

Pacific Journal of Mathematics

EXTENDING HOMEOMORPHISMS

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Theorem 1 of this paper establishes a necessary and sufficient condition that a locally flat imbedding $f: B^k \rightarrow R^n$ of a k -cell in euclidean n -space R^n admits an extension to a homeomorphism $F: R^n \rightarrow R^n$ onto R^n such that $F| (R^n - B^k)$ is a diffeomorphism which is the identity outside some compact set in R^n . An analogous result for locally flat imbeddings of a euclidean $(n-1)$ -sphere into R^n is proved. A lemma which generalizes a theorem of Huebsch and Morse concerning Schoenflies extensions without interior differential singularities is also established.

Let the points of euclidean n -space R^n be written $x = (x^1, \dots, x^n)$, and provide R^n with the usual euclidean norm $\|x\| = [\sum (x^i)^2]^{1/2}$. We set $S_r = \{x \in R^n \mid \|x\| = r\}$, (and $S = S_1$). If M is a topological $(n-1)$ -sphere in R^n , we denote the bounded component of $R^n - M$ by \mathring{M} , and the closure of \mathring{M} in R^n by \bar{M} . We refer the reader to §1 of [2] for the definition of the terms admissible cone K_z , conical point, axis of singular approach, and cone $K_z(\Sigma)$, where Σ is a euclidean $(n-1)$ -sphere in R^n .

LEMMA 1. *Let z be an arbitrary point of S and φ a sense-preserving homeomorphism into R^n of an open neighborhood N of S such that φ carries points inside S to points inside $\varphi(S)$, and $\varphi|(N - S)$ is a C^m -diffeomorphism. There then exists a homeomorphism Φ of R^n onto R^n and a cone K_z (resp. \check{K}_z) with axis interiorly normal (resp. exteriorly normal) to S at z , such that if $X \subset N$ is a sufficiently small open neighborhood of S ,*

$$\Phi(x) = \varphi(x) \quad [x \in X - \{K_z(S) \cup \check{K}_z\}] ,$$

$\Phi|(R^n - S)$ is a C^m -diffeomorphism, and Φ is the identity outside some compact set in R^n .

REMARK. We note that a direct application of the proof of Theorem 1.2 of [2] will yield the conclusions of Lemma 1 except for single differential singularities in each component of $R^n - S$.

Proof of Lemma 1. The proof of Lemma 1 will be a variation of the proof of Theorem 1.2 of [2]. We can assume that $0 \in \mathring{\varphi}(S)$. Let $\delta \in (\frac{1}{2}, 1)$ be a constant so near 1 that $S_\delta \subset N$. Using Theorem 1.1 of [2], there is a homeomorphism $f: JS \rightarrow R^n$ into R^n such that

$$f | (JS - \dot{J}S_\delta) = \varphi | (JS - \dot{J}S_\delta) ,$$

and $f | (\dot{J}S - 0)$ is a C^m -diffeomorphism. We can also assume that $f(0) = 0$. We now apply Lemma 5.3 of [2] to $f | (\dot{J}S \cap N)$, S_δ , and the fixed point $y = \delta z \in S_\delta$, and conclude that if $\rho > 1$ is a sufficiently large constant, there exists a homeomorphism θ of R^n onto R^n such that $\theta(x) = f(x)$ for $x \in Y \cup JS_\delta$, where Y is a suitable neighborhood of S_δ , $\theta | (R^n - \{0 \cup \rho z\})$ is a C^m -diffeomorphism, where the point ρz is a conical point of θ with cone $K_{\rho z}$ of singular approach to ρz whose axis is interiorly normal to S_ρ at ρz , and if a constant $\nu \in (0, 1)$ is sufficiently near 1, θ reduces to the identity on $R^n - \{B_{\nu\rho} \cup K_{\rho z}(S_\rho)\}$.

