FORMS OF THE AFFINE LINE AND ITS ADDITIVE GROUP

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Let $k$ be a field, $X_0$ an object (e.g., scheme, group scheme) defined over $k$. An object $X$ of the same type and isomorphic to $X_0$ over some field $K \supset k$ is called a form of $X_0$. If $k$ is not perfect, both the affine line $A_1$ and its additive group $G_a$ have nontrivial sets of forms, and these are investigated here. Equivalently, one is interested in $k$-algebras $R$ such that $K \otimes_k R \cong K[t]$ (the polynomial ring in one variable) for some field $K \supset k$, where, in the case of forms of $G_a$, $R$ has a group (or co-algebra) structure $s : R \to R \otimes_k R$ such that $(K \otimes s)(t) = t \otimes 1 + 1 \otimes t$. A complete classification of forms of $G_a$ and their principal homogeneous spaces is given and the behaviour of the set of forms under base field extension is studied.

If $k$ is perfect, all forms of $A_1$ and $G_a$ are trivial, as is well known (cf. 1.1). So assume $k$ is not perfect of characteristic $p > 0$. Then a nontrivial example (cf. [5], p. 46) of a form of $G_a$ is the subgroup of $G_a^\sharp = \text{Spec} \, k[x, y]$ defined by $y^p = x + ax^p$ where $a \in k$, $a \notin k^p$. We show that this example is quite typical (cf. 2.1): Every form of $G_a$ is isomorphic to a subgroup of $G_a^\sharp$ defined by an equation $y^{p^n} = a_0 x + a_1 x^p + \cdots + a_m x^{p^m}$, $a_i \in k$, $a_0 \neq 0$. Analyzing the equivalence relation induced on the right hand side polynomials by isomorphism of the groups which they define, we obtain a description of the set of forms of $G_a$ split by $k^p$ as, essentially, the quotient of an infinite direct sum of copies of $k/k^p$ under a certain group action (cf. 2.5).

If $G$ is a nontrivial form of $G_a$, we show that $\text{End}_k G$ is a finite field (cf. 3.1). This allows one to compute the set of $k_s/k$-forms of $G$ ($k_s$ a separable algebraic closure of $k$) using Galois cohomology. This set is nontrivial in general, in contrast to the same situation for $G_a$.

A form $X$ of $A_1$ may fail to have a group structure for two reasons. First, and this is the serious failure, $X_{k_s}$ may not have enough (i.e., infinitely many) automorphisms. As an example, with the identity as the only automorphism, one may take $P^1 - \{q\}$, where $P^1$ is the projective line and $q$ is a purely inseparable point of degree $p^n > 2$. The general case here seems to be rather complex. Secondly, $X_{k_s}$ may have enough automorphisms, but $X$ may not have a rational point. We show that then $X$ is a principal homogeneous space for a form of $G_a$ (cf. 4.1). This gives a new interpretation of a result of Rosenlicht ([4], p. 10, theorem) on curves with exceptionally many automorphisms (cf. 4.2).
Throughout this paper \( k \) will be a fixed base field, \( \overline{k} \) an algebraic closure of \( k \), \( k_i = k^{p^i} \) (\( p = \text{char} \ k \)) the perfect and \( k \), the separable closure of \( k \) in \( \overline{k} \). Reference to \( k \) will usually be omitted.

It is well known (cf. [5], p. 34 and [6], p. 108) that a form \( G \) of \( G_\alpha \) is split by \( k \), that is, \( G_{k_i} \cong G_{k^{p^i}} \). The same is true for forms \( X \) of \( A^1 \). For the sake of completeness, and to establish some notation, we briefly outline the argument. The idea is to investigate the complete regular curve \( P \) determined by \( X \). As a matter of terminology, we call a scheme \( Y \) regular if all its local rings are regular, and non-singular if \( Y_K \) is regular for any \( K \supset k \). As is well known, \( Y_K \) non-singular implies \( Y \) nonsingular, and \( Y \) is nonsingular if and only if \( Y_{k^{p-1}} \) is regular. The existence of forms of \( A^1 \) is closely connected with the divergence of these notions if \( k \) is not perfect. If \( Y \) is a curve, we denote by \( \tilde{Y}_K \) the regular curve obtained by normalizing \( Y_K \).

