-2}|)p - p \right\| \\
\leq \sum_{i=1}^{m} \left\| \prod_{j=1}^{m} L(s'_j, |s_{2j} - s_{2j-2}|)p - \prod_{j=1}^{m} L(s'_j, |s_{2j} - s_{2j-2}|)p \right\| \\
\leq \sum_{i=1}^{m} \left\| L(s'_i, |s_{2i} - s_{2i-2}|)p - p \right\| \\
= \sum_{i=1}^{m} \left\| L(s'_i, |s_{2i} - s_{2i-2}|)p - L(s'_i, |s_{2i} - s_{2i-2}|)(I - |s_{2i} - s_{2i-2}|A(s'_i))p \right\| \\
\leq \sum_{i=1}^{m} |s_{2i} - s_{2i-2}| \cdot \left\| A(s'_i)p \right\| .
\]

**Lemma 1.3.** If \( M \) is a bounded subset of \( D, a, b \geq 0, \gamma > 0, \) and \( \varepsilon > 0, \) there exists \( \delta > 0 \) such that if \( u, v \in [a, b], |u - v| < \delta, \) \( 0 \leq x < \gamma, \) and \( z \in M, \) then \( \left\| T(u, x)z - T(v, x)z \right\| \leq x \cdot \varepsilon. \)

**Proof.** Let \( M' = \{ \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \mid z \in M, v \in [a, b], 0 \leq x < \gamma, \) and \( \{s_i\}_{i=0}^{2m} \) is a chain from 0 to \( x \}. \) Let \( z_0 \in M, \) let \( z \in M, \) let \( v \in [a, b], \) let \( 0 \leq x < \gamma, \) and let \( \{s_i\}_{i=0}^{m} \) be a chain from 0 to \( x. \) Then,
\[
\left\| \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z - \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z_0 \right\| \leq \left\| z - z_0 \right\| .
\]

Further, by Lemma 1.2,
\[
\left\| \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z_0 - z_0 \right\| \leq x \cdot \max_{u \in [0, x]} \left\| A(u)z_0 \right\| .
\]

Then, \( \left\| \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \right\| \leq \left\| z - z_0 \right\| + \left\| z_0 \right\| + x \cdot \max_{u \in [0, x]} \left\| A(u)z_0 \right\| \) and so \( M' \) is bounded. There exists \( \delta > 0 \) such that if \( u, v \in [a, b], |u - v| < \delta, \) and \( z \in M', \) then \( \left\| A(u)z - A(v)z \right\| < \varepsilon. \) Then if \( 0 \leq x < \gamma, \) \( z \in M, \) \( \{s_i\}_{i=0}^{2m} \) is a chain from 0 to \( x, u, v \in [a, b], \) and \( |u - v| < \delta, \)
\[
\left\| \prod_{i=1}^{m} L(u, s_{2i} - s_{2i-2})z - \prod_{i=1}^{m} L(v, s_{2i} - s_{2i-2})z \right\| \\
\leq \sum_{i=1}^{m} \left\| \prod_{j=1}^{m} L(u, s_{2j} - s_{2j-2}) \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z \right\| \\
- \prod_{j=1}^{m} L(u, s_{2j} - s_{2j-2}) \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z \right\| \\
\leq \sum_{i=1}^{m} \left\| L(u, s_{2i} - s_{2i-2}) \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z \right\| \\
- \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z \right\| \\
\leq \sum_{i=1}^{m} \left\| \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z \right\| \\
- (I - (s_{2i} - s_{2i-2})A(u)) \prod_{k=1}^{i-1} L(v, s_{2k} - s_{2k-2})z \right\| .
\]
\[
= \sum_{i=1}^{m} (s_{2i} - s_{2i-2}) A(v) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2}) z \\
- A(u) \prod_{k=1}^{i} L(v, s_{2k} - s_{2k-2}) z \\
< \sum_{i=1}^{m} (s_{2i} - s_{2i-2}) \varepsilon \\
= x \cdot \varepsilon.
\]

