

# Pacific Journal of Mathematics

## **PREASSIGNING THE SHAPE OF A FACE**

DAVID WILMOT BARNETTE AND BRANKO GRÜNBAUM

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**THEOREM.** Let  $G$  be a 3-connected planar graph,  $F$  a nonseparating  $n$ -circuit of  $G$  with nodes  $A_1, \dots, A_n$  (in a cyclic order of  $F$ ), and let  $F'$  be a convex  $n$ -gon with vertices  $A'_1, \dots, A'_n$  (in cyclic order). Then there exists a 3-polytope  $P$  which realizes  $G$ , such that  $F'$  is a face of  $P$  and that the vertex  $A'_i$  of  $P$  corresponds to the node  $A_i$  of  $G$  for  $i = 1, 2, \dots, n$ .

A remarkable theorem of Steinitz (see [2, p. 77], [3, p. 192], [1, p. 235]) asserts that a graph  $G$  is realizable by a 3-polytope (i.e., isomorphic to the graph of the vertices and edges of a 3-dimensional convex polytope  $P$ ) if and only if  $G$  is planar and 3-connected. Moreover, this realization is unique up to combinatorial equivalence of the 3-polytopes, the boundaries of the 2-faces of each polytope realizing  $G$  being determined by nonseparating circuits in  $G$ . (The regions of the plane determined in any imbedding of  $G$  in the plane by the nonseparating circuits of  $G$  will be called *faces* of  $G$ .)

Combinatorially equivalent polytopes may—intuitively speaking—have various shapes. Among the different problems concerning the freedom of choice of 3-polytopes realizing a given graph  $G$ , one may ask whether the shape of one  $n$ -gonal 2-face  $F$  of  $P$  can be required to be any preassigned (convex)  $n$ -gon  $F'$  (see [1, p. 244]). The solution of this problem is trivially affirmative if  $F$  is a triangle or a quadrangle, since in this case there exists an affine, respectively projective transformation of any realization  $P$  of  $G$  carrying  $F$  onto any predetermined  $F'$ . However, the problem ceases to be trivial already for a pentagonal face  $F$ .

It is the aim of the present note to establish the affirmative solution of the above problem for any face  $F$ . More precisely, we shall prove the theorem enunciated at the outset. By the remark made above, we shall without loss of generality assume that  $n \geq 5$ .

The proof of the theorem consists mainly of a repetition of the proof of Steinitz's theorem as given in [1, pp. 236–242], with modifications resulting from the desire to interfere with  $F$  as little as possible and, if such interference is unavoidable, to conduct it under special precautions. Since a repetition of the whole argument would lead to a needless duplication, we shall restrict ourselves to a summary of the proof of Steinitz's theorem, indicating the main ideas and the necessary changes in the argumentation. We shall use the terminology and notation employed in [1].

The proof of Steinitz's theorem proceeds by induction on the

number of edges of  $G$ , and two cases have to be considered. In each case, the graph  $G$  is "reduced" to a 3-connected planar graph  $G^*$  by a change of such a nature that from each realization of  $G^*$  by a 3-polytope  $P^*$  a realization of  $G$  by a 3-polytope  $P$  may be constructed. The "reductions" needed for the proof are indicated in Figure 1.

*Case 1.* The graph contains a nonseparating circuit with three edges (i.e., a triangular face, or *triangle*) one node of which has valence 3. Then one of the reductions  $\eta_1, \eta_2, \eta_3$  may be applied, yielding a  $G^*$  with 1, 2, or 3 edges less than  $G$ . In the proof of Steinitz's theorem, there is nothing to be added in this case, since induction takes over. In the proof of our theorem one has to require that  $G^*$  be realized by a 3-polytope  $P^*$  having  $F'$  as a face, unless the circuit  $F$  contains one of the edges—marked by one, two, or three stars in Figure 1—of the triangle in question. In those exceptional cases the circuit  $F^*$  of  $G^*$  (corresponding to the circuit  $F$  of  $G$ , and having also  $n$  edges in cases denoted by one star,  $n - 1$  edges in cases denoted by two stars, and  $n + 1$  edges in cases denoted by three stars) has to correspond in  $P^*$  to a face  $F^{**}$  with vertices  $A_1^*, A_2^*, \dots$  different from  $F'$ . The construction of the polygon  $F^{**}$  from  $F'$  is indicated

