A CLASSIFICATION OF CENTER-FOCI

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The main purpose of this paper is to classify center-foci
according to isomorphisms. Necessary and sufficient conditions
are found for an isomorphism to exist in terms of properties
on the cycles in suitable neighborhoods of the center-foci. In
the last section $o +$ stable isolated critical points are classified
according to isomorphisms.

This paper was motivated by discussions with Otomar Hájek and
Taro Ura.

Throughout this paper R^1, R^+, R^-, and R^2 will denote the reals,
the nonnegative reals, the nonpositive reals, and the plane respectively.

Let F be a family of curves filling a region R of the plane. F
will be said to be regular at a point p of R if F is locally home-
omorphic with parallel lines at p. F is called regular in R if it is
regular at each point of R. A cross-section of F (through the point
p of R) shall mean an arc T (of which p is a nonend-point) which
lies in R and is such that each curve of F intersects T at most once.

Let (X, π) be a dynamical system on X, i.e., X is a topological
space and π is a mapping of $X \times R^1$ onto X satisfying the following
axioms: (where $x\pi t = \pi(x, t)$ for $(x, t) \in X \times R^1$)

   (1) $x\pi 0 = x$ for $x \in X$

   (2) $(x\pi t)\pi s = x\pi(t + s)$ for $x \in X$ and $t, s \in R^1$

   (3) π is continuous.

For $x \in X$, $x\pi R^1$ is called the trajectory through x and is denoted by
$C(x)$. If $C(x) = \{x\}$, x is called a critical point. If there exists $t \in R^1$,
$t \neq 0$, such that $x\pi t = x$, then x is called a periodic point. If x is a
periodic point and not a critical point, $C(x)$ is called a cycle. A
subset A of X is said to be invariant if $A\pi R^1 = A$, or $+$ invariant if
$A\pi R^+ = A$. A subset B of X is said to be

   (1) orbitally stable (o stable) if B has arbitrarily small invariant
neighborhoods.

   (2) orbitally $+$ stable ($o +$ stable) if B has arbitrarily small $+$ in-
variant neighborhoods.

   (3) asymptotically orbitally $+$ stable ($ao +$ stable) if B is
$o +$ stable and for some neighborhood U of x, $L^+(x) \subset \bar{B}$ for every
$x \in U$, where $L^+(x)$ is the positive limit set of x.

Let (X, π) be a dynamical system on a metric space X. The
positive prolongational limit set, denoted by $J^+(x)$, of a point $x \in X$ is
given by $J^+(x) = \{y$: there exist sequences $\{x_i\} \subset X$ and $\{t_i\} \subset R^1$ such
that $x_i \rightarrow x$, $t_i \rightarrow + \infty$, and $x_i\pi t_i \rightarrow y\}$. The negative prolongational limit

set, $J^-(x)$, is defined analogously. For any $x \in X$, $J^+(x)$ and $J^-(x)$ are invariant. If A and B are subsets of X such that $A \subset J^+(B)$ $(A \subset J^-(B))$, we will write AJ^+B (AJ^-B).

1. **Definition and properties of a center-focus.** In the following (R^2, π) will denote a dynamical system on R^2 and M the set of non-critical periodic points of (R^2, π).

DEFINITION 1.1. A cycle $C(x)$ of (R^2, π) decomposes R^2 into two components, one bounded and the other unbounded. int $C(x)$ and ext $C(x)$ will denote the bounded and unbounded components, respectively, of $R^2 - C(x)$.

PROPOSITION 1.2. *Let $C(x)$ be a cycle in (R^2, π). Then* int $C(x)$ *and* ext $C(x)$ *are invariant.*

Proof. The components of an invariant set are invariant.

In [1, VII, 4.8] it is proved that

PROPOSITION 1.3. *If $C(x)$ is a cycle in (R^2, π), then* int $C(x)$ *contains a critical point.*

DEFINITION 1.4. An isolated critical point p of (R^2, π) is called a center-focus if and only if there exist both cycles and noncyclic trajectories arbitrarily near p.