**Lemma 1.1.** Let \( X \) be a form of \( A^1 \) and \( P \supset X \) a complete regular curve.

(i) \( P - X \) is a point purely inseparable over \( k \).

(ii) There is a unique minimal field \( k' \supset k \) such that \( X_{k'} \cong A^1_{k'} \), and \( k' \) is purely inseparable of finite degree over \( k \).

**Proof.** The genus of \( \tilde{P}_{k_i} \) is zero since this is so after suitable base field extension and since, \( k_i \) being perfect, the genus does not change under base field extension (cf. [1], V, § 5, Th. 5). Since \( \tilde{P}_{k} \) has a rational point, \( \tilde{P}_{k} \cong P^1_k \). An open subscheme of \( P^1_K \) (\( K \) any field) is a form of \( A^1_k \) if and only if it is the complement of a purely in separable point. Hence \( \tilde{P}_{k_i} - X_{k_i} \) is a point, and a fortiori \( P - X \) (resp. \( P_{k_i} - X_{k_i} \)) is a point purely inseparable over \( k \) (resp. rational over \( k_i \)). In particular, \( \tilde{P}_{k_i} \cong P_{k_i} \) and \( X_{k_i} \cong A^1_{k_i} \). If \( K \supset k \) is any field such that \( X_K \cong A^1_K \), then \( \tilde{P}_K - X_K \) is a point rational over \( K \) and \( K \) contains (up to unique isomorphism) the residue field \( k_i \) of \( P - X \). Now pass to \( X_{k_i} \) and continue this process. After finitely many steps, we reach a field \( k' \subset K, k \subset k' \subset k_i \), such that \( \tilde{P}_{k'} \cong P_{k'} \), and \( \tilde{P}_{k'} - X_{k'} \) is rational over \( k' \). Then \( X_{k'} \cong A^1_{k'} \).

\( A^1 = \text{Spec} \ k[t] \) admits, up to choice of origin, a unique group structure (given by \( s(t) = t \otimes 1 + 1 \otimes t \) if the origin is at \( t = 0 \)), and any automorphism of \( A^1 \) sending the origin to the origin is a group homomorphism. Let \( G \) and \( G' \) be groups with origins \( q \) and \( q' \) and \( \psi \) an isomorphism of the underlying schemes, supposed to be forms of \( A^1 \), such that \( \psi(q) = q' \). Then \( \psi \) is a homomorphism of groups after base field extension, which means that a certain diagram of morphisms (over \( k \)) commutes after base extension and so is commutative to begin with. Hence \( \psi \) is an isomorphism of groups. This gives:
**Lemma 1.2.** Let $X$ be a form of $\mathbb{A}^1$. Then any group scheme $G$ with underlying scheme $X$ is a form of $G_\kappa$. The group structure (if it exists) is unique up to choice of origin. If $X_\kappa \cong \mathbb{A}_\kappa$, then $G_\kappa \cong G_{\kappa K}$.

We assume from now on that $\text{char } k = p > 0$. We denote by $\Theta^n$ the base change functor deduced from

$$\varphi^n: k \longrightarrow k$$

$$a \longmapsto a^{p^n}.$$ 

For any scheme $X$ there is a canonical morphism $F^n_X: X \to \Theta^n X$. If $X$ is a group scheme, so is $\Theta^n X$ and $F^n_X$ is a homomorphism. Referring to [3], p. I. 1-5 for more details, we remark only that if $X = \text{Spec } R$ is affine, then $\Theta^n X = \text{Spec } ((k, \varphi^n) \otimes_k R)$ where $(k, \varphi^n) = k$ considered as a right $k$-algebra via $\varphi^n$ and as a left $k$-algebra in the usual way, and that $F^n_X$ is deduced from

