UNKNOTTING UNIONS OF CELLS

THOMAS BENNY RUSHING
In this note we consider the problem of determining whether the union of cells is nicely embedded in the $n$-sphere if each of the cells is nicely embedded. This question is related to many embedding problems. For instance, the $n$-dimensional Annulus Conjecture (now known to be true for $n \neq 4$) is a special case. Cantrell and Lacher have shown that an affirmative answer implies local flatness of certain submanifolds. Also, this question is related to the conjecture that an embedding of a complex into the $n$-sphere which is locally flat on open simplexes is $\varepsilon$-tame in codimension three.

The problem mentioned above was first investigated by Doyle [9] [10] in the three dimensional case and by Cantrell [2] in high dimensions and later by Lacher [15], Cantrell and Lacher [3][4], Kirby [13], Černavskii [5][6] and the author [17]. Also, Sher [21] has generalized a construction of Debrunner and Fox [8] to obtain counterexamples in certain cases. Since the $n$-dimensional Annulus Conjecture, $n \neq 4$, is now known to be true [14], only two results of § 7 of [17] remain of interest. First we will prove a strengthened form of one of those results and we greatly simplify the proof by employing the powerful tools now available. In particular we prove the following theorem.

**Theorem 1.** If $D^n_1$ and $D^n_2$ are cells in $S^n$, $n > 5$, of dimensions $m_1$ and $m_2$, respectively, and if $D^n_1 \cap D^n_2 = \partial D^n_1 \cap \partial D^n_2 = D$ is a $k$-cell (possibly empty), $n - k \geq 4$, which is locally flat in $\partial D^n_1$, in $\partial D^n_2$ and in $S^n$ and is such that $D^n_1 - D$ and $D^n_2 - D$ are locally flat, then there is an ambient isotopy $e_i$ of $S^n$ such that $e_i(D^n_1)$ and $e_i(D^n_2)$ are simplexes and $e_i(D^n_1 \cap D^n_2)$ is a face of each.

**Remark.** If the above theorem is modified by requiring $n - k = 3$, then counterexamples can be constructed for any $m_1$ and $m_2$ (see [21]).

**Proof of Theorem 1.** Every orientation preserving homeomorphism of $S^n$, $n \geq 5$, is stable [14], hence isotopic to the identity. It will then suffice to construct an orientation preserving homeomorphism $e_i$ satisfying the conclusion of the theorem. By Theorem 5.2 of [1], we may assume that $D^n_1$ and $D^n_2$ are locally flat. For $i = 1, 2$, it is easy to construct a homeomorphism $f_i : S^n \to S^n$ such that $f_i(D^n_i, D) = (\Delta^m_i, \Delta^k)$ where $\Delta^m_i$ is an $m_i$-simplex and $\Delta^k$ is a $k$-face. Thus, by using $f_i$, $i = 1, 2$, and Lemma 3.6 of [18], we can construct locally flat $n$-cells $D^n_i$ and
$D_i^n$ satisfying the following conditions,

1. $D_i^n \cap D_j^n = \partial D_i^n \cap \partial D_j^n = D$,
2. $D$ is locally flat in $\partial D_i^n$ and $\partial D_j^n$, and
3. $(D_i^n, D_j^n)$ is a trivial cell pair, $i = 1, 2$.

Let $A_i^n$ and $A_j^n$ be $n$-simplexes in $S^n$ such that $A_i^n \cap A_j^n = \Delta$ is a $k$-face of each. We will now construct an orientation preserving homeomorphism $h$ of $S^n$ such that $h((D_i^n, D_j^n, D)) = (A_i^n, A_j^n, \Delta)$. It is easy to obtain an orientation preserving homeomorphism $h_1$ of $S^n$ such that $h_1((D_i^n, D_j^n, D)) = (A_i^n, D)$. Let $A_o$ be the $n$-simplex having as vertices the midpoints of the segments which join the vertices of $A_i^n$ with the barycenter of $A_i^n$. Let $f: I^k \to \Delta$ be a PL-homeomorphism and define $F: P \times I \to A_i^n$ by extending linearly on each segment $\{x\} \times I, x \in I^k$, the map which takes $(x, 0)$ to $f(x)$ and $(x, 1)$ to the midpoint of the segment joining $f(x)$ and the barycenter of $A_i^n$. Then, $E = F(I^k \times \{1\})$ is a $k$-face of $A_o$. Now, by using the Annulus Theorem, it is easy to get an orientation preserving homeomorphism $h_2$ of $S^n$ such that

1. $h_2((D_i^n, D_j^n, D)) = (A^n, J^n, \Delta)$, and
2. $h_2 \vert \partial A_o \cup E = 1$.

