A CONJECTURE AND SOME PROBLEMS ON PERMANENTS

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Let $A = [a_{ij}]$ denote an $n \times n$ matrix and let $E$ be the $n \times n$ identity matrix. We will designate by $\det A$ and $\perm A$ the determinant and the permanent of $A$ respectively. The polynomial $\varphi(z) = \det (zE - A)$ plays a fundamental role in matrix theory. Similarly we can consider the polynomial $f(z) = \perm (zE - A)$ which has been object of several studies recently, particularly when $A$ is a doubly stochastic matrix. The aim of the present paper is to give some results on the existence of matrices satisfying certain conditions involving the roots of this polynomial.

Let $M_n$ and $\mathcal{M}_n$ be the regions defined as follows: $z \in M_n$ if and only if there exists a stochastic matrix of order $n$ with $z$ as characteristic root; $(z_1, \ldots, z_n) \in \mathcal{M}_n$ if and only if there exists a stochastic matrix of order $n$ whose $n$ characteristic roots are the complex numbers $z_1, \ldots, z_n$.

Similarly we define the regions $D_n$ and $\mathcal{D}_n$ respectively when 'stochastic' is replaced by 'doubly stochastic'. $M_n$ was determined by Karpelevič [3] but the determination of the other three regions seems to be a very difficult problem and has not yet been solved (see [7], [8], [9]).

Replacing in the definitions of $M_n$, $\mathcal{M}_n$, $D_n$ and $\mathcal{D}_n$ 'characteristic root' by 'root of the polynomial $f(z) = \perm (zE - A)$' we can define four other regions which we shall denote by $M^*_n$, $\mathcal{M}^*_n$, $D^*_n$ and $\mathcal{D}^*_n$ respectively. To our knowledge no attempt has been made to determine these regions. Their determination is likely to be a much harder problem than the determination of $M_n$, $\mathcal{M}_n$, $D_n$ and $\mathcal{D}_n$.

Some problems dealing with the characteristic values of a matrix (like some of the problems mentioned in [6]) can be replaced by similar problems dealing with the roots of $f(z) = \perm (zE - A)$.

Examples: (1) find a necessary and sufficient condition for the numbers $a_1, \ldots, a_n$ and $z_1, \ldots, z_n$ to be the principal elements of a symmetric $A$ and the roots of $f(z) = \perm (zE - A)$ respectively; (2) find a necessary and sufficient condition for the numbers $\lambda_1, \ldots, \lambda_n$ and $z_1, \ldots, z_n$ to be the characteristic roots of an $n \times n$ matrix $A$ and the roots of $f(z) = \perm (zE - A)$ respectively. In the sequel we give some results on problems of this nature.

2. Let
\[ J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & \ddots \\ & & 1 \\ 0 & & \lambda_i \end{bmatrix} \text{ (of type } s_i \times s_i), \]

\[ X_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{is_i} \end{bmatrix}, \quad Y_i = [y'_1, \ldots, y'_{i}] \]

and

\[ C = \begin{bmatrix} J_1 & 0 & \cdots & 0 & X_1 \\ 0 & J_2 & \cdots & 0 & X_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_m & X_m \\ Y_1 & Y_2 & \cdots & Y_m & q \end{bmatrix} \]

**Lemma.** If \( C \) is the matrix described above and \( E \) denotes the appropriate identity matrix then

\[
\text{perm} (zE - C) = \sum_{i=1}^{m} \sum_{h=0}^{s_i-1} b_{ih}(z - \lambda_i)^h \prod_{j=1}^{m} (z - \lambda_j)^{s_j} + (z - q) \prod_{j=1}^{m} (z - \lambda_j)^{s_j},
\]

where

\[
b_{ih} = (-1)^{s_i+h+1} \sum_{j=1}^{s_i} y'_j x'_{j+s_i+1-h} \quad (h = 0, \ldots, s_i - 1). \]

**Proof.** Let

\[ C_i = \begin{bmatrix} J_i & 0 & \cdots & 0 & X_i \\ 0 & J_{i+1} & \cdots & 0 & X_{i+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_m & X_m \\ Y_i & Y_{i+1} & \cdots & Y_m & q \end{bmatrix} \]