$$F^n_X: (k, \varphi^n) \otimes_k R \longrightarrow R$$

$$a \otimes x \longmapsto a x^{p^n}.$$ 

$\Theta^n$ accomplishes, up to isomorphism, the same as the base change $k \subset k^{p^n}$. More precisely, if $K$ is purely inseparable of exponent $\leq n$ over $k$ (that is, $K^{p^n} \subset k$), there is a commutative diagram

$$\begin{array}{ccc} 
& k & \longrightarrow K \\
\varphi^n & \downarrow & \varphi \\
k & \longrightarrow & k \\
\end{array}$$

and we have $\Theta^n X \cong (k, \varphi) \otimes_K X_\kappa$ for any scheme $X$ over $k$.

**Lemma 1.3.** Let $X$ be a form of $\mathbb{A}^1$. For any integer $n \geq 0$, $F^n_X$ is a purely inseparable morphism of degree $p^n$. For any morphism $\psi: X \to Y$ of finite degree, there is a unique factorization $\psi = \tilde{\psi} F^n_X$ where $p^n$ is the inseparable degree of $\psi$ and $\tilde{\psi}$ is a separable morphism. Finally, there is an integer $n \geq 0$ such that $\Theta^n X \cong \mathbb{A}^1$.

**Proof.** The last statement follows from 1.1 and the remark above. The function field $\kappa(X)$ of $X$ is separable of transcendence degree one over $k$ and so has, for each $n$, a unique subfield $\supset k$ over which it is purely inseparable of degree $p^n$, namely

$$k(\kappa(X)^{p^n}) \cong (k, \varphi^n) \otimes_k \kappa(X) = \kappa(\mathcal{O}^n X)$$

(cf. [2], p. 186, Th. 19 and p. 179, corollary). This proves the first statement and the second follows in view of the fact that $\Theta^n X$ is normal.
1.4. Let $X$ be a form of $A^1$. We let $n(X)$ be the least $n$ such that $\Theta^n X \cong A^1$ or, equivalently, the least $n$ such that $X$ has a splitting field of exponent $n$ over $k$.

The point of 1.3 is that the affine ring $R$ of $X$ has a unique maximal subring of the form $S = k[x]$ such that $R^{p^n} \subset S$ for some $n$, and that the only other subrings with this property are the rings $k[x^{p^m}]$, $m \geq 0$. Note, however, that $n(X)$ need not be the least $n$ such that $k(\Theta^n X) \cong k(t)$ or, equivalently, that $\Theta^n X \subset P^1$. $Y = P^1 - \{q\}$, $q$ purely inseparable and not rational over $k$, is one example and, giving $Y$ some further twist, one can find $X$ such that $\Theta^n X \cong Y$ and $n > 1$.

2. Since $G_a$ is defined over the prime field, we may identify $G_a$ and $\theta G_a$. Then $F = F G_a \in A = \text{Hom}_k(G_a, G_a)$. It is well known that $A = k[F]$, a ring of noncommutative polynomials with relations $F a = a^p F$ for $a \in k$. We define the power series ring $\hat{A} = k[[F]]$ in the same way. Let $\varepsilon : A \to k$ be the natural augmentation. We let $A^* = \varepsilon^{-1}(k^*)$ and $A^{**} = \varepsilon^{-1}(1)$ and make corresponding definitions for $\hat{A}$. By truncation we obtain groups $U_n = \hat{A}^* / \hat{A} F^n \cong A^* / AF^n$. $\tau = \sum_{i=0}^p a_i F^i \in A, a_m \neq 0$, has degree $p^m$ as a morphism $\tau : G_a \to G_a$, and we also give it degree $p^m$ in the graded ring $k[F]$. Note that $A^* \subset A$ is the subset of separable homomorphisms. An endomorphism $\lambda : k \to k$ commutes with $p$-th powers and so extends to an endomorphism $\lambda : A \to A$

$$\sum a_i F^i \longmapsto \sum \lambda(a_i) F^i.$$

In the particular case $\lambda = \varphi^n$ we put $\lambda(\tau) = \tau^{(n)}$ for $\tau \in A$ and $\lambda(A) = A^{(n)}$. $\tau^{(n)}$ is characterized by $F^n \tau = \tau^{(n)} F^n$.