Let $A$ denote $C_1(S^n - (A_o \cup A_i^n))$. Then, the embedding $h_3 F: I^k \times I \to A$ satisfies the hypotheses of Theorem 1 of [19]; hence, by that theorem there is a homeomorphism $h_4$ of $A$ such that $h_4 \vert \partial A_o \cup \partial A_i^n = 1$ and $h_3 h_2 F: I^k \times I \to A$ is PL. Extend $h_4$ to all of $S^n$ by way of the identity. Consider the two PL embeddings $F \vert \partial I^k \times I: \partial I^k \times I \to A$ and $h_3 h_2 F \vert \partial I^k \times I: \partial I^k \times I \to A$. These two embeddings clearly satisfy the hypotheses of Theorem 4 of [11]; therefore, by that theorem there is a PL homeomorphism $h_5$ of $A$ such that $h_5 h_4 h_3 h_2 F = F$ and $h_5 \vert \partial A_o \cup \partial A_i^n = 1$. Extend $h_5$ to $S^n$ by the identity. Now, the PL embeddings $h_3 h_5 h_3 h_2 F: I^k \times I \to A$ and $F: I^k \times I \to A$ satisfy the hypothesis of Theorem 4 of [11] and so by another application of that theorem we get a PL homeomorphism $h_6$ of $A$ such that $h_6 h_5 h_3 h_2 F = F$ and $h_6 \vert \partial A_o \cup \partial A_i^n = 1$. Extend, $h_6$ to $S^n$ by the identity.

Let $p: S^n \to S^n$ be a map such that

1. $p(A_o) = A_i^n$,
2. $p \vert h_i(D_j^n) \cup A_i^n = 1$, and
3. $p \vert S^n - F(I^k \times I)$ is one-to-one, and $p(F([x] \times I)) = F(x, 0)$

for each $x \in I^k$.

It is now easy to check that $h = p h_6 h_5 h_3 h_2 p^{-1} h_4$ is the desired homeomorphism that flattens the pair $D_i^n \cup D_j^n$.

Let $A_i^{n-1}$ be a face of $A_i^n$ of dimension $m_i - 1$ which has $A$ as a face. Let $\delta_i$ denote the face of $A_i^n$ dual to $A_i^{n-1}$ and let $\tilde{\delta}_i$ denote the barycenter of $\delta_i$. Now, let $A_i^{n-1}$ be the $m_i$-simplex $A_i^{n-1} \ast \tilde{\delta}_i$. Then, it is easy to get a homeomorphism $g_i: A_i^n \to A_i^n$ such that $g_i h(D_i^n) = A_i^{n-1}$ and $g_i \vert A = 1$. Furthermore, we may assume that $g_i \vert \partial D_i^n$ is orientation
preserving for if it is not we may follow $g_i$ by an appropriate reflection of $\mathcal{A}_i$. Let $\mathcal{A}_i$, $i = 1, 2$, be an annulus pinched at $\mathcal{A}$, in particular, $\mathcal{A}_i = (\partial \mathcal{A}_i \times I) / \sim$ where $(x, t) \sim (x, 0)$ if $x \in \mathcal{A}$, $t \in I$. Let $C_i: \mathcal{A}_i \to S^n$, $i = 1, 2$, be homeomorphisms satisfying the following conditions:

1. $C_i(\mathcal{A}_i) \subset S^n - (\text{int } \mathcal{A}_i \cup \text{int } \mathcal{A}_i)$,
2. $C_i((x, 1)) = x$ for $x \in \partial \mathcal{A}_i$, and
3. $C_1(\mathcal{A}_1) \cap C_2(\mathcal{A}_2) = \mathcal{A}$.
(Thus, $C_i(\mathcal{A}_i)$ is a certain pinched collar of $\partial \mathcal{A}_i$.)

It follows from [20] that $g_i: \mathcal{A}_i \to \mathcal{A}_i$ can be extended to $\mathcal{A}_i \cup C_i(\mathcal{A}_i)$ such that $g_i | \partial(\mathcal{A}_i \cup C_i(\mathcal{A}_i)) = 1$. Let $g$ be the homeomorphism taking $\bigcup_{i=1,2} (\mathcal{A}_i \cup C_i(\mathcal{A}_i))$ onto itself which is $g_i$ on $\mathcal{A}_i \cup C_i(\mathcal{A}_i)$. Then, $g$ can be extended to $S^n$ by way of the identity and it is clear that $e = gh$ is the desired orientation preserving homeomorphism which flattens the pair $D^{m_1}_i \cup D^{m_2}_i$ since $gh(D^{m_1}_i \cup D^{m_2}_i) = \mathcal{A}_i \cup \mathcal{A}_i$.

**Theorem 2.** Let $\{\mathcal{A}_i\}$, $i = 1, 2, \ldots, p$ be simplexes such that $\mathcal{A}_i$ is of dimension $m_i$ and such that $\bigcap_{i=1}^p \mathcal{A}_i = \Delta$ is a $k$-face of each $\mathcal{A}_i$. Let $f, g: \bigcup_{i=1}^p \mathcal{A}_i \to \text{int } Q^n$ be PL embeddings into the connected $n$-dimensional PL manifold $Q^n$, $n \geq m_i + 3$, $i = 1, 2, \ldots, p$. Then, there is a PL isotopy $e_\gamma$ of $Q$ such that $e_0 = 1$ and $e_\gamma f = g$.