Then, since \( T(u, x)z = \prod_{i=1}^{x} L(u, dI)z \) and \( T(v, x)z = \prod_{i=1}^{x} L(v, dI)z \) (see Remark 1.1), \( \| T(u, x)z - T(v, x)z \| \leq x \cdot \varepsilon. \)

**Proof of Part (1) of Theorem 1.** Let \( p \in D, \) let \( a, b \geq 0, \) and let \( \varepsilon > 0. \) Let \( M = \{ \prod_{i=1}^{m} T(r_{2i-1}, r_{2i-2}) \mid p \mid x \in [a, b] \) and \( \{ r_i \}_{i=0}^{2m} \) is a chain from \( a \) to \( x \}. \) Then \( M \) is a bounded subset of \( D \) by Lemma 1.1. There exists \( \delta > 0 \) such that if \( u, v \in [a, b], \) \( \| u - v \| < \delta, 0 \leq x \leq 1 \) and \( z \in M, \) then \( \| T(u, x)z - T(v, x)z \| \leq \varepsilon \cdot x. \) Let \( \{ s_i \}_{i=0}^{2m} \) be a chain from \( a \) to \( b \) with norm \( < \min \{ \delta, 1 \} \) and let \( \{ t_i \}_{i=0}^{2m} \) be a refinement of \( \{ s_i \}_{i=0}^{2m}, \) i.e., there is an increasing sequence \( \{ u_i \} \) such that \( u_0 = 0, u_m = n, \) and if \( 1 \leq i \leq m, \) let \( K_i = T(s_{2i-1}, s_{2i} - s_{2i-1}) \) and let \( J_i = \prod_{j=1}^{x} T(t_{2i-1}, t_{2i} - t_{2i-2} \} . \) Then,

\[
\left\| \prod_{i=1}^{m} T(t_{2i-1}, t_{2i} - t_{2i-2} \} p - \prod_{i=1}^{m} T(s_{2i-1}, s_{2i} - s_{2i-2} \} p \right\|
\leq \sum_{i=1}^{m} \| J_i \prod_{j=1}^{x} K_k p - \prod_{j=1}^{x} J_j \prod_{k=1}^{i} K_k p \|
\leq \sum_{i=1}^{m} \| J_i \prod_{k=1}^{i} K_k p - \prod_{k=1}^{i} K_k \| \|
\leq \sum_{i=1}^{m} \prod_{j=1}^{x} T(t_{2i-1}, t_{2j} - t_{2j-2} \} \prod_{k=1}^{i-1} K_k p
- \prod_{j=1}^{x} T(s_{2i-1}, s_{2j} - s_{2j-2} \} \prod_{k=1}^{i-1} K_k p
\leq \sum_{i=1}^{m} \sum_{j=1}^{x} \| T(t_{2i-1}, t_{2j} - t_{2j-2} \} \prod_{k=1}^{i-1} K_k p
- T(t_{2j-1}, t_{2j} - t_{2j-2} \} \prod_{k=1}^{i-1} K_k p \|
\leq b - a \cdot \varepsilon.
\]

Hence, \( \prod_{i=1}^{x} T(I, dI) p \) exists. Further, using the fact that \( D \) is dense
in S and T(u, x) is nonexpansive for u, x ≥ 0 one sees that if p ∈ S, a, b ≥ 0, then Π戈 T(I, dI)p exists and thus part (1) of Theorem 1 is proved.

Before proving part (2) of Theorem 1 three lemmas will be proved each under the hypothesis of Theorem 1.

**Lemma 1.4.** If p, q ∈ S, a, c ≥ 0, and b ∈ [a, c], then the following are true:

(i) \[ ||\prod_{n} T(I, dI)p - \prod_{n} T(I, dI)q|| \leq ||p - q||.\]

(ii) \[ \prod_{n} T(I, dI) \prod_{n} T(I, dI)p = \prod_{n} T(I, dI)p.\]

(iii) If p ∈ D then \[ ||\prod_{n} T(I, dI)p - p|| \leq \|b - a\| \cdot \max_{u \in [a, b]} ||A(u)p||.\]

**Proof.** Parts (i) and (ii) follow from the nonexpansive property of T(u, x), u, x ≥ 0. Part (iii) follows from Lemma 1.1.