EXAMPLE 1.5. Take any closed set F in $[0, 1] \subset R^+$ and choose any mapping $f \colon R^+ \to [-1, 1]$ of class C^1 having F as its zero-point set. Then define the differential dynamical system by means of the equation.

$$\frac{dz}{d\theta} = z(i + f \mid z \mid) \, .$$

This corresponds to the "polar" equation $(dr/d\theta) = rf(r)$. The origin is an isolated critical point and the cycles of the dynamical system are precisely those circles $\mid z \mid = r$ for which $f(r) = 0$. We can now make the origin a center-focus by choosing F such that $0 \in F$ and $0 \in \overline{[0, 1] - F}$, e.g.,

$$F = \{0\} \cup \left\{ \frac{1}{m} \colon m \text{ is a positive integer} \right\} \, .$$

In [1, VIII, 4.1], it is proved that

PROPOSITION 1.6. *Let p be a center-focus of (R^2, π). Then p is o stable.*

PROPOSITION 1.7. *Let p be a center-focus of (R^2, π). Then for any $x \in M$ sufficiently near p, $p \in \text{int } C(x)$.*

Proof. Let U be a disc neighborhood of p which contains no critical points other than p. Since p is o stable, there exists a neighborhood V of p such that $V\pi R^1 \subset U$. For any $x \in M \cap V$, $C(x) \subset U$. Then $\text{int } C(x) \subset U$ since U is simply connected. By Proposition 1.3, $\text{int } C(x)$ contains a critical point. This critical point must be p since $U - \{p\}$ contains no critical points.

COROLLARY 1.8. *Let p be a center focus of (R^2, π). Then there exists a cycle $C(x)$ such that*
    (1) $p \in \text{int } C(x)$
    (2) $p \in \text{int } C(y)$ *for every* $y \in \text{int} C(x) \cap M$.

We may now reformulate Definition 1.4 as

DEFINITION 1.4'. An isolated critical point p of (R^2, π) is called a center-focus if and only if there exists a cycle $C(x)$ such that
    (1) p is the unique critical point in $\text{int } C(x)$
    (2) there exist both cycles and noncyclic trajectories arbitrarily close to p.

We choose a fixed $C(x_0)$ satisfying these conditions and denote $\text{int } C(x_0)$ by U.

REMARK. The set $\{y: y = p \text{ or } y \in M \cap \bar{U}\}$ is closed.

The proofs of the following relationships between the cycles in U are the same as the proofs of [4, 1.10] and [4, 1.11].

PROPOSITION 1.9. *Let p be a center-focus and U a neighborhood of p as described in Definition 1.4'. If C_1 and C_2 are distinct cycles in U, then $C_1 \subset \text{int } C_2$ or $C_2 \subset \text{int } C_1$.*

COROLLARY 1.10. *If C_1 and C_2 are distinct cycles in U such that $C_1 \subset \text{ext } C_2$, then $C_2 \subset \text{int } C_1$.*

2. Isomorphism of dynamical systems.

DEFINITION 2.1. Let (X, π) and (Y, ρ) be dynamical systems. A homeomorphism $h: X \to Y$ is said to be trajectory preserving if and only if $h(x\pi R^1)$ is a trajectory of (Y, ρ) for every $x \in X$. If such a homeomorphism exists, (X, π) and (Y, ρ) are said to be equivalent.

DEFINITION 2.2. Let (X, π) and (Y, ρ) be dynamical system. An isomorphism of (X, π) onto (Y, ρ) is a pair of mappings (h, φ) which satisfies the following conditions:

(1) h is a homeomorphism of X onto Y

(2) φ is a continuous mapping of $X \times R^1$ onto R^1 such that for every fixed $x \in X$, $\varphi(x, \cdot)$ is a homeomorphism of R^1 onto itself with $\varphi(x, 0) = 0$

(3) $h(x\pi t) = h(x)\rho\varphi(x, t)$ for $x \in X$ and $t \in R^1$.

