

# Pacific Journal of Mathematics

**UNKNOTTING UNIONS OF CELLS**

THOMAS BENNY RUSHING

## UNKNOTTING UNIONS OF CELLS

T. B. RUSHING

In this note we consider the problem of determining whether the union of cells is nicely embedded in the  $n$ -sphere if each of the cells is nicely embedded. This question is related to many embedding problems. For instance, the  $n$ -dimensional Annulus Conjecture (now known to be true for  $n \neq 4$ ) is a special case. Cantrell and Lacher have shown that an affirmative answer implies local flatness of certain submanifolds. Also, this question is related to the conjecture that an embedding of a complex into the  $n$ -sphere which is locally flat on open simplexes is  $\varepsilon$ -tame in codimension three.

The problem mentioned above was first investigated by Doyle [9] [10] in the three dimensional case and by Cantrell [2] in high dimensions and later by Lacher [15] Cantrell and Lacher [3][4], Kirby [13], Černavskii [5][6] and the author [17]. Also, Sher [21] has generalized a construction of Debrunner and Fox [8] to obtain counterexamples in certain cases. Since the  $n$ -dimensional Annulus Conjecture,  $n \neq 4$ , is now known to be true [14], only two results of § 7 of [17] remain of interest. First we will prove a strengthened form of one of those results and we greatly simplify the proof by employing the powerful tools now available. In particular we prove the following theorem.

**THEOREM 1.** *If  $D_1^{m_1}$  and  $D_2^{m_2}$  are cells in  $S^n$ ,  $n > 5$ , of dimensions  $m_1$  and  $m_2$ , respectively, and if  $D_1^{m_1} \cap D_2^{m_2} = \partial D_1^{m_1} \cap \partial D_2^{m_2} = D$  is a  $k$ -cell (possibly empty),  $n - k \geq 4$ , which is locally flat in  $\partial D_1^{m_1}$ , in  $\partial D_2^{m_2}$  and in  $S^n$  and is such that  $D_1^{m_1} - D$  and  $D_2^{m_2} - D$  are locally flat, then there is an ambient isotopy  $e_t$  of  $S^n$  such that  $e_1(D_1^{m_1})$  and  $e_1(D_2^{m_2})$  are simplexes and  $e_1(D_1^{m_1} \cap D_2^{m_2})$  is a face of each.*

**REMARK.** If the above theorem is modified by requiring  $n - k = 3$ , then counterexamples can be constructed for any  $m_1$  and  $m_2$  (see [21]).

*Proof of Theorem 1.* Every orientation preserving homeomorphism of  $S^n$ ,  $n \geq 5$ , is stable [14], hence isotopic to the identity. It will then suffice to construct an orientation preserving homeomorphism  $e_1$  satisfying the conclusion of the theorem. By Theorem 5.2 of [1], we may assume that  $D_1^{m_1}$  and  $D_2^{m_2}$  are locally flat. For  $i = 1, 2$ , it is easy to construct a homeomorphism  $f_i: S^n \rightarrow S^n$  such that  $f_i(D_i^{m_i}, D) = (\Delta^{m_i}, \Delta^k)$  where  $\Delta^{m_i}$  is an  $m_i$ -simplex and  $\Delta^k$  is a  $k$ -face. Thus, by using  $f_i$ ,  $i = 1, 2$ , and Lemma 3.6 of [18], we can construct locally flat  $n$ -cells  $D_1^n$  and

$D_2^n$  satisfying the following conditions,

- (1)  $D_1^n \cap D_2^n = \partial D_1^n \cap \partial D_2^n = D$ ,
- (2)  $D$  is locally flat in  $\partial D_1^n$  and  $\partial D_2^n$ , and
- (3)  $(D_i^n, D_i^{m_i})$  is a trivial cell pair,  $i = 1, 2$ .