Let ω be a radial C^∞ -diffeomorphism of R^n onto R^n such that $\omega(x) = x$ for $\|x\| \leq \varepsilon$, and $\omega(x) = \rho x$ for $\|x\| \geq 1 - \varepsilon$, where $0 < \varepsilon < \frac{1}{2}$. We then set $\Phi(x) = f\omega^{-1}\theta^{-1}\omega(x)$ for $x \in JS$. If $\zeta: R^n \rightarrow R^n$ is the C^∞ -diffeomorphism $\zeta(x) = \rho x$, we set $K_x = \zeta^{-1}(K_{\rho x})$ and $\hat{X} = \zeta^{-1}(JS_\rho - B_{\nu\rho})$. We can assume that ν is so near 1 that $1 - \varepsilon < \nu$ and $\delta < \nu$, so that $\zeta^{-1}(JS_\rho - B_{\nu\rho}) = \omega^{-1}(JS_\rho - B_{\nu\rho})$ and $\hat{X} \subset JS - \dot{J}S_\delta$. Then we see that $\Phi | (\hat{X} - K_x(S)) = \varphi | (\hat{X} - K_x(S))$ and $\Phi | \dot{J}S$ is a C^m -diffeomorphism (which reduces to the identity on a neighborhood of 0). We have therefore defined $\Phi | JS$ with the desired properties. We then define Φ on $R^n - \dot{J}S$ in an analogous manner to satisfy the conclusions of Lemma 1 by regarding R^n , with the "point at infinity" added, as an n -sphere, and using the geometry of inversion. This completes the proof of Lemma 1.

REMARK. As the proof of Lemma 1 shows, we also could state corresponding "one-sided" lemmas in which the differentiability is only assumed either outside or inside of S . For example, if only $\varphi | (N - JS)$ is a C^m -diffeomorphism, then \check{K}_x and Φ exist where $\Phi | (R^n - JS)$ is a C^m -diffeomorphism which is the identity outside some compact set in R^n .

We now fix the integer n , and for any integer $k \leq n$, we regard $R^k \subset R^n$ as consisting of those points $x = (x^1, \dots, x^n)$ with $x^{k+1} = \dots = x^n = 0$. We denote the unit k -cell in $R^k \subset R^n$ by B^k . For convenience, we assume in what follows that "diffeomorphism" means " C^∞ -diffeomorphism."

THEOREM 1. *Let $f: B^k \rightarrow R^n$ be a homeomorphism into R^n such that each point $x \in B^k$ has an open neighborhood V_x in R^n and a sense-preserving homeomorphism $f_x: V_x \rightarrow R^n$ into R^n satisfying*

$$f_x | (V_x \cap B^k) = f | (V_x \cap B^k)$$

and $f_x | (V_x - B^k)$ is a diffeomorphism. Then there exists a homeomorphism F of R^n onto R^n such that

- (i) $F|B^k = f$,
- (ii) $F|(R^n - B^k)$ is a diffeomorphism,
- (iii) F is the identity outside some compact set in R^n .

Proof. An examination and easy modification of the proof of Proposition C in [1] shows that there exists an open neighborhood N of B^k in R^n and a sense-preserving homeomorphism $\varphi: N \rightarrow R^n$ into R^n such that $\varphi|B^k = f$ and $\varphi|(N - B^k)$ is a diffeomorphism. Let $J\mathcal{S} \subset N$ be a smooth convex n -cell in R^n , where \mathcal{S} is a smooth $(n - 1)$ -sphere in R^n such that $B^k \subset \mathcal{S}$, and let z be an arbitrary point in B^k . Using the remark following Lemma 1, there exists an open neighborhood Y of \mathcal{S} in R^n , a cone K_z with axis exteriorly normal to \mathcal{S} at z , and a homeomorphism Φ of R^n onto itself such that $\Phi(x) = \varphi(x)$ for $x \in (Y - K_z)$, and $\Phi|(R^n - J\mathcal{S})$ is a diffeomorphism which is the identity outside some compact subset of R^n . We then define $F: R^n \rightarrow R^n$ by

$$\begin{aligned} F(x) &= \varphi(x) && [x \in (Y - K_z) \cup J\mathcal{S}] \\ F(x) &= \Phi(x) && [x \in Y \cup (R^n - J\mathcal{S})]. \end{aligned}$$

It is clear that F satisfies the conclusions of Theorem 1.