If such a pair of mappings exists, (X, π) and (Y, ρ) are said to be isomorphic.

REMARKS. 1. If (h, φ) is an isomorphism of (X, π) onto (Y, ρ), then h is a trajectory preserving homeomorphism of (X, π) onto (Y, ρ). Hence, if (X, π) and (Y, ρ) are isomorphic, then they are equivalent.

2. " Equivalent to " is evidently an equivalence relation in the family of all dynamical systems.

3. In [5], it is proved that "isomorphic to" is an equivalence relation in the family of all dynamical systems.

The following theorem, [6, 3, Th. 3], states conditions for "equivalent to" and "isomorphic to" to be identical.

THEOREM 2.3. *Let (X, π) and (Y, ρ) be dynamical systems on completely regular Hausdorff spaces X and Y and let S_π and S_ρ denote the set of critical points of (X, π) and (X, p) respectively. Then if h is a trajectory preserving homeomorphism of (X, π) onto (Y, ρ) there exists a mapping $\varphi: (X - S_\pi) \times R^1 \to R^1$ such that $(h \mid X - S_\pi, \varphi)$ is an isomorphism of $(X - S_\pi, \pi)$ onto $(Y - S_\rho, \rho)$.*

Let (h, φ) be an isomorphism of (X, π) onto (Y, ρ). If X is connected, it can be shown that $\varphi(x, \cdot)$ is either strictly increasing for every $x \in X$ or strictly decreasing for every $x \in X$.

DEFINITION 2.4 Let (X, π) and (Y, ρ) be dynamical systems on connected spaces X and Y and let (h, φ) be an isomorphism of (X, π) onto (X, ρ). If $\varphi(x, \cdot)$ is strictly increasing for every $x \in X$, (h, φ) is called proper. If $\varphi(x, \cdot)$ strictly decreasing for every $x \in X$, (h, φ) is called dual.

The following proposition displays the connection between isomorphisms and prolongational limit sets.

PROPOSITION 2.5. *Let (X, π) and (Y, ρ) be dynamical systems on connected metric spaces X and Y and let (h, φ) be an isomorphism of (X, π) onto (Y, ρ). For $x \in X$*

(1)   *if* (h, φ) *is proper, then* $h(J_\pi^+(x)) = J_\rho^+(h(x))$

(2)   *if* (h, φ) *is dual, then* $h(J_\pi^+(x)) = J_\rho^-(h(x))$,

where the subscript π *or* ρ *denotes the dynamical system with respect to which the positive or negative prolongational limit set was taken.*

Proof.   The proof follows from the definitions of proper and dual.

3.   **Curve families of Kaplan.**   In [3] Kaplan classifies families in a neighborhood of a singularity.   In particular, if we consider the trajectories of a planar dynamical system as curves, he classifies the dynamical system in a neighborhood of an isolated critical point.   For a singularity surrounded by closed curves, the classification is done as follows.

DEFINITION 3.1.   Let A and B be curve families filling the planar sets U and V respectively. A and B are said to be equivalent if there exists an orientation preserving homeomorphism $h: U \to V$ such that for any curve C of A, $h(C)$ is a curve of B.

DEFINITION 3.2.   An annular family shall mean a curve family F in the plane which has the following properties:

(1) F fills a closed region A, whose boundary consists of two disjoint closed curves D_1, and D_2.

(2) F is regular at every point of A.

A half open annular family shall mean a family which can be written as a union $\bigcup_{i=1}^{\infty} F_i$ of closed annular families F_i, where $F_i \cap F_{i+1}$ is the common bounding curve of F_i and F_{i+1} for each i and $F_i \cap F_j$ is the empty set for $|i - j| > 1$.