**Lemma 1.5.** If M is a bounded subset of D, a, b ≥ 0, and ε > 0, there exists δ > 0 such that if u, v ∈ [a, b], |v - u| < δ, w ∈ [u, v], and z ∈ M, then

\[ ||\prod_{n} T(I, dI)z - T(w, |v - u|)z|| \leq |v - u| \cdot \varepsilon.\]

**Proof.** Let M' = \{\prod_{n} T(s_{2i-1}, s_{2i} - s_{2i-2}|)z | z ∈ M, y ∈ [a, b], \{s_i\}_{i=0}^{\infty} is a chain from y to x\}. Then M' is a bounded subset of D by Lemma 1.1. By Lemma 1.3 there exists δ > 0 such that if u, v ∈ [a, b], |u - v| < δ, z ∈ M' and 0 ≤ x ≤ 1, then \[ ||T(u, x)z - T(v, x)z|| \leq x \cdot \varepsilon.\] Let u, v ∈ [a, b], |v - u| < min {δ, 1}, w ∈ [u, v], z ∈ M, and let \{s_{i}\}_{i=0}^{\infty} be a chain from u to v. Then,

\[ ||\prod_{n} T(s_{2i-1}, s_{2i} - s_{2i-2}|)z - T(w, |v - u|)z|| = \leq \sum_{i=1}^{\infty} ||T(s_{2i-1}, s_{2i} - s_{2i-2}|)z - \prod_{n} T(s_{2i-1}, s_{2i} - s_{2i-2}|)z|| \]

\[ \leq \sum_{i=1}^{\infty} \left( T(s_{2i-1}, s_{2i} - s_{2i-2}|)z - \prod_{n} T(s_{2i-1}, s_{2i} - s_{2i-2}|)z \right) \]

\[ = |v - u| \cdot \varepsilon.\]

Thus, \[ ||\prod_{n} T(I, dI)z - T(w, |v - u|)z|| \leq |v - u| \cdot \varepsilon.\]

**Lemma 1.6.** If M is a bounded subset of D, a, b ≥ 0, and ε > 0, there exists δ > 0 such that if u, v ∈ [a, b], w ∈ [u, v], |v - u| < δ, z ∈ M,
and \( \{s_i\}_{i=0}^{2m} \) is a chain from \( u \) to \( v \), then
\[
\left\| \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^{m} L(w, |s_{2i} - s_{2i-2}|)z \right\| \leq |v - u| \cdot \varepsilon.
\]