LEMMA 2. *Let $f: R^{n-1} \rightarrow R^n$, $n \geq 4$ be an imbedding of R^{n-1} as a closed subset of R^n . Suppose for each $x \in R^{n-1}$ there is a neighborhood V_x of x in R^n and a homeomorphism $f_x: V_x \rightarrow R^n$ into R^n such that $f_x|(V_x \cap R^{n-1}) = f|(V_x \cap R^{n-1})$, and $f_x|(V_x - R^{n-1})$ is a diffeomorphism. Then there is a homeomorphism F of R^n onto R^n such that $F|R^{n-1} = f$ and $F|(R^n - R^{n-1})$ is a diffeomorphism.*

PROOF. As in the proof of Lemma 2, (cf. Proposition C_1 of [1]), there is an open neighborhood U of R^{n-1} in R^n and a homeomorphism $\Phi: U \rightarrow R^n$ into R^n such that $\Phi|R^{n-1} = f$ and $\Phi|(U - R^{n-1})$ is a diffeomorphism. Let $\mathcal{R}_1^{n-1}, \mathcal{R}_2^{n-1}$ be diffeomorphs (under good C^1 -approximations to the inclusion) of R^{n-1} as closed subsets of R^n such that \mathcal{R}_1^{n-1} and \mathcal{R}_2^{n-1} are contained in opposite components of $R^n - R^{n-1}$, and if V denotes the component of $R^n - \{\mathcal{R}_1^{n-1} \cup \mathcal{R}_2^{n-1}\}$ which contains R^{n-1} , then $\bar{V} = V \cup \mathcal{R}_1^{n-1} \cup \mathcal{R}_2^{n-1} \subset U$. Let V_1 (resp. V_2) denote that component of $R^n - \mathcal{R}_1^{n-1}$ (resp. $R^n - \mathcal{R}_2^{n-1}$) which does not contain \mathcal{R}_2^{n-1} (resp. \mathcal{R}_1^{n-1}). Applying the corollary to Theorem 1 of [3], there are diffeomorphisms θ_1, θ_2 of R^n onto R^n such that $\theta_i|\mathcal{R}_i^{n-1} = \Phi|\mathcal{R}_i^{n-1}$, $i = 1, 2$. Since any orientation-preserving diffeomorphism of R^{n-1} on itself is diffeotopic to the identity, we may assume that $\theta|0_i = \Phi|0_i$, where 0_i is an open neighborhood of \mathcal{R}_i^{n-1} in $R^n - R^{n-1}$, $i = 1, 2$. Then $F: R^n \rightarrow R^n$ defined by

$$\begin{aligned}
 F(x) &= \theta_1(x) & [x \in O_1 \cup V_1], \\
 F(x) &= \theta_2(x) & [x \in O_2 \cup V_2], \\
 F(x) &= \Phi(x) & [x \in O_1 \cup O_2 \cup V]
 \end{aligned}$$

satisfies the conclusions of Lemma 2.

Using one point compactification and stereographic projection, Theorem 2 below is obtained readily from Lemmas 2 and 1.

THEOREM 2. *Let $f: S \rightarrow R^n$ be a homeomorphism into R^n , $n \geq 4$, and let p be an arbitrary point in S . Suppose each point $x \in S - p$ has an open neighborhood V_x in R^n and a sensepreserving homeomorphism $f_x: V_x \rightarrow R^n$ into R^n such that $f_x|(V_x \cap S) = f|(V_x \cap S)$, $f|(V_x - S)$ is a diffeomorphism, and f_x carries points inside S to points inside $f(S)$. Then there is a homeomorphism F of R^n onto itself such that $F|S = f$, and $F|(R^n - S)$ is a diffeomorphism which is the identity outside some compact subset in R^n .*

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