Let  $\Delta_1^n$  and  $\Delta_2^n$  be  $n$ -simplexes in  $S^n$  such that  $\Delta_1^n \cap \Delta_2^n = \Delta$  is a  $k$ -face of each. We will now construct an orientation preserving homeomorphism  $h$  of  $S^n$  such that  $h((D_1^n, D_2^n, D)) = (\Delta_1^n, \Delta_2^n, \Delta)$ . It is easy to obtain an orientation preserving homeomorphism  $h_1$  of  $S^n$  such that  $h_1((D_2^n, D)) = (\Delta_2^n, \Delta)$ . Let  $\Delta_0$  be the  $n$ -simplex having as vertices the midpoints of the segments which join the vertices of  $\Delta_2^n$  with the barycenter of  $\Delta_2^n$ . Let  $f: I^k \rightarrow \Delta$  be a  $PL$ -homeomorphism and define  $F: I^k \times I \rightarrow \Delta_2^n$  by extending linearly on each segment  $\{x\} \times I, x \in I^k$ , the map which takes  $(x, 0)$  to  $f(x)$  and  $(x, 1)$  to the midpoint of the segment joining  $f(x)$  and the barycenter of  $\Delta_2^n$ . Then,  $E = F(I^k \times \{1\})$  is a  $k$ -face of  $\Delta_0$ . Now, by using the Annulus Theorem, it is easy to get an orientation preserving homeomorphism  $h_2$  of  $S^n$  such that

- (1)  $h_2((\Delta_0, h_1(D_1^n))) = (\Delta_0, \Delta_1^n)$ , and
- (2)  $h_2|_{\Delta \cup E} = 1$ .

Let  $A$  denote  $C1(S^n - (\Delta_0 \cup \Delta_1^n))$ . Then, the embedding  $h_2F: I^k \times I \rightarrow A$  satisfies the hypotheses of Theorem 1 of [19]; hence, by that theorem there is a homeomorphism  $h_3$  of  $A$  such that  $h_3|_{\partial \Delta_0 \cup \partial \Delta_1^n} = 1$  and  $h_3h_2F: I^k \times I \rightarrow A$  is  $PL$ . Extend  $h_3$  to all of  $S$  by way of the identity. Consider the two  $PL$  embeddings  $F|_{\partial I^k \times I}: \partial I^k \times I \rightarrow A$  and  $h_3h_2F|_{\partial I^k \times I}: \partial I^k \times I \rightarrow A$ . These two embeddings clearly satisfy the hypotheses of Theorem 4 of [11]; therefore, by that theorem there is a  $PL$  homeomorphism  $h_4$  of  $A$  such that  $h_4h_3h_2F|_{\partial I^k \times I} = F|_{\partial I^k \times I}$  and  $h_4|_{\partial \Delta_0 \cup \partial \Delta_1^n} = 1$ . Extend  $h_4$  to  $S^n$  by the identity. Now, the  $PL$  embeddings  $h_4h_3h_2F: I^k \times I \rightarrow A$  and  $F: I^k \times I \rightarrow A$  satisfy the hypothesis of Theorem 4 of [11] and so by another application of that theorem we get a  $PL$  homeomorphism  $h_5$  of  $A$  such that  $h_5h_4h_3h_2F = F$  and  $h_5|_{\partial \Delta_0 \cup \partial \Delta_1^n} = 1$ . Extend,  $h_5$  to  $S^n$  by the identity.

Let  $p: S^n \rightarrow S^n$  be a map such that

- (1)  $p(\Delta_0) = \Delta_2^n$ ,
- (2)  $p|_{h_1(D_1^n) \cup \Delta_1^n} = 1$ , and
- (3)  $p|_{S^n - F(I^k \times I)}$  is one-to-one, and  $p(F(\{x\} \times I)) = F(x, 0)$

for each  $x \in I^k$ .

It is now easy to check that  $h = ph_5h_4h_3h_2p^{-1}h_1$  is the desired homeomorphism that flattens the pair  $D_1^n \cup D_2^n$ .