REMARK.   If F is a curve family defined by a planar dynamical system without critical points, then F is regular.   This is an immediate consequence of the existence of a transversal through each point of the dynamical system. [1, VI, 2.12; VIII, 1.6]

DEFINITION 3.3.   An annular family F will be called simple if it contains no closed curves besides D_1 and D_2.

The possibilities for a simple annular family are of two fundamental types: the type of spirals which are asymptotic to the boundary curves in the opposite direction; the type of spirals which are asymptotic to the boundary curves in the same direction.

Kaplan shows that a simple annular family may be mapped by an orientation preserving homeomorphism onto one of the following four annular families.

(i)   the family of Figure 1.

(ii) the family obtained from (i) by reflecting in a diameter of the larger circle.

(iii) the family of Figure 2.

(iv) the family obtained from (iii) by reflecting in a diameter of the larger circle.
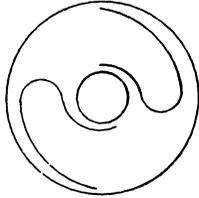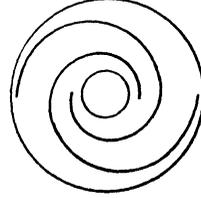

Figure 1.   Figure 2.

Families which are homeomorphic to (i) or (ii) are called bent. An annular family can contain only a finite number of bent families. Families which are homeomorphic to (iii) or (iv) are called smooth. A smooth annular family is an annular family which contains no bent families.

Every half open annular family can be associated with a sequence of letters such as:

$$bsbbbsb \cdots \quad \text{or} \quad sbsbsb \cdots$$

corresponding to successive rings of bent or smooth types. The sequence is made unique by writing the letters corresponding to the rings in order of nearness of the ring to the singular point.

With each of the b's is associated a bent annular family. This bent annular family has an associated canonical family (i) or (ii). We now insert after each b in the sequence the corresponding symbol i or ii as superscripts, so that the symbol might appear

$$b^i s b^i b^{ii} s b^{ii} s \cdots .$$

DEFINITION 3.4. Let h_1 and h_2 be two orientation preserving homeomorphisms of $0 \leq y \leq 1$ onto itself. h_1 and h_2 are called similar if and only if the following conditions holds:

(1) If E_i is the (necessarily closed) set of fixed points of $h_i (i = 1, 2)$, then there exists a homeomorphism $f: E_1 \to E_2$ which is a monotone increasing function.

(2) If y'_i and y''_i being to E_i, but $y \notin E_i$ for $y'_i \leq y \leq y''_i$ and $f(y'_1) = y'_2, f(y''_1) = y''_2$, then $h_1(y) - y$ and $h_2(y) - y$ have the same sign in the respective intervals $y'_i \leq y \leq y''_i$.

To each smooth annular family there exists a cross-section, [3, Corollary 1, p. 27]. If a smooth annular family (bounded by D_1 and D_2)

is cut open along a cross section and mapped onto a rectangular family $M = \{0 \leqq x \leqq 1, 0 \leqq y \leqq 1$, the "curves" being the lines $y = $ a constant$\}$ so that D_1 corresponds to $y = 0$ and D_2 to $y = 1$ and the direction from $(0, 0)$ to $(1, 0)$ on $y = 0$ is taken to be the positive orientation on D_1, then the corresponding identification is determined up to a class of similar homeomorphisms.

Suppose the set of all orientation preserving homeomorphisms of $0 \leqq y \leqq 1$ onto itself is subdivided into similarily classes. To each class assign a different index α, so that a collection I of indices is formed in one-to-one correspondence with the similarily classes. Then to each smooth annular family there corresponds a unique index of I. Moreover, two smooth annular families have the same index if and only if they are equivalent. We now insert into our already modified symbol for the smooth annular family the corresponding index of I after each s, so that the symbol might appear

$$b^i s(\alpha) b^{i} b^i s(\beta) \cdots .$$

Let F be an annular family. Then $bs(F)$ will denote the symbol corresponding to F. The theorem of Kaplan in which we are interested is ([3, Th. 22a])

THEOREM 3.5. *Let F and G be two half-open annular families. Then F and G are equivalent if and only if $bs(F) = bs(G)$.*

4. **Classification of center-foci.** Kaplan looked at the behavior of all the curves to obtain his classification. We will now apply his theorem to show that it is necessary and sufficient to consider relations between the cycles of two center-foci in order to obtain a classification.