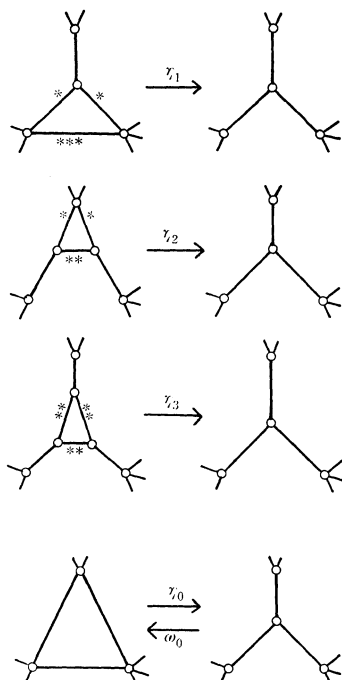


FIGURE 1

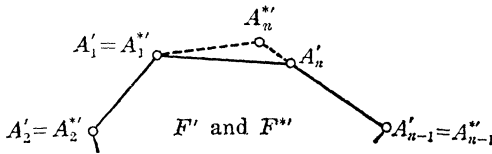
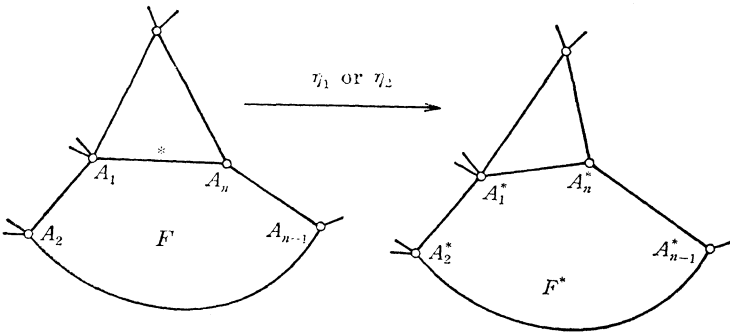


FIGURE 2

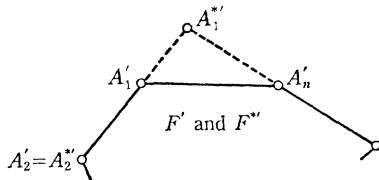
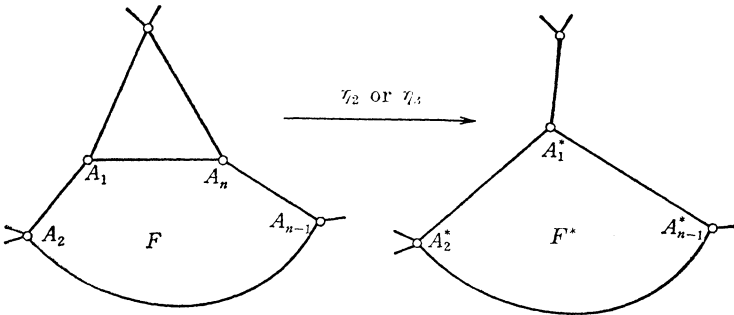


FIGURE 3

in Figures 2 and 3 for the first two cases; in case of the edge denoted by three stars,  $F^{**}$  may be any  $(n + 1)$ -gon obtained as the convex hull of  $F'$  and a point  $A_{n+1}^*$  near the edge  $A'_n A'_1$ . (Note that in the case represented in Figure 3 it may be necessary to start not with  $F'$  itself but—in order to guarantee the existence of  $F^{**}$ —with a suitable projective transformation of  $F'$ . However, this does not impair the construction since a suitable inverse projective transformation of the polytope realizing  $G$  will restore  $F'$  as a face of  $P'$ .)

This completes the proof in Case 1.