Let  $\Delta_i^{m_i-1}$  be a face of  $\Delta_i^n$  of dimension  $m_i - 1$  which has  $\Delta$  as a face. Let  $\delta_i$  denote the face of  $\Delta_i^n$  dual to  $\Delta_i^{m_i-1}$  and let  $\hat{\delta}_i$  denote the barycenter of  $\delta_i$ . Now, let  $\Delta_i^{m_i}$  be the  $m_i$ -simplex  $\Delta_i^{m_i-1} * \hat{\delta}_i$ . Then, it is easy to get a homeomorphism  $g_i: \Delta_i^n \rightarrow \Delta_i^n$  such that  $g_i h(D_i^{m_i}) = \Delta_i^{m_i}$  and  $g_i|_{\Delta} = 1$ . Furthermore, we may assume that  $g_i|_{\partial \Delta_i^n}$  is orientation

preserving for if it is not we may follow  $g_i$  by an appropriate reflection of  $\Delta_i^n$ .

Let  $A_i, i = 1, 2$ , be an annulus pinched at  $\Delta$ , in particular,  $A_i = (\partial\Delta_i^n \times I)/\sim$  where  $(x, t) \sim (x, 0)$  if  $x \in \Delta, t \in I$ . Let  $C_i: A_i \rightarrow S^n, i = 1, 2$ , be homeomorphisms satisfying the following conditions:

- (1)  $C_i(A_i) \subset S^n - (\text{int } \Delta_1^n \cup \text{int } \Delta_2^n)$ ,
- (2)  $C_i((x, 1)) = x$  for  $x \in \partial\Delta_i^n$ , and
- (3)  $C_1(A_1) \cap C_2(A_2) = \Delta$ .

(Thus,  $C_i(A_i)$  is a certain pinched collar of  $\partial\Delta_i^n$ .)

It follows from [20] that  $g_i: \Delta_i^n \rightarrow \Delta_i^n$  can be extended to  $\Delta_i^n \cup C_i(A_i)$  such that  $g_i|_{\partial(\Delta_i^n \cup C_i(A_i))} = 1$ . Let  $g$  be the homeomorphism taking  $\bigcup_{i=1,2} (\Delta_i^n \cup C_i(A_i))$  onto itself which is  $g_i$  on  $\Delta_i^n \cup C_i(A_i)$ . Then,  $g$  can be extended to  $S^n$  by way of the identity and it is clear that  $e_1 = gh$  is the desired orientation preserving homeomorphism which flattens the pair  $D_1^{m_1} \cup D_2^{m_2}$  since  $gh(D_1^{m_1} \cup D_2^{m_2}) = \Delta_1^{m_1} \cup \Delta_2^{m_2}$ .

**THEOREM 2.** *Let  $\{\Delta_i^{m_i}, i = 1, 2, \dots, p$  be simplexes such that  $\Delta_i^{m_i}$  is of dimension  $m_i$  and such that  $\bigcap_{i=1}^p \Delta_i^{m_i} = \Delta$  is a  $k$ -face of each  $\Delta_i^{m_i}$ . Let  $f, g: \bigcup_{i=1}^p \Delta_i^{m_i} \rightarrow \text{int } Q^n$  be PL embeddings into the connected  $n$ -dimensional PL manifold  $Q^n, n \geq m_i + 3, i = 1, 2, \dots, p$ . Then, there is a PL isotopy  $e_t$  of  $Q$  such that  $e_0 = 1$  and  $e_1 f = g$ .*

If one can tame certain clusters of cells, then Theorem 2 can be used to unknot them. For instance, the following corollary follows from Theorem 1' of [7].

**COROLLARY.** *Let  $\{\Delta_i^{m_i}, i = 1, 2, \dots, p$  be simplexes in the interior of the connected  $n$ -dimensional PL manifold  $Q^n, m_i < (2/3)n - 1, i = 1, 2, \dots, p$ , such that  $\bigcap_{i=1}^p \Delta_i^{m_i} = \Delta$  is a  $k$ -face of each  $\Delta_i^{m_i}$ . Let  $f: \bigcup_{i=1}^p \Delta_i^{m_i} \rightarrow \text{int } Q$  be an embedding which is locally flat on the open faces of  $\Delta_i^{m_i}, i = 1, 2, \dots, p$ . Then, there is an isotopy  $e_t$  of  $Q$  such that  $e_0 = 1$  and  $e_1 f$  is the inclusion of  $\bigcup_{i=1}^p \Delta_i^{m_i}$  into  $Q$ .*