For $i = 1, 2$, let (R^2, π_i) be a dynamical system, p_i a center-focus and U_i a neighborhood of p_i as described in Definition 1.4'. By restricting the mapping π_i to U_i we may obtain a dynamical system $(U_i, \pi_i \mid U_i)$ which will be denoted by (U_i, π_i). For $x \in R^2$, $C_i(x)$ and $J_i^+(J_i^-)$ will denote the trajectory and the positive (negative) prolongational limit set with respect to (R^2, π_i), $i = 1, 2$. M_i will denote the family of cycles of (U_i, π_i), $i = 1, 2$.

LEMMA 4.1. *Let (V, π) be a dynamical system whose curves form a simple annular family with D_1 and D_2 as boundings cycles. Then $D_1 J^+ D_2$ or $D_2 J^+ D_1$.*

Proof. Let x be any point in the interior of V. Then $C(x)$ is asymptotic to both D_1 and D_2. Let $x\pi R^+$ be asymptotic to D_1 and $x\pi R^-$ be asymptotic to D_2. If the reverse is true, the proof is similar.

For $i = 1, 2$, if z_i is any point of D_i, then there exists a sequence $t_m^i \in R^1$ such that $x\pi t_m^i \to z_i$. Moreover $t_m^1 \to +\infty$ and $t_m^2 \to -\infty$. Then $t_m^1 - t_m^2 \to +\infty$, $x\pi t_m^2 \to z_2$, and

$$(x\pi t_m^2)\pi(t_m^1 - t_m^2) = x\pi t_m^1 \to z_1 .$$

Thus $z_1 \in J^+(z_2)$ and $D_1 J^+ D_2$.

DEFINITION 4.2. A homeomorphism $f: M_1 \to M_2$ is said to preserve the order of cycles if and only if f satisfies the following property for $C_1, C_2 \in M_1$:

$$C_1 \subset \operatorname{int} C_2 \text{ if and only if } f(C_1) \subset \operatorname{int} f(C_2) .$$

DEFINITION 4.3. The orientation of the cycles of M_i, $i = 1, 2$, will be in the direction of movement with respect to R^+.

PROPOSITION 4.4. $(U_1 - p_1, \pi_1)$ and $(U_2 - p_2, \pi_2)$ are isomorphic if there exists a homeomorphism $f: M_1 \to M_1$ such that
 (1) f preserves the order of cycles
 (2) f is orientation preserving
 (3) $C_1 J_1^+ C_2$ if and only if $f(C_1) J_2^+ f(C_2)$ for $C_1, C_2 \in M_1$.

Proof. We will first show that to each simple annular family of U_1 there corresponds an equivalent simple annular family of U_2.

Let F be a simple annular family of U_1 with D_1 and D_2, $D_1 \subset \operatorname{int} D_2$, as the cycles which bound F. For definiteness let F be equivalent to (i) and $D_2 J_1^+ D_1$. If F is equivalent to one of the other canonical families or if $D_1 J_1^+ D_2$, the proof is similar.

D_1 and D_2 are oriented in opposite directions since the spirals of F are asymptotic to D_1 and D_2 in opposite directions. Since F is equivalent to (i), we will, for simplicity, assume F is (i). Then the spirals of F are asymptotic to D_1, in a counter-clockwise direction, i.e., if we move along one of the spirals in a counter-clockwise direction we can come arbitrarily close to D_1. Since $D_2 J_1^+ D_1$, the orientation of D_1 must be in a clockwise direction. The orientation of D_2 must then be in a counter-clockwise direction.