Now we expand \( \text{perm} (zE_i - C_i) \) (where \( E_i \) is the identity matrix of the same order as \( C_i \)) in terms of its first \( s_i \) rows. The submatrices contained in these rows with permanent nonnecessarily zero are: \( zE_i - J_i \) (\( E^{(i)} \) denotes the identity matrix of the same order as \( J_i \)) and the submatrices obtained from \( zE^{(i)} - J_i \) by striking out the \( \rho^{th} \) column (\( \rho = 1, \ldots, s_i \)) and bordering on the right hand side with the column \(-X_i\). We denote this submatrix by \( H_\rho \). It is not difficult to see that
\[
\text{perm } H_\rho = \sum_{r=0}^{s_\rho-1} (-1)^{s_\rho+r+1} x_{\rho+r}(z - \lambda_i)^{s_i-r-1}.
\]

Let \( \tilde{H}_\rho \) denote the complementary submatrix of \( H_\rho \) in \( zE_i - C_i \). It can be easily seen that
\[
\text{perm } \tilde{H}_\rho = -y_\rho^i \prod_{j=i+1}^{m} (z - \lambda_j)^{s_j}.
\]

We can now write
\[
\text{perm } (zE_i - C_i) = \sum_{\rho=1}^{s_i} \text{perm } H_\rho \text{ perm } \tilde{H}_\rho
+ \text{perm } (zE_{i+1} - J_i) \text{ perm } (zE_{i+1} - C_{i+1})
= \sum_{\rho=1}^{s_i} \sum_{r=0}^{s_i-\rho} (-1)^{s_i+r+1} y_\rho^i x_{\rho+r}(z - \lambda_i)^{s_i-r-1} \prod_{j=i+1}^{m} (z - \lambda_j)^{s_j}
+ (z - \lambda_i)^{s_i} \text{ perm } (zE_{i+1} - C_{i+1}).
\]

Interchanging the order of the first two sums we get
\[
\text{perm } (zE_i - C_i) = \sum_{\rho=1}^{s_i-1} \sum_{r=1}^{s_i-\rho} (-1)^{s_i+r+1} y_\rho^i x_{\rho+r}(z - \lambda_i)^{s_i-r-1} \prod_{j=i+1}^{m} (z - \lambda_j)^{s_j}
+ (z - \lambda_i)^{s_i} \text{ perm } (zE_{i+1} - C_{i+1})
= \sum_{\rho=1}^{s_i-1} \sum_{r=1}^{s_i-\rho} (-1)^{s_i+r+1} y_\rho^i x_{\rho+r}(z - \lambda_i)^{s_i-r-1} \prod_{j=i+1}^{m} (z - \lambda_j)^{s_j}
+ (z - \lambda_i)^{s_i} \text{ perm } (zE_{i+1} - C_{i+1}).
\]

We now set \( i = 1 \), use induction, and after some manipulation we obtain the formula stated in the lemma.

We proceed to our main result.

**Theorem 1.** Given any \( n \) complex numbers \( a_1, \ldots, a_n \) and a polynomial \( f(z) = z^n - cz^{n-1} + \cdots \), there exists a square matrix \( A \) of order \( n \) with \( a_1, \ldots, a_n \) as principal elements and such that \( f(z) = \text{perm } (zE - A) \) if and only if \( a_1 + \cdots + a_n = c \). If this condition is satisfied and both \( a_1, \ldots, a_n \) and the coefficients of \( f(z) \) are real, \( A \) can be chosen real.

**Proof.** We prove first the 'if' part. If we perform a permutation on the rows of a square matrix \( A \) and then the same permutation on its columns, the roots of \( f(z) = \text{perm } (zE - A) \) are not altered. Hence we can, without loss of generality, take the numbers \( a_1, \ldots, a_n \) in any order. Thus we will assume that the first \( s_1 \) numbers from among \( a_1, \ldots, a_{n-1} \) have the common value \( \lambda_1 \), the following \( s_2 \) numbers have the common value \( \lambda_2 \), \ldots, the last \( s_m \) numbers have the common value \( \lambda_m \) and that \( \lambda_i \neq \lambda_j \) for \( i \neq j \). Consider now the matrix \( C \) of the
Lemma with \( q = a_n \) and all the \( x_i = 1 \). We will show that we can choose \( Y_1, \ldots, Y_m \) such that \( \text{perm} (zE - C) = f(z) \).