*Case 2.* Now we assume that  $G$  contains no triangle possessing at least one 3-valent node. Since in any case Euler's relation implies that  $G$  contains either a triangle, or a 3-valent vertex, it follows that now one of the reductions  $\omega_0$  or  $\eta_0$  may be applied to  $G$ . However, those reductions do not reduce the number of edges and may thus be deemed useless. The depth of Steinitz's proof lies in establishing that by a judicious choice of a finite sequence of  $\omega_0$  or  $\eta_0$  reductions (not exceeding in total number the number of edges of  $G$ ) one may transform  $G$  into a graph  $\tilde{G}$  which is covered by Case 1. In order to prove our theorem we shall show that there is enough freedom in the choice of the  $\omega_0$  and/or  $\eta_0$  reductions to reach  $\tilde{G}$  without interfering (in any of the stages) with the circuit  $F$  of  $G$  (or the corresponding circuits in the intermediate stages). The derived realization of  $\tilde{G}$  and of all the intermediate graphs will still have  $F'$  as the preassigned polygon.

Before proving those assertions, we have to introduce a number of notions. For each 3-connected planar graph  $G$ , we define a graph  $I(G)$  as follows. The vertices of  $I(G)$  correspond to edges of  $G$ ; two vertices of  $I(G)$  are connected by an edge if and only if the corresponding edges of  $G$  have a common node and belong to the same nonseparating circuit. (In Figure 4, a graph  $G$  is shown by heavy lines, while  $I(G)$  is indicated by thin lines.) It is easily seen that  $I(G)$  is planar and 3-connected. Note that  $I(G)$  is 4-valent, and that each  $n$ -gonal region of  $I(G)$  corresponds either to an  $n$ -gonal region of  $G$  or to an  $n$ -valent vertex of  $G$ . A "geodesic arc" in  $I(G)$  is a path in  $I(G)$  in which each two adjacent edges separate the other two edges of  $I(G)$  issuing from their common node. A "lens" of  $I(G)$  is

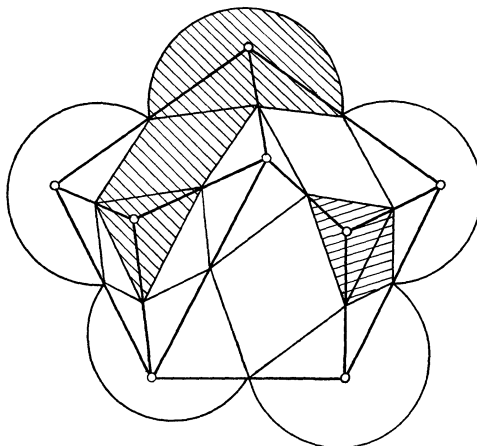


FIGURE 4

a region of plane having as boundary two geodesic arcs of  $I(G)$ , such that at their common endpoints (the "poles" of the lens) the remaining two edges of  $I(G)$  do not belong to the lens. (For example, each of the shaded regions in Figure 4 is a lens, one with poles  $A$  and  $B$ , the other with poles  $C$  and  $D$ ; the second lens is clearly the simplest possible type, consisting of only two triangles; we shall denote it by  $L_0$ .)

An important role is played by "irreducible lenses", that is, lenses which do not contain any proper sublenses. The relevant facts about irreducible lenses are (see [1, pp. 239-242]):

(1) Each region of the plane which has a boundary consisting either of a (closed) geodesic arc, or of two geodesic arcs, contains an irreducible lens. This implies, among other consequences, the existence of irreducible lenses in each  $I(G)$ .

(2) Each irreducible lens contains at least two triangles incident to the boundary of the lens; unless the lens is  $L_0$ , they have no edge in common.

(3) If an irreducible lens  $L \neq L_0$  of  $I(G)$  is given, and if an  $\omega_0$  or  $\eta_0$  reduction is performed on a 3-valent node or on a triangle of  $G$  corresponding to a triangle of  $L$  having an edge in the boundary of  $L$ , the resulting graph  $G^*$  has the following property: There is a lens (and hence, by (1), an irreducible lens  $L^*$ ) of  $I(G^*)$  having fewer faces than  $L$ , such that its faces correspond to some of the faces of  $L$ .