Let G be the family of curves bounded by $f(D_1)$ and $f(D_2)$. By (1), G can contain no cycles other than $f(D_1)$ and $f(D_2)$. G is a simple annular family. By (2), $f(D_1)$ and $f(D_2)$ are oriented in the same directions as D_1 and D_2 respectively. Thus $f(D_1)$ and $f(D_2)$ are oriented in a clockwise and counter-clockwise direction respectively. Then the spirals of G are asymptotic to D_1 and D_2 in opposite directions, i.e., G is a bent region. If the spirals of G are asymptotic to $f(D_1)$ and $f(D_2)$ in the same directions that $f(D_1)$ and $f(D_2)$ are oriented, G is

equivalent to family represented by Figure 4. If the spirals of G are asymptotic to $f(D_1)$ and $f(D_2)$ in the opposite directions that $f(D_1)$ and $f(D_2)$ are oriented, G is equivalent to the family represented in Figure 3. Figure 3 represents (i) and Figure 4 represents (ii).
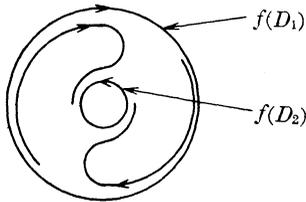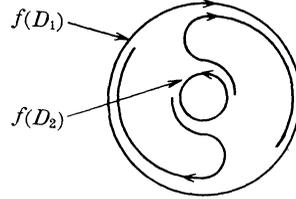
Figure 3.   Figure 4.

$f(D_1) \subset \text{int } f(D_2)$ and $f(D_1)J_2^+ f(D_2)$ by (1) and (3) respectively. Since $f(D_1)J_2^+ f(D_2)$, the spirals must be asymptotic to $f(D_1)$ and $f(D_2)$ in the opposite directions that $f(D_1)$ and $f(D_2)$ are oriented. Hence G is equivalent to (i) and therefore equivalent to F.

We will now show that to each smooth annular family in U, there corresponds an annular family in U_2 with the same index. Let F be a smooth annular family in U_1 with bounding cycles D_3 and D_4, $D_3 \subset \text{int } D_4$. All the cycles in F have the same orientation. Otherwise F would contain a bent annular family. Let G be the annular family bounded by $f(D_3)$ and $f(D_4)$. G is smooth since all cycles are oriented in the same direction. Let S_1 and S_2 be cross-sections of F and G respectively. We may define a mapping $T_i : S_i \to S_i$ as follows:

$$T_i(x) = \begin{cases} x & \text{if} \quad C_i(x) \in M_i \\ x\pi_i t(x) & \text{if} \quad C_i(x) \in M_i \end{cases}$$

where $t(x) \in R^+$ is such that $x\pi_i t(x) \in S_i$ and $x\pi_i(0, t(x)) \cap S_i = \phi$. Evidently T_i is one-to-one and onto. The continuity of T_i follows from the fact that for a given $x \in S_i$, subarcs of S_i containing x as a nonend-point generate arbitrarily small neighborhoods of x, i.e., if A is any subarc of S_i containing x as a nonend-point, $A\pi_i(-a, a)$ is a neighborhood of x for $a > 0$. Hence T_i is a homeomorphism of S_i onto itself whose fixed points correspond to the intersection of the cycles of (U_i, π_i) with S_i, $i = 1, 2$.