**Proof.** An argument similar to the one in Lemma 1.3 proves Lemma 1.6.

**Proof of Part (2) of Theorem 1.** Let \( p \in D, a, b \geq 0 \), and \( \varepsilon > 0 \). Let \( M = \{\prod_{i} T(I, dI)p \mid x \in [a, b]\} \). Then \( M \) is a bounded subset of \( D \) by Lemma 1.4. By Lemmas 1.5 and 1.6 there exists \( \delta > 0 \) such that if \( u, v \in [a, b], w \in [u, v], |u - v| < \delta, z \in M \), and \( \{s_i\}_{i=0}^{2m} \) is a chain from \( u \) to \( v \), then
\[
\left\| \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)z - \prod_{i=1}^{m} L(w, |s_{2i} - s_{2i-2}|)z \right\| \leq |v - u| \cdot \varepsilon/3 | b - a |
\]
and \( \| \prod_{i} T(I, dI)z - T(w, |v - u|)z \| \leq |v - u| \cdot \varepsilon/3 | b - a | \). Let \( \{r_i\}_{i=0}^{2m} \) be a chain from \( a \) to \( b \) with norm \( < \delta \). Let \( \{s_i\}_{i=0}^{2m} \) be a refinement of \( \{r_i\}_{i=0}^{2m} \) such that there exists an increasing sequence \( u \) such that \( u_0 = 0, u_q = m \), if \( 1 \leq i \leq q r_{2i} = s_{2m+i} \), and if \( 1 \leq i \leq q \) and \( \{t_{ij}\}_{j=0}^{2m} \) is a refinement of \( \{s_{ij}\}_{i=0}^{2m} \), then
\[
\left\| \prod_{k=1}^{n} L(r_{2i-1}, |t_{2k} - t_{2k-2}|) \prod_{a} T(I, dI)p - T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_{a} T(I, dI)p \right\| \leq |r_{2i} - r_{2i-2}| \cdot \varepsilon/3 | b - a | \cdot
\]
(Note that if \( 1 \leq i \leq q T(r_{2i-1}, |r_{2i} - r_{2i-2}|) \prod_{a} T(I, dI)p = \prod_{a} L(r_{2i-1}, dI) \prod_{a} T(I, dI)p = \prod_{a} L(r_{2i}, dI) \prod_{a} T(I, dI)p \)
—see Remark 1.1). Let \( \{t_{ij}\}_{i=0}^{2m} \) be a refinement of \( \{s_{ij}\}_{i=0}^{2m} \) and let \( v \) be an increasing sequence such that \( v_0 = 0, v_m = n \), and if \( 1 \leq i \leq m s_{2i} = t_{2i} \). Then,
\[
\left\| \prod_{i=1}^{n} L(t_{2i-1}, |t_{2i} - t_{2i-2}|)p - \prod_{a} T(I, dI)p \right\|
= \left\| \prod_{i=1}^{q} \prod_{j=a_{i-1}+1}^{v_{i+1}} L(t_{2k-1}, |t_{2k} - t_{2k-2}|)p - \prod_{a} T(I, dI)p \right\|
\leq \sum_{i=1}^{q} \left\| \prod_{j=a_{i-1}+1}^{v_{i+1}} L(t_{2k-1}, |t_{2k} - t_{2k-2}|) \right\| T(I, dI)p
- \prod_{a} T(I, dI) \prod_{a} T(I, dI)p \]
\[ \sum_{i=1}^{q} \left| r_{2i} - r_{2i-1} \right| \cdot \frac{\varepsilon}{3} |b - a| \\
+ \sum_{i=1}^{q} \left| \prod_{j=m_{i-1}+1}^{m_{i}} \prod_{k=v_{j-1}+1}^{v_{j}} L(r_{2i-1} - t_{2k} - t_{2k-1}) \prod_{a}^{r_{2i-2}} T(I, dI)p \right| \\
- T(r_{2i-1} - r_{2i-2}) \prod_{a}^{r_{2i-2}} T(I, dI)p \right| \\
+ \sum_{i=1}^{q} \left| r_{2i} - r_{2i-2} \right| \cdot \frac{\varepsilon}{3} |b - a| \leq \varepsilon . \]

Thus, \( \prod_{a}^{b} L(I, dI)p \) exists and is \( \prod_{a}^{b} T(I, dI)p \) for \( p \in D \). Further, using the fact that \( D \) is dense in \( S \) and \( L(u, x) \) is nonexpansive for \( u, x \geq 0 \) one sees that \( \prod_{a}^{b} L(I, dI)p = \prod_{a}^{b} T(I, dI)p \) for all \( p \in S \).

Define \( U(b, a)p = \prod_{a}^{b} T(I, dI)p \) for \( p \in S \) and \( a, b \geq 0 \).