If $G = \text{Spec} R$ is an affine group with group operation $s : R \to R \otimes_k R, \text{Hom}_k(G, G_a)$ may be identified with

$$\{r \mid s(r) = r \otimes 1 + 1 \otimes r\} \subset R \cong \text{Hom}(k[t], R).$$

In particular, $A$ is identified with the set of $p$-polynomials

$$f(t) = a_0 t + a_1 t^p + \cdots + a_m t^{p^m} \in k[t].$$

**Theorem 2.1.** Let $G$ be a form of $G_a$. Then $G$ is isomorphic to a subgroup $\text{Spec} k[x, y] / I$ of $G_a = \text{Spec} k[x, y]$ where $I$ is generated by a polynomial $y^{p^n} - (a_0 x + a_1 x^p + \cdots + a_m x^{p^m}), a_0 \neq 0$. Equivalently, $G$ is a fiber product.
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\[
\begin{align*}
G & \xrightarrow{\xi} G_a \\
\eta & \downarrow \quad \downarrow \tau \\
G_a & \xrightarrow{\tau} G_a
\end{align*}
\]

where \( \tau = a_0 + a_1 F + \cdots + a_m F^m \in A^* \). Conversely, any \( G \) defined that way is a form of \( G_a \).

**Proof.** Let \( G = \text{Spec } R, s: R \to R \otimes_k R \) the group operation,

\[
\bar{s}: (k, \varphi^n) \otimes_k R \to (k, \varphi^n) \otimes_k R \otimes_k R \cong ((k, \varphi^n) \otimes_k R) \otimes_k ((k, \varphi^n) \otimes_k R)
\]

the induced group operation for \( \Theta^* G \). By 1.3, we have \( \Theta^* G \cong G_a \) for some \( n \), so that \( (k, \varphi^n) \otimes_k R \cong k[t] \) where we can choose \( t \) such that \( \bar{s}(t) = t \otimes 1 + 1 \otimes t \). Write \( t = \sum a_i \otimes y_i \) with \( a_i \in k \) and \( y_i \in R \). Then

\[
\bar{s}(t) = t \otimes 1 + 1 \otimes t = \sum a_i \otimes y_i \otimes 1 + \sum a_i \otimes 1 \otimes y_i = \sum \bar{s}(a_i \otimes y_i) = \sum a_i \otimes s(y_i).
\]

If we choose the \( a_i \) linearly independent in \( k \) considered as a vector space over \( k \) via \( \varphi^n \), i.e., linearly independent over \( k \varphi^n \), this implies \( s(y_i) = y_i \otimes 1 + 1 \otimes y_i \). Hence the \( y_i (1 \otimes y_i) \) define homomorphisms \( \gamma_i: G \to G_a (\Theta^n \gamma_i: G_a \to G_a) \). As observed above, this implies \( 1 \otimes y_i = f_i(t) \) where \( f_i \) is a \( p \)-polynomial. Applying \( F^n_k \) and putting \( x = F^n_k(t) \), we obtain \( y^n_i = f_i(x) \). Clearly the \( y_i \) generate \( R \) over \( k \) and one of them, call it \( y \), is a separating variable for \( \kappa(G) \). Then \( y^n = f(x) = a_0 x + a_1 x^p + \cdots + a_m x^{p^m} \), with \( a_0 \neq 0 \) since \( x \) is separable over \( k(y) \). This shows that \( k[x, y] \subseteq R \) is integrally closed. \( \kappa(G) \) is separable and purely inseparable over \( k(x, y) \), so \( k(x, y) = \kappa(G) \) and \( R = k[x, y] \). This proves the first statement. The next follows letting \( \eta \) be the homomorphism corresponding to \( y \) and \( \xi = F^n_k \) the homomorphism corresponding to \( x \). Finally, let \( R = k[x, y] \) where \( y^n = f(x) \). Then \( s: R \to R \otimes_k R, s(x) = x \otimes 1 + 1 \otimes x, s(y) = y \otimes 1 + 1 \otimes y, \) is well defined and gives a group structure on \( R \). Taking \( a_0 = 1 \) for simplicity, we have