Combining (1), (2) and (3) the proof of Steinitz's theorem is completed by starting from any region bounded by one or two geodesic arcs, finding an irreducible lens  $L$  in it, and applying a suitable  $\omega_0$  or  $\eta_0$  reduction. Since  $L$  contains only finitely many faces, by repeated application of (3) we necessarily reach a graph  $G$  containing a lens of type  $L_0$ . But the presence of such a lens in  $I(G)$  means that  $G$  contains a triangle with a trivalent node; hence to  $G$  the Case 1 is applicable. (In the proof of Steinitz's theorem  $L$  is chosen to contain the least number of faces among all lenses of  $I(G)$ , hence there is no necessity to use the italicized part of (3); however, this part of (3) is evident from the proof of the first part [1, p. 242].)

In view of the above, in order to complete the proof of our theorem it is sufficient to exhibit, for each  $G$  which is not covered by Case 1, a region  $R$  bounded by one or two geodesic arcs of  $I(G)$  and such that at most one of the triangles of  $R$  corresponds to a node of  $G$  belonging to  $F$ . Indeed, this property will be inherited by any irreducible lens  $L$  contained in  $R$ , and hence  $L$  will contain at least one other triangle such that the 3-valent node, or triangle, of  $G$  corresponding to it is disjoint from  $F$ . Thus an appropriate reduction  $\omega_0$  or  $\eta_0$  will be applicable without interference with  $F$ , and  $F'$  may be chosen as the preassigned face of a 3-polytope realizing the reduced