**Proof of Theorem 2.** Parts (1), (2), and (3) of Theorem 2 follow from Lemma 1.4. Suppose that \( p \in S, 0 \leq a \leq t, \) and \( U(t, a)p \in D \). Let \( \varepsilon > 0 \). There exists \( \delta_i > 0 \) such that if \( 0 < h < \delta_i \)

\[ \left\| A(t)T(t, h)U(t, a)p - A(t)U(t, a)p \right\| < \varepsilon/2 \]

(see Definition 1.1, part (4)). By Lemma 1.5 there exists \( \delta_2 > 0 \) such that if \( 0 < h < \delta_2 \) \( \left\| U(t + h, t)U(t, a)p - T(t, h)U(t, a)p \right\| < h \cdot \varepsilon/2 \). Then, if \( 0 < h < \min \{ \delta_i, \delta_2 \} \),

\[ \left\| (1/h)(U(t + h, a)p - U(t, a)p) - A(t)U(t, a)p \right\| \\
= \left\| (1/h)(U(t + h, t)U(t, a)p - U(t, a)p) - A(t)U(t, a)p \right\| \\
< \varepsilon/2 + \left\| (1/h)(T(t, h)U(t, a)p - U(t, a)p) - A(t)U(t, a)p \right\| \\
= \varepsilon/2 + \left\| 1/h \int_{0}^{h} [A(t)T(t, u)U(t, a)p - A(t)U(t, a)p]du \right\| < \varepsilon . \]

Hence, \( \partial^+ U(t, a)p/\partial t = A(t)U(t, a)p \). Suppose that \( p \in S, 0 < s \leq b, \) and \( U(s, b)p \in D \). Let \( \varepsilon > 0 \). There exists \( \delta_i > 0 \) such that if \( 0 < h < \delta_i \), then \( 0 \leq s - h \) and \( \left\| (A(s)T(s, h)U(s, b)p - A(s)U(s, b)p \right\| < \varepsilon/2 \). By Lemma 1.5 there exists \( \delta_2 > 0 \) such that if \( 0 < h < \delta_2 \)

\[ \left\| U(s - h, s)U(s, b)p - T(s, h)U(s, b)p \right\| < h \cdot \varepsilon/2 . \]

Then, if \( 0 < h < \min \{ \delta_i, \delta_2 \} \)

\[ \left\| (1/h)(U(s - h, b)p - U(s, b)p) - (-A(s)U(s, b)p) \right\| \\
= \left\| (1/h)(U(s - h, s)U(s, b)p - U(s, b)p) - A(s)U(s, b)p \right\| \\
< \varepsilon/2 + \left\| (1/h)(T(s, h)U(s, b)p - U(s, b)p) - A(s)U(s, b)p \right\| \\
= \varepsilon/2 + \left\| 1/h \int_{0}^{h} [A(s)T(s, u)U(s, b)p - A(s)U(s, b)p]du \right\| < \varepsilon . \]

Hence, \( \partial^- U(s, b)p/\partial s = -A(s)U(s, b)p \).
3. **Product integral representation in the uniform case.** For Theorem 3 $A$ is required to satisfy, in addition to conditions (I), (II), (III) of § 1, the following:

(IV) For each $t \geq 0$ $A(t)$ has domain all of $S$.

(V) If $0 \leq a \leq b$, $M$ is a bounded subset of $S$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \in [a, b], z, w \in M$, and $|| z - w || < \delta$, then

$$|| A(u)z - A(u)w || < \varepsilon.$$ 

**Theorem 3.** Let $A$ satisfy conditions (I)—(V) and define

$$M(u, v) = (I + vA(u))$$

for $u, v \geq 0$. If $p \in S$ and $a, b \geq 0$, then $\prod_{i=1}^{\infty} M(I, dI)p = U(b, a)p$.

Before proving Theorem 3, three lemmas will be proved each under the hypothesis of Theorem 3.