\[
1 \otimes x = (1 \otimes y^{p-1} - (a_1^{p-1} \otimes x + \cdots + a_m^{p-1} \otimes x^{p^{m-1}}))' = t_1^n
\]

in \( (k, \varphi^n) \otimes_k R \). Replacing \( 1 \otimes x \) by \( t_1^n \) on the right hand side and continuing that way, we find \( t \in (k, \varphi^n) \otimes_k R \) such that \( 1 \otimes x = t^n \) and \( 1 \otimes y^n = (f(t))^n \). Spec \( R \) is nonsingular, so \( (k, \varphi^n) \otimes_k R \) is reduced. Hence \( 1 \otimes y = f(t) \), showing that \( (k, \varphi^n) \otimes_k R = k[t] \).

2.2. We write \( G = (F^n, \tau) \) (with \( \tau \in A^* \)) for a fiber product as in the theorem. Note that \( G \) can be so written if and only if \( \Theta^* G \cong G_a \).
PROPOSITION 2.3. Let $G = (F^n, \tau), G_1 = (F^{n_1}, \tau_1)$ and assume $n_1 \leq n$. Then $G \cong G_1$ if and only if there exist elements $\rho \in A^*, \sigma \in A$ and $c \in k^*$ such that 

$$\tau_1^{(n-n_1)} = (\rho^{(n)}\tau + F^n\sigma)c^{-1}.$$ 

$\rho$ may be chosen of degree $\leq p^{n-1}$.

Proof. The monomorphism $(\xi, \eta): G \to G_x^*$ induces an epimorphism of $A$-modules $A \oplus A \to \text{Hom}_k(G_x^*, G_a) \to \text{Hom}_k(G, G_a)$ (cf. [6], p. 102, proposition). Hence $\text{Hom}_k(G, G_a) = A\eta + A\xi$ with $F^n\eta = \tau\xi$ as a defining relation. Since $G$ is reduced and irreducible, $\text{Hom}_k(G, G_a)$ is torsion free.

Let $\psi: G \to G_1$ be an isomorphism and consider the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\psi} & G_1 \\
\downarrow{\gamma'} & & \downarrow{\xi_1} \\
G_a & \xrightarrow{F^n1} & G_a \\
\end{array}
$$

Now $\gamma' = \eta_1\psi = \rho\eta + \sigma\xi$ for some $\rho, \sigma \in A$, and we must have $\rho \in A^*$ since $\gamma'$ is separable. Also, if $\rho = \rho_1 + \rho_2 F^n$, then $\rho\eta = \rho\eta + \rho_2\tau\xi$. So we can choose $\rho$ of degree $< p^n$. Assume first $n = n_1$. Then $\xi' = \xi_1\psi$ is purely inseparable of degree $p^n$. By 1.3, $\xi_1\psi = c\xi$ with $c \in A$ a separable and purely inseparable homomorphism, that is, $c \in k^*$. Now

$$\tau_1\xi_1\psi = F^n\eta_1\psi = F^n\rho\eta + F^n\sigma\xi = \rho^{(n)}F^n\eta + F^n\sigma\xi = (\rho^{(n)}\tau + F^n\sigma)c^{-1}\xi_1\psi,$$

giving $\tau_1 = (\rho^{(n)}\tau + F^n\sigma)c^{-1}$. Conversely, define $\xi', \eta' \in \text{Hom}_k(G, G_a)$ by $\xi' = c\xi$ and $\eta' = \rho\eta + \sigma\xi$. Then $F^n\eta' = \tau_1\xi'$, and we obtain a homomorphism $\psi: G \to G_1$ such that $\xi' = \xi_1\psi$ and $\eta' = \eta_1\psi$. Now $\rho$ is invertible in $A$ and we can write $\rho^{-1} = \rho_1 + \sigma_1 F^n$ with $\rho_1 \in A^*$. Then $\tau = (\rho_1^{(n)}\tau_1 + F^n\sigma_1)c$ with $\sigma_1 = (\sigma_1^{(n)}\tau - \rho_1\sigma)c^{-1} \in A$. Reversing the roles of $G$ and $G_1$ we get $\psi_1: G_1 \to G$ inverting $\psi$.