If one can tame certain clusters of cells, then Theorem 2 can be used to unknot them. For instance, the following corollary follows from Theorem 1' of [7].

**Corollary.** Let $\{\mathcal{A}_i\}$, $i = 1, 2, \ldots, p$ be simplexes in the interior of the connected $n$-dimensional PL manifold $Q^n$, $m_i < (2/3)n - 1$, $i = 1, 2, \ldots, p$, such that $\bigcap_{i=1}^p \mathcal{A}_i = \Delta$ is a $k$-face of each $\mathcal{A}_i$. Let $f: \bigcup_{i=1}^p \mathcal{A}_i \to \text{int } Q$ be an embedding which is locally flat on the open faces of $\mathcal{A}_i$, $i = 1, 2, \ldots, p$. Then, there is an isotopy $e_\gamma$ of $Q$ such that $e_0 = 1$ and $e_\gamma f$ is the inclusion of $\bigcup_{i=1}^p \mathcal{A}_i$ into $Q$.

**Proof of Theorem 2.** Let $\{v_j\}_{j=0}^{m_i}$ denote the vertices of $\mathcal{A}_i$ and let $\{v_j\}_{j=0}^{m_i}$ denote the vertices of $\Delta$. Let $\mathcal{A}_i^{k-q}$ be the face of $\mathcal{A}_i$ spanned by the vertices $\{v_j\}_{j=0}^{m_i} - \{v_j\}_{j=k-q+1}^{m_i}$ and let $\Delta^{k-q}$ be the face of $\Delta$ spanned by $\{v_j\}_{j=0}^{k-q}$. Thus, for $0 \leq q \leq k$, $\Delta^{k-q}$ and $\mathcal{A}_i^{k-q}$, $i = 1, 2, \ldots, p$, are cones over $\Delta^{k-(q+1)}$ and $\mathcal{A}_i^{k-(q+1)}$, $i = 1, 2, \ldots, p$, respectively, with vertex $v_{k-q}$.

We will work with the following inductive statement.

**q**-**Inductive Statement.** Let $f, g: \bigcup_{i=1}^p \mathcal{A}_i^{k-q} \to \text{int } Q^n$ (n arbitrary) be PL embeddings. Then, there is a PL isotopy $e_\gamma$ of $Q^n$ such
that \( e_0 = 1 \) and \( e_1 f = g \).

The case \( q = k + 1 \) can be proved easily by using uniqueness of regular neighborhoods. Now we assume the \((q + 1)\)-inductive statement, where \( 0 \leq q \leq k \), and will establish the \( q^{\text{th}} \) inductive statement. Let \( N \) be a regular neighborhood of \( f(\bigcup_{i=1}^{p} A^{m_i-q}) \mod f(\bigcup_{i=1}^{p} A^{m_i-(q+1)}) \) in \( Q \) (see [12]), and let \( N_* \) be a regular neighborhood of \( g(\bigcup_{i=1}^{p} A^{m_i-q}) \mod g(\bigcup_{i=1}^{p} A^{m_i-(q+1)}) \) in \( Q \). Then, there is a PL isotopy \( e_i^* \) of \( Q \) such that \( e_0^* = 1 \) and \( e_1^* f = e_1^* N_* \). But, \( \partial(N_*) \) is a PL \((n-1)\)-sphere and \( e_1^* f \big|_{\bigcup_{i=1}^{p} A^{m_i-(q+1)}} \) and \( g \big|_{\bigcup_{i=1}^{p} A^{m_i-(q+1)}} \) are PL embeddings into \( \partial(N_*) \).

Hence, by the inductive assumption, there is a PL isotopy \( e_i^* \) of \( \partial(N_*) \) such that \( e_0^* = 1 \) and \( e_1^* e_i^* f \big|_{\bigcup_{i=1}^{p} A^{m_i-(q+1)}} = g \big|_{\bigcup_{i=1}^{p} A^{m_i-(q+1)}} \). It is now easy to extend \( e_i^* \) over \( Q \) so that it is the identity at the zero level by using a PL bicollar of \( \partial(N_*) \) in \( Q \). Then,

\[
e_0^* e_1^* f : \bigcup_{i=1}^{p} A^{m_i-q} \rightarrow N_* \quad \text{and} \quad g : \bigcup_{i=1}^{p} A^{m_i-q} \rightarrow N_*
\]

are proper embeddings (in the sense of [16]) which agree on \( \bigcup_{i=1}^{p} A^{m_i-(q+1)} \) and so by Theorem 2 of [16] there is a PL isotopy \( e_i^* \) of \( N_* \) which is the identity on \( \partial(N_*) \) such that \( e_0^* = 1 \) and \( e_1^* e_i^* f = g \). Hence, we can extend \( e_i^* \) to \( Q \) by way of the identity and we see that \( e_i = e_i^* e_i^* \) is the desired isotopy of \( Q \).

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