For $i = 1, 2$, let f_i be a homeomorphism of S_i onto $[0, 1]$ such that $f_1(D_3 \cap S_1) = 0$ and $f_2(f(D_3) \cap S_2) = 0$. Let

$$x_1, x_2 \in M_1 \cap S_1, \ C_1(x_2) \subset \text{int } C_1(x_1) \ ,$$

and $y \notin M_1 \cap S_1$ for y in the subarc of S_1 with end-points x_1 and x_2. Then $C_1(x_1) \cup C_1(x_2)$ forms the boundary of a simple annular family. Therefore $C_1(x_1)J_1^+ C_1(x_2)$ or $C_1(x_2)J_1^+ C_1(x_1)$. For definiteness let

$C_1(x_1)J_1^+C_1(x_2)$. The proof is similar if $C_1(x_2)J_1^+C_1(x_1)$. Then $y\pi_i t_i(y)$ $(=T_1(y))$ intersects S_1 between y and x_1 by the Monotonicity Theorem [1, VIII, 4.13]. Then $f_1T_1(y) - f_1(y)$ has constant sign for y in the subarc of S_1 with end-points x_1 and x_2. Similarly $f_2T_2(y) - f_2(y)$ has the same sign for all y in the subarc of S_2 with end-points $f(C_1(x_1)) \cap S_2$ and $f(C_2(x_2)) \cap S_2$. Since $D_3 \subset \mathrm{int}\, D_4$, $f_1(D_3 \cap S_1) = 0$, and $f_1(D_4 \cap S_1) = 1$, f increases from x_2 to x_1 along S_1. Similarly f_2 increases from $f(C_2(x_1)) \cap S_2$ to $f(C_1(x_2)) \cap S_2$ along S_2. Then $f_1T_1(y) - f_1(y)$ and $f_2T_2(y) - f_2(y)$ are both positive for y in the appropriate subarc. Hence T_1 and T_2 are similar. F and G are equivalent.

We have shown that to each bent or smooth annular family of U_1 there corresponds an equivalent bent or smooth annular family of U_2. The ordering (with respect to nearness to the critical point) of the annular families of U_1 is the same as the ordering of the equivalent annular families of U_2 because the order of the boundaries of the annular families is preserved by (1). Hence $bs(U_1) = bs(U_2)$. By Theorem 3.5, (U_1, π_1) and (U_2, π_2) are equivalent. Theorem 2.3 yields the desired result.

REMARKS. 1. If we replace (2) and (3) by (2') and (3') the conclusion of Proposition 4.4 remains valid.

(2') f is orientation reversing

(3') $C_1, J_1^+C_2$ if and only if $f(C_1)J_2^-f(C_2)$ for $C_1, C_2 \in M_1$.

The proof is analogous to that of Proposition 4.3.

2. Conditions (2) and (3) are independent as the following examples show. Let F and G be the simple annular families indicated by Figures 5 and 6.
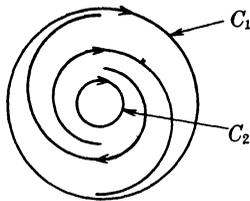
Figure 5.
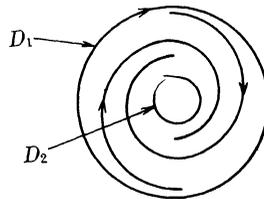
Figure 6.
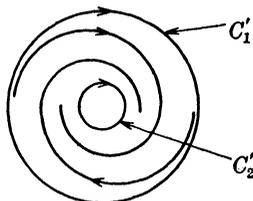
Figure 7.
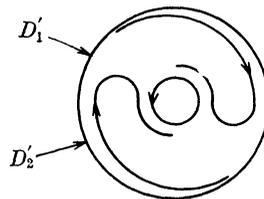
Figure 8.

Let $f(C_i) = D_i$. Then (2) is satisfied and (3) is not.

Now let F' and G' be the simple annular families indicated by Figures 7 and 8.

Let $f(C_i') = D_i'$. Then (3) is satisfied and (2) is not.

THEOREM 4.5. $(U_1 - \{p_1\}, \pi_1)$ and $(U_2 - \{p_2\}, \pi_2)$ are isomorphic if and only if either (1), (2) and (3) or (1), (2'), and (3') are satisfied.