**Lemma 3.1.** Let $p \in S$ and let $a, b \geq 0$. There is a neighborhood $N_{p, \varepsilon}$ about $p$, a positive number $\gamma$, and a positive number $K$ such that if $q \in N_{p, \varepsilon}$, $x, y \in [a, b], |y - x| < \gamma$, and $\{s_{i}\}_{i=0}^{m}$ is a chain from $x$ to $y$, then

$$\left| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - q \right| \leq |y - x| \cdot K.$$

**Proof.** There exists a positive number $K$ such that if $u \in [a, b]$ and $q \in N_{p, \varepsilon}$, then $|| A(u)q || \leq K$. Let $\delta = 1/2$ and let $\gamma = 1/2K$. Let $q \in N_{p, \varepsilon}$, $x, y \in [a, b], |y - x| < \gamma$, $\{s_{i}\}_{i=0}^{m}$ a chain from $x$ to $y$, $1 \leq j \leq m - 1$, and suppose that $|| \prod_{i=1}^{j} M(s_{2i+1}, s_{2i} - s_{2i-2})q - q || \leq |s_{2j+1} - s_{2j} - s_{0}| \cdot K$. Then, $\prod_{i=1}^{j} M(s_{2i+1}, s_{2i} - s_{2i-2})q \in N_{p, \varepsilon}$ and so

$$\left| \prod_{i=1}^{j+1} M(s_{2i+1}, s_{2i} - s_{2i-2})q - q \right|$$

$$\leq \left| \prod_{i=1}^{j} M(s_{2i+1}, s_{2i} - s_{2i-2})q - q \right|$$

$$+ |s_{2j+2} - s_{2j+1}| \cdot \left| A(s_{2j+1}) \prod_{i=1}^{j} M(s_{2i+1}, s_{2i} - s_{2i-2})q \right|$$

$$\leq |s_{2j+2} - s_{2j+1}| \cdot K.$$

**Lemma 3.2.** If $p \in S$ and $a \geq 0$ then $U(t, a)p$ is continuous in $t$.

**Proof.** Let $p \in S$ and $a, b \geq 0$. In a manner similar to Lemma 3.1 one proves the following: There is a neighborhood $N_{q, \varepsilon}$ about $q = U(b, a)p, \gamma > 0$, and $K > 0$ such that if $z \in N_{q, \varepsilon}$, $x, y \in [a, b], |y - x| < \gamma$, and $\{s_{i}\}_{i=0}^{m}$ is a chain from $x$ to $y$ then
\[ \left\| \prod_{i=1}^{m} \left( I - s_{2i} - s_{2i-2} |A(s_{2i-1})| z - z \right) \right\| \leq |y - x| \cdot K. \]

Let \( \varepsilon > 0 \), let \( x \in [a, b] \) such that \( |x - b| < \gamma \), let \( \{s_i\}_{i=0}^{2m} \) be a chain from \( a \) to \( b \) and \( k \leq m \) an integer such that \( s_{2k} = x \) and

\[ \left\| U(b, a)p - \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p \right\| < \min \{\varepsilon, \delta\} \]

and

\[ \left\| U(x, a)p - \prod_{i=1}^{k} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p \right\| < \varepsilon. \]

Then,

\[
\begin{align*}
\left\| U(x, a)p - U(b, a)p \right\| &< 2\varepsilon + \left\| \prod_{i=1}^{m} \left( I - s_{2i} - s_{2i-2} |A(s_{2i-1})| \right) \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p \\
&\quad - \prod_{i=1}^{m} L(s_{2i-1}, |s_{2i} - s_{2i-2}|)p \right\| \\
&< 2\varepsilon + |b - x| \cdot K.
\end{align*}
\]

Then, \( \lim_{x \to b} U(x, a)p = U(b, a)p \) for \( x \in [a, b] \). Further, by Lemma 1.4 \( \lim_{x \to b} U(x, a)p = U(b, a)p \) for \( x \in [a, b] \).