*Proof of Theorem 2.* Let  $\{v_j^i\}_{j=0}^{m_i}$  denote the vertices of  $\Delta_i^{m_i}$  and let  $\{v_j\}_{j=0}^k$  denote the vertices of  $\Delta$ . Let  $\Delta_i^{m_i-q}$  be the face of  $\Delta_i^{m_i}$  spanned by the vertices  $\{v_j^i\}_{j=0}^{m_i} - \{v_j\}_{j=k-q+1}^k$  and let  $\Delta^{k-q}$  be the face of  $\Delta$  spanned by  $\{v_j\}_{j=0}^q$ . Thus, for  $0 \leq q \leq k, \Delta^{k-q}$  and  $\Delta_i^{m_i-q}, i = 1, 2, \dots, p$ , are cones over  $\Delta^{k-(q+1)}$  and  $\Delta_i^{m_i-(q+1)}, i = 1, 2, \dots, p$ , respectively, with vertex  $v_{k-q}$ .

We will work with the following inductive statement.

**$q^{\text{th}}$  INDUCTIVE STATEMENT.** *Let  $f, g: \bigcup_{i=1}^p \Delta_i^{m_i-q} \rightarrow \text{int } Q^n$  ( $n$  arbitrary) be PL embeddings. Then, there is a PL isotopy  $e_t$  of  $Q^n$  such*

that  $e_0 = 1$  and  $e_1 f = g$ .

The case  $q = k + 1$  can be proved easily by using uniqueness of regular neighborhoods. Now we assume the  $(q + 1)$ -inductive statement, where  $0 \leq q \leq k$ , and will establish the  $q^{\text{th}}$  inductive statement. Let  $N$  be a regular neighborhood of  $f(\mathbf{U}_{i=1}^p \Delta_i^{m_i-q}) \bmod f(\mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)})$  in  $Q$  (see [12]), and let  $N_*$  be a regular neighborhood of  $g(\mathbf{U}_{i=1}^p \Delta_i^{m_i-q}) \bmod g(\mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)})$  in  $Q$ . Then, there is a  $PL$  isotopy  $e_i^1$  of  $Q$  such that  $e_0^1 = 1$  and  $e_i^1(N) = N_*$ . But,  $\partial(N_*)$  is a  $PL$   $(n - 1)$ -sphere and  $e_i^1 f | \mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)}$  and  $g | \mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)}$  are  $PL$  embeddings into  $\partial(N_*)$ . Hence, by the inductive assumption, there is a  $PL$  isotopy  $e_i^2$  of  $\partial(N_*)$  such that  $e_0^2 = 1$  and  $e_i^2 e_i^1 f | \mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)} = g | \mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)}$ . It is now easy to extend  $e_i^2$  over  $Q$  so that it is the identity at the zero level by using a  $PL$  bicollar of  $\partial(N_*)$  in  $Q$ . Then,

$$e_i^2 e_i^1 f: \bigcup_{i=1}^p \Delta_i^{m_i-q} \rightarrow N_* \quad \text{and} \quad g: \bigcup_{i=1}^p \Delta_i^{m_i-q} \rightarrow N_*$$

are proper embeddings (in the sense of [16]) which agree on  $\mathbf{U}_{i=1}^p \Delta_i^{m_i-(q+1)}$  and so by Theorem 2 of [16] there is a  $PL$  isotopy  $e_i^3$  of  $N_*$  which is the identity on  $\partial(N_*)$  such that  $e_0^3 = 1$  and  $e_i^3 e_i^2 e_i^1 f = g$ . Hence, we can extend  $e_i^3$  to  $Q$  by way of the identity and we see that  $e_i = e_i^3 e_i^2 e_i^1$  is the desired isotopy of  $Q$ .

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