Proof. *If*: This is Proposition 4.4 and Remark 1 following Proposition 4.4.

Only If: Let (h, φ) be an isomorphism of $(U_1 - \{p_1\}, \pi_1)$ onto $(U_2 - \{p_2\}, \pi_2)$. Then h defines a homeomorphism $h': M_1 \to M_2$. h may be extended to a homeomorphism of U_1 onto U_2 by defining $h(p_1) = p_2$. Hence we may assume h is a homeomorphism of U_1 onto U_2. Let C_1, $C_2 \in M_1$, $C_1 \subset \text{int } C_2$. Then $\partial h(\text{int } C_2) = h(C_2)$. This and Corollary 1.10 imply $h(C_1) \subset \text{int } h(C_2)$. Since $h'(C_i) = h(C_i)$, $i = 1, 2$, h' satisfies (1).

We now consider the case that (h, φ) is proper. Then $\varphi(x, \cdot)$ is an increasing function implies the orientation of C and $h'(C)$ are the same for $C \in M_1$. h' satisfies (2). By applying Proposition 2.5, h' satisfies (3).

If (h, φ) is dual, then $\varphi(x, \cdot)$ is a decreasing function and the orientation of C and $h'(C)$ are reversed for any $C \in M_1$. Proposition 2.5 implies h' satisfies (3').

Thus if (h, φ) is proper, (1), (2), and (3) are satisfied and if (h, φ) is dual, (1), (2'), and (3') are satisfied.

5. **Classification of $o +$ stable isolated critical points.** The behavior of trajectories in a sufficiently small neighborhood of an $o +$ stable isolated critical point has been completely determined, [1, VII, 4.1]. The theorem states

THEOREM 5.1. *Let p be an $o +$ stable isolated critical point of a dynamical system (X, π). Then p satisfies precisely one of the following conditions*:
  (1) *p is asymptotically orbitally $+$ stable (a focus).*
  (2) *Some neighborhood of p consists entirely of cycles (a center).*
  (3) *p is a center-focus.*
Furthermore p is an $o +$ stable if and only if (1) holds, and o stable if and only if (2) or (3) hold.

We have just classified center-foci. Centers are classified in [4]. To complete the classification of an $o +$ stable critical point, we must classify foci.

In [1, VII, 4.3] it is proved that

PROPOSITION 5.1. *Each $o +$ stable isolated critical point, not a center, possesses arbitrarily small neighborhoods bounded by closed transversal curves.*

For $i = 1, 2$, let p_i be a focus of (R^2, π_i), A_i be the region of attraction ([1, IV, 4.9]) for p_i, and S_i be a closed transversal curve contained in A_i. Then it can be shown that $S_i \pi_i R^1 = A_i - \{p_i\}$, $i = 1, 2$, i.e., $A_i - \{p_i\}$ is parallelizable.

PROPOSITION 5.2. (A_1, π_1) *and* (A_2, π_2) *are isomorphic.*

Proof. Let $f \colon S_1 \to S_2$ be a homeomorphism. Then for each $x \in A_1 - \{p_1\}$ there exists a unique $t(x)$ such that $x \pi_1 t(x) \cap S_1$ (1, VII, 4.4]). Moreover since $A_i - \{p_i\}$ is parallelizable, $t(\cdot)$ is a continuous mapping of $A_1 - p_1$ onto R^1.

It can be verified that

$$h(x) = \begin{cases} f(x\pi_1 t(x))\pi_2 - t(x) & \text{if } x \ne p_1 \\ p_2 & \text{if } x = p_1 \end{cases}$$

is a homeomorphism. Noting that $t(x\pi_1 s) = t(x) - s$ we get that $h(x\pi_1 s) = h(x)\pi_2 s$ for $x \in A_1$ and $s \in R^1$. If we denote the identity mapping of R^1 onto itself by i, (h, i) is an isomorphism of (A_1, π_1) onto (A_2, π_2).

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