Suppose now $n - n_1 = n_2 \geq 0$. In the commutative diagram

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\xi_1} & G_a \\
\downarrow{\eta_1} & & \downarrow{\tau_1} \\
G_a & \xrightarrow{F^{n_1}} & G_a \\
\end{array}
$$

and

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\xi_1} & G_a \\
\downarrow{\eta_1} & & \downarrow{\tau_1} \\
G_a & \xrightarrow{F^{n_1}} & G_a \\
\end{array}
$$

Now $\gamma' = \eta_1\psi = \rho\eta + \sigma\xi$ for some $\rho, \sigma \in A$, and we must have $\rho \in A^*$ since $\gamma'$ is separable. Also, if $\rho = \rho_1 + \rho_2 F^n$, then $\rho\eta = \rho\eta + \rho_2\tau\xi$. So we can choose $\rho$ of degree $< p^n$. Assume first $n = n_1$. Then $\xi' = \xi_1\psi$ is purely inseparable of degree $p^n$. By 1.3, $\xi_1\psi = c\xi$ with $c \in A$ a separable and purely inseparable homomorphism, that is, $c \in k^*$. Now

$$\tau_1\xi_1\psi = F^n\eta_1\psi = F^n\rho\eta + F^n\sigma\xi = \rho^{(n)}F^n\eta + F^n\sigma\xi = (\rho^{(n)}\tau + F^n\sigma)c^{-1}\xi_1\psi,$$

giving $\tau_1 = (\rho^{(n)}\tau + F^n\sigma)c^{-1}$. Conversely, define $\xi', \eta' \in \text{Hom}_k(G, G_a)$ by $\xi' = c\xi$ and $\eta' = \rho\eta + \sigma\xi$. Then $F^n\eta' = \tau_1\xi'$, and we obtain a homomorphism $\psi: G \to G_1$ such that $\xi' = \xi_1\psi$ and $\eta' = \eta_1\psi$. Now $\rho$ is invertible in $A$ and we can write $\rho^{-1} = \rho_1 + \sigma_1 F^n$ with $\rho_1 \in A^*$. Then $\tau = (\rho_1^{(n)}\tau_1 + F^n\sigma_1)c$ with $\sigma_1 = (\sigma_1^{(n)}\tau - \rho_1\sigma)c^{-1} \in A$. Reversing the roles of $G$ and $G_1$ we get $\psi_1: G_1 \to G$ inverting $\psi$.

Suppose now $n - n_1 = n_2 \geq 0$. In the commutative diagram

$$
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G_a & \xrightarrow{F^{n_1}} & G_a \\
\end{array}
$$

and

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\xi_1} & G_a \\
\downarrow{\eta_1} & & \downarrow{\tau_1} \\
G_a & \xrightarrow{F^{n_1}} & G_a \\
\end{array}
$$
both the left and right square are cartesian. So the big square is cartesian, and consequently $G_i = (F^n, \tau_{(i,n)}^2)$. Now the previous argument applies.

Since $(F^n, \tau) \cong (F^n, \tau \epsilon(-\tau))$, any $G$ can be written with $\epsilon \in A^{**}$. This normalizes $\tau$ to some extent:

**Corollary 2.3.1.** Let $G = (F^n, \tau)$. Then $G \cong G_a$ if and only if $\tau \epsilon \in A^{(m)}$ for some $\epsilon \in k^*$. If $\tau = 1 + a_1 F + \cdots + a_m F^m \in A^{**}$, then $k' = k(a_1^{-1}, \ldots, a_m^{-1})$ is the minimal splitting field for $G$.