**Lemma 3.3.** Let \( p \in S \) and \( a \geq 0 \). There exists a neighborhood \( N_{p, \delta} \) about \( p \) and \( \gamma > 0 \) such that the following are true:

1. If \( \varepsilon > 0 \) there exists \( \alpha > 0 \) such that if \( q \in N_{p, \delta}, a \leq x \leq a + \gamma \), and \( \{s_i\}_{i=0}^{2m} \) is a chain from \( a \) to \( x \) with norm \( < \alpha \), then

\[ \left\| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - U(x, a)q \right\| < \varepsilon. \]

and

2. If \( \varepsilon > 0 \) there exists \( \alpha > 0 \) such that if \( q \in N_{p, \delta}, \max \{0, a - \gamma\} \leq x \leq a \), and \( \{s_i\}_{i=0}^{2m} \) is a chain from \( a \) to \( x \) with norm \( < \alpha \), then

\[ \left\| \prod_{i=1}^{m} M(s_{2i-1}, |s_{2i} - s_{2i-2}|)q - U(x, a)q \right\| < \varepsilon. \]

**Proof.** By Lemma 3.1 there exists \( \delta > 0 \) and \( \gamma > 0 \) such that if \( q \in N_{p, \delta}, a \leq x \leq a + \gamma \), and \( \{s_i\}_{i=0}^{2m} \) is a chain from \( a \) to \( x \) then

\[ \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q \in N_{p, 2\delta}. \]

Let \( \varepsilon > 0 \). By Lemma 1.5 there exists \( \alpha_i > 0 \) such that if

\[ u, v \in [a, a + \gamma], 0 \leq v - u < \alpha_i, u \leq w \leq v, \]
and \( q \in N_{p,2} \), then \( ||U(v, u)q - T(w, v - u)q|| \leq (v - u) \cdot \varepsilon / 2\gamma \). There exists \( \alpha_2 > 0 \) such that if \( q \in N_{p,2} \), \( u \in [a, a + \gamma] \), and \( 0 \leq x < \alpha_2 \), then \( ||A(u)T(u, x)q - A(u)q|| < \varepsilon / 2\gamma \) (Note that
\[
|| T(u, x)q - q || = \left| \int_0^x A(u)T(u, t)q \, dt \right| \leq x \cdot ||A(u)q|| \leq x \cdot (\max \| A(t)z \|, t \in [a, a + \gamma], z \in N_{p,2})) .
\]
Let \( \alpha = \min \{ \alpha_1, \alpha_2 \} \), let \( q \in N_{p,2} \), let \( a \leq \alpha < a + \gamma \), and let \( \{ s_i \}_{i=0}^{2m} \) be a chain from \( a \) to \( x \) with norm \( \alpha \). Then,
\[
\left| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - U(x, a)q \right|
= \left| \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})q - \prod_{i=1}^{m} U(s_{2i}, s_{2i-2})q \right|
\leq \sum_{i=1}^{m} \left| U(s_{2i}, s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q - M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right|
= \varepsilon / 2 + \sum_{i=1}^{m} \left| T(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q - M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q \right|
= \varepsilon / 2 + \sum_{i=1}^{m} \left| \int_0^{s_{2i}-s_{2i-2}} [A(s_{2i-1}) T(s_{2i-1}, t) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q - A(s_{2i-1}) \prod_{j=1}^{i-1} M(s_{2j-1}, s_{2j} - s_{2j-2})q] \, dt \right|
< \varepsilon / 2 + \sum_{i=1}^{m} (s_{2i} - s_{2i-2}) \cdot \varepsilon / 2\gamma < \varepsilon .
\]
A similar argument proves part (2) of the lemma.

**Proof of Theorem 3.** Let \( p \in \mathcal{S} \) and \( 0 \leq a < b \). Suppose that if \( a \leq x < b \prod_{i=1}^{\ell} M(I, dI)p \) exists and is \( U(x, a)p \). Let \( a \leq x < b \), let \( \{ s_i \}_{i=0}^{2m} \) be a chain from \( a \) to \( b \), and let \( j < m \) such that \( s_{2j} = x \). One uses the inequality
\[
\| U(b, a)p - \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})p \|
\leq \| U(b, a)p - \prod_{i=1}^{\ell} M(I, dI)p \|
+ \| \prod_{i=1}^{\ell} M(I, dI)p - \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \|
+ \| \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \|
- \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \|
\]
and Lemmas 3.1 and 3.2 to show $\prod_{i} M(I, dI)p$ exists and is $U(b, a)p$.
Suppose now that for $a \leq x \leq b$ $\prod_{i} M(I, dI)p = U(x, a)p$. Let $b < x$, let $\{s_{i}\}_{i=0}^{m}$ be a chain from $a$ to $x$, and let $j < m$ such that $s_{2j} = b$. One uses the inequality

