

# Pacific Journal of Mathematics

**ON THE GLOBAL DIMENSION OF RESIDUE RINGS**

KENNETH LEWIS FIELDS

# ON THE GLOBAL DIMENSION OF RESIDUE RINGS

K. L. FIELDS

**Techniques of Eilenberg, Nakayama, and Nagao are used to obtain results on the global dimension of the ring  $R/I$  where  $I$  is an eventually idempotent ideal. We also consider the cases when  $I$  is contained in the socle of  $R$ , and when  $I$  has nonzero annihilator.**

In [3] the global dimension of  $R/I$  is calculated for  $R$  left hereditary and  $I$  eventually idempotent (i.e.,  $I^n = I^{n+1}$  for some  $n$ ).<sup>1</sup> These results are generalized by the following theorem:

**THEOREM 1.** *Let  $I$  be an ideal of  $R$  which is either projective as a left  $R$ -module or flat as a right  $R$ -module. If  $I^m = I^{m+1}$  then for any  $R/I$ -module  $M$*

$$\text{l.h.d.}_{R/I} M \leq \text{l.h.d.}_R M + (m - 1) \text{l.h.d.}_R I + 2m - 2.$$

The proof depends on the following lemmas whose proofs we leave to the reader:

**LEMMA 1.** *Let  $0 \rightarrow M_m \rightarrow \dots \rightarrow M_1 \rightarrow M_0 \rightarrow M \rightarrow 0$  be an exact sequence of  $S$ -modules such that  $\text{l.h.d.}_S M_j \leq n \forall j$ . Then  $\text{l.h.d.}_S M \leq n + m$ .*

**LEMMA 2.** *If the ideal  $I$  is left  $S$ -projective and  $P$  is a projective left  $S$ -module, then  $IP$  is left  $S$ -projective.*

**LEMMA 3.** *If the ideal  $I$  is right  $S$ -flat and  $K$  is a submodule of a free left  $S$ -module, then*

$$\text{l.h.d.}_S IK \leq \text{l.h.d.}_S I + \text{l.h.d.}_S K.$$

**REMARK.** All of the above may be stated with  $\text{l.h.d.}$  replaced by  $\text{l.wd.}$

*Proof of Theorem 1.* (a) Assume  $I$  is left projective. Proceed by induction on  $\text{l.h.d.}_R M = n$ .  $n = 0$  trivial.  $n = 1$ . Let  $0 \rightarrow P \rightarrow F \rightarrow M \rightarrow 0$  be an  $R$ -projective resolution. Consider the exact sequence (of [3]):

<sup>1</sup>  $R$  and  $S$  will always denote rings with unit elements;  $I$  will always denote a two-sided ideal. "Projective" means left projective.

$$0 = I^m F / I^{m+1} F \longrightarrow I^{m-1} P / I^m P \longrightarrow I^{m-1} F / I^m F \longrightarrow I^{m-2} P / I^{m-1} P \\ \longrightarrow \dots \longrightarrow IP / I^2 P \longrightarrow IF / I^2 F \longrightarrow P / IP \longrightarrow F / IF \longrightarrow M \longrightarrow 0 .$$

Now use Lemmas 1 and 2 to obtain  $\text{lhd}_{R/I} M \leq 2m - 1$  as required.  $n > 1$ . Consider  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$   $R$ -exact where  $F$  is  $R$ -free. Then  $\text{lhd}_R K = n - 1 \geq 1$ . Since  $IM = 0$  we may consider the  $R/I$  exact sequence

$$0 \longrightarrow K / IF \longrightarrow F / IF \longrightarrow M \longrightarrow 0 .$$

Then  $\text{lhd}_R K / IF = n - 1$  so by induction

$$\text{lhd}_{R/I} K / IF \leq n - 1 + 2m - 2$$

and so  $\text{lhd}_{R/I} M \leq n + 2m - 2$ .

(b) Assume  $I$  is right flat. Let  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$  be  $R$ -exact, where  $F$  is  $R$ -free. Again, consider the  $R/I$ -exact sequence

$$0 \longrightarrow I^{m-1} K / I^m K \longrightarrow I^{m-1} F / I^m F \longrightarrow I^{m-2} K / I^{m-1} K \longrightarrow \dots \\ \longrightarrow IK / I^2 K \longrightarrow IF / I^2 F \longrightarrow K / IK \longrightarrow F / IF \longrightarrow M \longrightarrow 0$$

and apply Lemmas 1 and 3, making use of the fact that if  $N$  is a submodule of a free left  $R$ -module then  $\forall n \geq 0 \text{Tor}_{1+n}^R(R/I, N) = 0$  and so  $\text{lhd}_{R/I} N / IN \leq \text{lhd}_R N$ .

REMARK. Conversely, if  $I$  is left projective and  $\text{lhd}_{R/I^2} R/I = m - 1$ , then  $I^m = I^{m+1}$ .