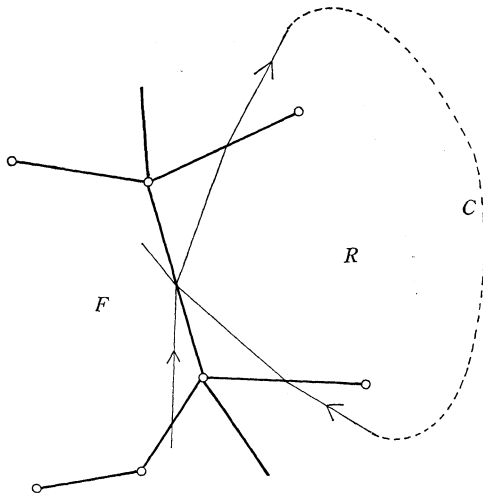


FIGURE 5

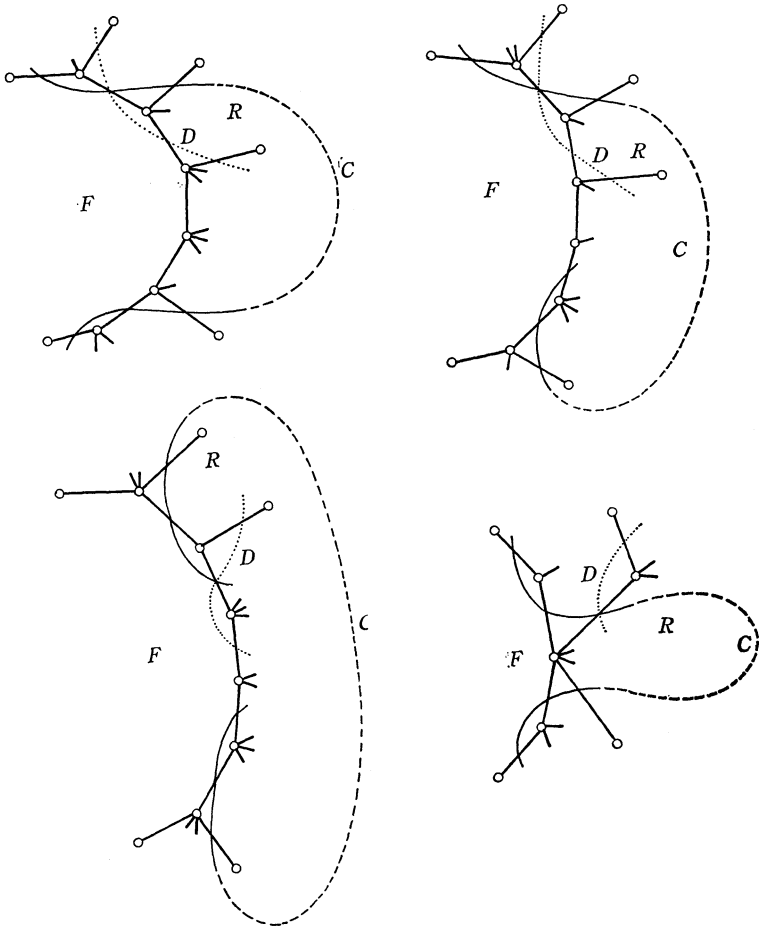


FIGURE 6

graph.

But the existence of a region  $R$  of the type required is easy to establish. Proceeding on any geodesic arc (starting with an edge of  $I(G)$  having both endpoints on edges of  $F$ ) we note that if  $C$  intersects itself prior to its return to  $F$ , or if it self-intersects while reentering  $F$ , that region enclosed by  $C$  which does not contain  $F$  may be taken as  $R$ . (Compare the schematic drawing in Figure 5.) Hence we are left only with the case in which each geodesic arc is free of self-intersections outside  $F$ . Among all the (simple) geodesic arcs of  $I(G)$  having endpoints at edges of  $F$ , we choose one bridging the smallest possible number of edges of  $F$  and denote it by  $C$ . The possible situations are schematically indicated in Figure 6. In the first three cases the minimality of  $C$  prevents  $D$  from returning to  $F$  before meeting  $C$  again, and thus parts of  $C$  and  $D$  will determine a region  $R$  of the type required. In the fourth case  $D$  is any geodesic arc crossing  $C$  at a relatively interior point, and  $D$  either meets  $C$  before meeting  $F$ , or it meets  $C$  and  $F$  at the same vertex, again producing a region of the type required.

This completes the proof in Case 2, and with it the proof of the theorem.

REMARK 1. *As an easy corollary of the theorem we have:*

*If  $P$  is a 3-polytope and if  $C$  is a simple closed circuit of edges of  $P$  such that no facet of  $P$  meets two edges of  $C$ , there exists a polytope  $P'$  combinatorially equivalent to  $P$  such that the circuit of  $P'$  corresponding to  $C$  is in a plane.*

2. By an obvious application of duality, it follows from the theorem that the shape of one vertex-figure of a 3-polytope may be arbitrarily prescribed. Probably, more elements of a 3-polytope may be arbitrarily prescribed; however, it is easy to see that it is not always possible to preassign the shape of two faces having a common edge. (For example, the two quadrilaterals of Figure 7 may not appear in any 3-sided prism.) It would be interesting to investigate the following

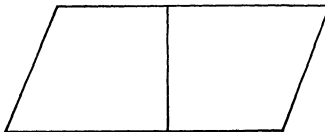


FIGURE 7

CONJECTURE. For any family  $\{F'_1, \dots, F'_n\}$  of disjoint faces of a 3-polytope  $P$ , and for any family  $\{F''_1, \dots, F''_n\}$  of polygons,  $F'_i$  being



of the same type as  $F_i$ , there exists a 3-polytope  $P'$  combinatorially equivalent to  $P$ , such that every face of  $P'$  corresponding to one of the faces  $F_i$  is projectively equivalent to  $F'_i$ .

3. It would be very interesting to determine to what extent the theorem holds in higher dimensions. The only known result in this direction seems to be M.A. Perles' example of an 8-dimensional polytope  $P$  with 12 vertices such that the shape of one of its 7-dimensional faces (with 10 vertices) may not be arbitrarily chosen within its combinatorial type ([1, p. 96, Exercise 3]). It may be conjectured that a similar failure of the theorem occurs already in four dimensions.

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