Let \( g(z) = \prod_{j=1}^n (z - \lambda_j)^{x_j} \). Using the formula of the lemma we can write

\[
\frac{\text{perm} (zE - C)}{g(z)} = \sum_{i=1}^m \sum_{h=0}^{x_i-1} \frac{b_{ih}}{(z - \lambda_i)^{x_i-h}} + z - q.
\]

Let us now resolve \( f(z)/g(z) \) into partial fractions. Bearing in mind that \( f(z) = z^n - (\sum_{i=1}^n a_i)z^{n-1} + \cdots \) we get

\[
\frac{f(z)}{g(z)} = \sum_{i=1}^m \sum_{h=0}^{x_i-1} \frac{d_{ih}}{(z - \lambda_i)^{x_i-h}} + z - q.
\]

Let us take \( b_{ih} = d_{ih} \). With this choice of the \( b_{ih} \) we have \( f(z) = \text{perm} (zE - C) \) as required. Now we compute the \( y_h^i \) by \( b_{ih} = (-1)^{x_i+h+1} \sum_{j=1}^h y_j^i (h = 0, \ldots, s_i - 1; i = 1, \ldots, m) \) which is a system of linear equations, always compatible.

If we suppose the numbers \( a_1, \ldots, a_n \) as well as the coefficients of \( f(z) \) real it follows from (I) that the \( d_{ih} \) and therefore the \( b_{ih} \) are also real. In this case \( C \) can, clearly, be chosen real.

The "only if" part of the theorem is an immediate consequence of the formula

\[
\text{perm} (zE - A) = z^n + \sum_{p=1}^n \sum_{1 \leq i_1 < \cdots < i_p \leq n} (-1)^p \text{perm} A(i_{i_1}, \ldots, i_{i_p}) z^{n-p}
\]

where \( A(i_{i_1}, \ldots, i_{i_p}) \) denotes the principal submatrix of \( A \) contained in the rows \( i_{i_1}, \ldots, i_{i_p} \).

Concerning the problem (1) mentioned in §1 of the present paper, we have been able to prove the following partial result.

**Theorem 2.** Let \( a_1, \ldots, a_n \) be real numbers and suppose that there exists an index \( i_0 \) such that \( i \neq j; i, j \neq i_0 \) implies \( a_i \neq a_j \). Let \( f(z) = z^n - cz^{n-1} + \cdots \) be a given polynomial with real coefficients such that \( c = \sum_{i=1}^n a_i \).

If \( f(a_j), \prod_{j=0}^n (a_j - a_i) \geq 0 \) \((j = 1, \ldots, n; j \neq i_0)\),

there exists an \( n \times n \) real symmetric matrix \( A \) with \( a_1, \ldots, a_n \) as principal elements and such that \( f(z) = \text{perm} (zE - A) \).

We omit the proof which follows closely the technique used in the proof of the Theorem 1.

3. We denote by \( \Omega_n \) the set of all doubly stochastic matrices of order \( n \). When \( A \in \Omega_n \), \( f(z) = \text{perm} (zE - A) \) enjoys some interesting
properties as for instance: the roots of $f(z)$ lie in or on the boundary of the unit disc $|z| \leq 1$ (see [1] and [4]). For the real roots of $f(z)$ it is known that they lie in the interval $0 < x \leq 1$. We have been led to the following

**Conjecture.** Let $A$ be an $n \times n$ doubly stochastic irreducible matrix. If $n$ is even, then $f(z) = \text{perm}(zE - A)$ has no real roots; if $n$ is odd, then $f(z) = \text{perm}(zE - A)$ has one and only one real root.

It can be seen by direct computation that the conjecture is true in the following cases:

(a) $A$ is a $2 \times 2$ real (not necessarily nonnegative) irreducible matrix all of whose row and column sums are 1.

(b) $A$ is a $3 \times 3$ real (not necessarily nonnegative) irreducible symmetric matrix all of whose row and column sums are 1.

(c) $A$ is the $n \times n$ matrix all of whose entries are equal to $1/n$.

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**References**


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