*Proof.* Consider the  $R/I^2$ -sequence

$$0 \longrightarrow I^{m-1} / I^m \longrightarrow I^{m-2} / I^m \longrightarrow I^{m-3} / I^{m-1} \longrightarrow \dots \\ \longrightarrow I / I^3 \longrightarrow R / I^2 \longrightarrow R / I \longrightarrow 0 .$$

By hypothesis,  $I^{m-1} / I^m$  is  $R/I^2$ -projective, and so

$$0 \longrightarrow I^m / I^{m+1} \longrightarrow I^{m-1} / I^{m+1} \longrightarrow I^{m-1} / I^m \longrightarrow 0$$

splits, hence  $I^m = I^{m+1}$ .

THEOREM 2. (a) Let  $I_1, \dots, I_n$  be ideals such that  $I_1 I_2 \dots I_n = 0$ . Then  $\text{lgld } R \leq \max_i \text{lgld } R / I_i + \text{lhd}_R R / I_i$

(b) Let  $I$  be any ideal contained in the left socle of  $R$ . Then  $\text{lgld } R \leq \text{lgld } R / I + \text{lhd}_R R / I$ .

*Proof.* (a) Let  $M$  be any left  $R$ -module. Consider the filtration  $0 = I_1 I_2 \dots I_n M \subset I_2 \dots I_n M \subset \dots \subset I_n M \subset M$ . By Auslander's lemma

$$\begin{aligned} \text{l.hd}_R M &\leq \max_i \text{l.hd}_R (I_{i+1} \cdots I_n M / I_i \cdots I_n M) \\ &\leq \max_i \text{l.hd}_R R / I_i + \text{lgld } R / I_i . \end{aligned}$$

(b)  $\text{lgld } R = \sup \text{l.hd}_R R / L$  where  $L$  runs through the left ideals of  $R$ . Since every left ideal is a direct summand of an essential left ideal (by Zorn), we may sup over only the essential left ideals. But  $I$  is contained in every essential left ideal, and so annihilates  $R / L$ .

**COROLLARY 1.** *If  $R$  is a semi-primary ring of finite global dimension, then the global dimension of  $R / I$  is finite for every projective ideal  $I$ .*

**COROLLARY 2.** *If  $R$  has finite global dimension and  $I$  is a projective nilpotent ideal, then  $\text{lgld } R / I^2 = n < \infty$ ,  $I^{n+1} = 0$ , and  $\text{lgld } R \leq n + 1$ .*

**COROLLARY 3.** *Let  $R$  have finite global dimension and let  $I$  be a projective ideal. Then  $\text{lgld } R / I^k < \infty \forall k$  if and only if  $I$  is eventually idempotent. This characterizes those rings for which  $\text{lgld } R / I$  is finite for every projective ideal  $I$  as being those rings of finite global dimension whose projective ideals are eventually idempotent.*

We remark that if  $I$  is any ideal of a ring  $R$  such that  $\text{lgld } R / I = 0$  and  $\text{lgld } R / I^k < \infty \forall k$ , then  $I$  is eventually idempotent by results of Chase [1].

**COROLLARY 4.** *If  $R$  is semi-prime, then*

$$\text{lgld } R / \text{l.Soc } R \leq \text{lgld } R \leq \text{lgld } R / \text{l.Soc } R + 1$$

**COROLLARY 5.** *Let  $T$  be an arbitrary triangular matrix ring [1], i.e.,  $T = \begin{pmatrix} R & M \\ O & S \end{pmatrix}$  where  $M$  is an  $(R, S)$ -bimodule. Then*

$$\max(\text{lgld } R, \text{lgld } S) \leq \text{lgld } T \leq \max \begin{cases} \text{lgld } S + \text{l.hd}_R M + 1 \\ \text{lgld } R \end{cases} .$$

(To obtain the lower bound, we observe that  $I = \begin{pmatrix} R & M \\ O & O \end{pmatrix}$  is right  $T$ -projective,  $I = I^2$ , and  $T / I \cong S$ ;  $J = \begin{pmatrix} O & M \\ O & S \end{pmatrix}$  is left  $T$ -projective,  $J = J^2$ , and  $T / J \cong R$ . For the upper bound, observe that  $J I = 0$  and apply Theorem 2 (a).)

A similar formula for  $\text{rtgld}$  and  $\text{wglgd}$  may be obtained. In particular, when  $S$  is semi-simple,

$$\text{lgld } T = \max \left\{ \begin{array}{l} 1 + \text{l.hd}_R M \\ \text{lgld } R \end{array} \right. , \quad \text{wgld } T = \max \left\{ \begin{array}{l} 1 + \text{l.wd}_R M \\ \text{wgld } R \end{array} \right.$$

and  $\text{rtgld } R \leq \text{rtgld } T \leq \text{rtgld } R + 1$ .

2. It is a well known result of Cartan and Eilenberg that the right global dimension of a right noetherian local ring is equal to the left weak dimension of its residue class field. Can “left weak” be replaced by “right homological”? The answer is no by the following example, which is based on a construction that has been recently used by P. M. Cohn [2] and Jategaonkar [4].

(This example also shows that the right global dimension of a right Noetherian ring may be greater than  $1 + \sup \text{rt.hd } L$  where  $L$  runs over the maximal right ideals. This is of course in contrast to the commutative case.)