\[
\left\| U(x, a)p - \prod_{i=1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\|
\]

\[
\leq \left\| U(x, b)U(b, a)p - U(x, b) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\|
\]

\[
+ \left\| U(x, b) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\|
\]

\[
- \prod_{i=j+1}^{m} M(s_{2i-1}, s_{2i} - s_{2i-2}) \prod_{i=1}^{j} M(s_{2i-1}, s_{2i} - s_{2i-2})p \right\|
\]

and Lemma 3.3 to show that there exists $\gamma > 0$ such that if $b \leq x < b + \gamma$ then $\prod_{i} M(I, dI)p$ exists and is $U(x, a)p$. Thus, if $p \in S$ and $0 \leq a \leq b$ then $\prod_{i} M(I, dI)p$ exists and is $U(b, a)p$. With a similar argument one shows that for $p \in S$ and $0 \leq a \leq b$ $\prod_{i} M(I, dI)p$ exists and is $U(a, b)p$.

4. Examples. In conclusion two examples will be given.

**Example 1.** Let $S$ be the Hilbert space and let $A$ be densely defined and $m$-monotone on $S$ (Definition 1.2). In M. Crandall and A. Pazy [2] and in T. Kato [6], it is shown that $B$ is the infinitesimal generator of a $\mathcal{C}$-semi-group on $S$ (Definition 1.1). Let $X$ be a function from $[0, \infty)$ to $S$ such that $X$ is continuous. Define $A(t)p = Bp + X(t)$ for $p \in \text{Domain}(B)$ and $t \geq 0$. Then $A$ satisfies conditions (I)—(III).

**Example 2.** Let $S$ be a Banach space and let $B$ be a mapping from $S$ to $S$ such that $B$ is $m$-monotone $S$ and uniformly continuous on bounded subsets of $S$. In [11] it is shown that $B$ is the infinitesimal generator of a $\mathcal{C}$-semi-group of mappings on $S$. Let $C$ be a continuous mapping from $[0, \infty)$ to $[0, \infty)$, let $D$ be a continuous mapping from $[0, \infty)$ to $(0, \infty)$, and let each of $E$ and $F$ be a continuous mapping from $[0, \infty)$ to $S$. Define $A(t)p = C(t)\cdot B(D(t)\cdot p + E(t)) + F(t)$ for $t \geq 0$ and $p \in S$. Suppose $t \geq 0$, $\varepsilon > 0$, and $p, q \in S$. Then,

\[
\left\| (I - \varepsilon A(t))p - (I - \varepsilon A(t))q \right\|
\]

\[
= (1/D(t)) \left\| (I - \varepsilon C(t)D(t)B)(D(t)p + E(t)) \right\|
\]

\[
- (I - \varepsilon C(t)D(t)B)(D(t)q + E(t)) \right\|
\]

\[
\geq (1/D(t)) \left\| (D(t)p + E(t)) - (D(t)q + E(t)) \right\|
\]

\[
= \left\| p - q \right\|
\]

and so $A(t)$ is monotone for $t \geq 0$. Suppose $t \geq 0$, $\varepsilon > 0$, and $p \in S$. Let $q'$ be in $S$ such that $(I - \varepsilon C(t)D(t)B)q' = D(t)p + E(t) + \varepsilon D(t)F(t)$.
Let \( q = (1/D(t))(q' - E(t)) \). Then \((I - \varepsilon A(t))q = p\) and so \(A(t)\) is \(m\)-monotone. Then \(A\) satisfies conditions (I)—(V).

**References**


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