Let  $R = F[[x]]$  where  $F$  is a field chosen large enough so that  $R$  possesses an injective endomorphism  $\sigma$  whose image is contained in  $F$ . Let  $S = R[[x_1, x_2; \sigma]]$ , i.e., the twisted right power series ring in two commuting variables over  $R$ . We claim that  $S$  is the required example:

(i)  $S$  is right noetherian.

*Proof.*  $T = R[[x_1; \sigma]]$  is a right P.I.D. ([2] or [4]). Now use the usual commutative proof of the Hilbert Basis Theorem to prove that  $T[x_2; \sigma]$  is right noetherian.

(Note: It is crucial that if  $f \in T$ ,  $f = \sum_{i \geq n} x_1^i r_i$ , then  $f \cdot x_2 = x_2 x_1^n u$  where  $u$  is a unit.)<sup>2</sup>

That  $S$  is right noetherian now follows as in the commutative case.

(ii)  $S$  is local and the maximal ideal of  $S$  is generated on the right by  $x$ .

(iii)  $\text{rt.gld } S = 2$ .

*Proof.* It is clear that  $\text{rt.hd}_S (x_1 S + x_2 S) = 1$ . Now  $Sx_2 S$  is left  $S$ -flat<sup>3</sup> and  $S/Sx_2 S$  is a right P.I.D.; hence by Small [5] we see that  $\text{rt.gld } S \leq 2$  and the proof is complete.

The author wishes to thank Professor L. W. Small for his invaluable advice and encouragement.

### BIBLIOGRAPHY

1. S. U. Chase, *Generalization of the ring of triangular matrices*, Nagoya Math. J. **18** (1961), 13-25.

<sup>2</sup> Neither  $R[x_1, x_2; \sigma]$  nor  $R[x_1, x_2, x_3; \sigma]$  need be right Noetherian—consider the right ideal generated by  $\{x_2^n(1+x_1x)\}$   $n = 0, 1, 2, \dots$ , and  $\{x_2^n(x_3 + x_1x)\}$ , respectively.

<sup>3</sup> Because  $Sx_2 S \cong S \otimes_T Tx_2 T$ .

2. P. M. Cohn, *Torsion modules over free ideal rings*, Proc. London Math. Soc. (3) **17** (1967), 577-599.
3. S. Eilenberg, T. Nakayama, H. Nagao, *Dimension of residue rings of hereditary rings*, Nagoya Math. J. **10** (1956), 87-95.
4. A. Jategaonkar, *Left principle ideal domains*, J. Algebra **8** (1968), 148-156.
5. L. W. Small, *Change of rings theorem*, Proc. Amer. Math. Soc. **19** (1968), 662-666.

Received March 4, 1969. The author is an NSF Graduate Fellow. These results are part of the authors doctoral dissertation.

UNIVERSITY OF CALIFORNIA, BERKELEY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON  
Stanford University  
Stanford, California 94305

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

RICHARD PIERCE  
University of Washington  
Seattle, Washington 98105

BASIL GORDON\*  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLE

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

\* Acting Managing Editor.



Harry P. Allen and Joseph Cooley Ferrar, <i>Jordan algebras and exceptional subalgebras of the exceptional algebra <math>E_6</math></i> .....	283
David Wilmot Barnette and Branko Grünbaum, <i>Preassigning the shape of a face</i> .....	299
Robert Francis Craggs, <i>Involutions of the 3-sphere which fix 2-spheres</i> .....	307
David William Dean, Bor-Luh Lin and Ivan Singer, <i>On <math>k</math>-shrinking and <math>k</math>-boundedly complete bases in Banach spaces</i> .....	323
Martin Engert, <i>Finite dimensional translation invariant subspaces</i> .....	333
Kenneth Lewis Fields, <i>On the global dimension of residue rings</i> .....	345
Howard Gorman, <i>The Brandt condition and invertibility of modules</i> .....	351
Benjamin Rigler Halpern, <i>A characterization of the circle and interval</i> .....	373
Albert Emerson Hurd, <i>A uniqueness theorem for second order quasilinear hyperbolic equations</i> .....	415
James Frederick Hurley, <i>Composition series in Chevalley algebras</i> .....	429
Meira Lavie, <i>Disconjugacy of linear differential equations in the complex domain</i> .....	435
Jimmie Don Lawson, <i>Lattices with no interval homomorphisms</i> .....	459
Roger McCann, <i>A classification of center-foci</i> .....	467
Evelyn Rupard McMillan, <i>On continuity conditions for functions</i> .....	479
Graciano de Oliveira, <i>A conjecture and some problems on permanents</i> .....	495
David L. Parrott and S. K. Wong, <i>On the Higman-Sims simple group of order 44, 352, 000</i> .....	501
Jerome L. Paul, <i>Extending homeomorphisms</i> .....	517
Thomas Benny Rushing, <i>Unknotting unions of cells</i> .....	521
Peter Russell, <i>Forms of the affine line and its additive group</i> .....	527
Niel Shilkret, <i>Non-Archimedean Gelfand theory</i> .....	541
Alfred Esperanza Tong, <i>Diagonal submatrices of matrix maps</i> .....	551