**Proof.** Since $G_a = (F^n, 1)$, the proposition gives $\tau \epsilon = \rho^{(n)} + F^n \sigma \in A^{(m)}$ if $G \cong G_a$. Conversely, let $\tau \epsilon \in A^*$ and we can write $1 = \rho \tau + \sigma F^n$. So $1 = (\rho^{(n)} \tau + F^{(m)} \sigma \epsilon^{-1}) \epsilon$ and $(F^n, 1) \cong (F^n, \tau)$. This proves the first statement, and the second follows since we can take $\epsilon = 1$ above if $\epsilon \in A^{**}$.

**Corollary 2.3.2.** Let $G = (F^n, \tau)$ and $0 \leq m \leq n$. Then

$$\theta^m G = (F^{n-m}, \tau).$$

**Proof.** Apply $\theta^m$ to the cartesian square defining $G$. Noting that $\theta^m \tau = \tau^{(m)}$, we get $\theta^m G = (F^n, \tau^{(m)}) \cong (F^{n-m}, \tau)$.

2.4. For any field $k \supseteq k$, we define $E(k)$ as the set of isomorphism classes of forms of $G_{a,k}$ and put $E(k, n) = \{G \in E(k) \mid \theta^* G \cong G_{a,k}\}$.

The rule $(\rho, \sigma, c) \cdot \tau = (\rho^{(n)} \tau + F^n \sigma) \epsilon^{-1}$ defines an action of

$$A^* \times A \times k^*,$$
edowed with a suitable semi-direct product structure, on $A^*$, and 2.3 states that $E(k, n)$ may be considered as the quotient of $A^*$ under this action. $A^*$ is not a group, but this inconvenience can be avoided by dividing out by $A$ first and passing to the group $U_n = A^*/AF^n$. Let $V_n = U_n \times k^*$. Then the map

\[ V_n \times A^*/F^n A \longrightarrow A^*/F^n A \]
\[ (\bar{\rho}, c) \times \bar{\tau} \longmapsto (\rho^{(n)} \tau \epsilon^{-1}) \]

(where $\epsilon$ denotes taking residue classes) is well defined and gives an action of $V_n$ on $A^*/F^n A$. Clearly all the operations involved are compatible with base field extension. Now 2.3 implies:

**Theorem 2.5.** The map

\[ A^* \longrightarrow E(k, n) \]
\[ \tau \longmapsto (F^n, \tau) \]
induces a bijection between the quotient of $A^*/F^nA$ by the action (*)
defined above and $E(k, n)$. This identification is compatible with
base field extension.

Similarly, we can define an action

$$U_n \times A^{**}/F^nA \longrightarrow A^{**}/F^nA$$

by $\bar{\rho} \cdot \bar{\tau} = (\rho^{(n)} \tau \in (\rho)^{-p^n})^{-}$. Since any $G$
can be written as $G = (F^n, \tau)$ with $\tau \in A^{**}$, the quotient
may again be identified with $E(k, n)$. As an example, let us work out
the case $n = 1$. Choose a complementary subspace $W_0$ for $k^p$ in $k$
and for each $i \geq 1$ let $W_i$ be a copy of $W_0$. Then $U_i = k^*$ acts on
$W = \bigoplus_{i=1}^{\infty} W_i$ by $c \cdot \sum a_i = \sum c^{p(i-1)p^i}a_i$. Letting $(F, 1 + \sum a_iF^i)$ correspond
to the class of $\sum a_i$, one identifies $E(k, 1)$ and $W/k^*$.

Let $A^*/F^{n+1}A \rightarrow A^*/F^nA$ be the natural map and define $V_{n+1} \rightarrow
V_n$ by $(\bar{\rho}, c) \mapsto (\rho^{(n)}, c)$. Then

$$
\begin{array}{ccc}
V_{n+1} \times A^*/F^{n+1}A & \longrightarrow & A^*/F^{n+1}A \\
\downarrow & & \downarrow \\
V_n \times A^*/F^nA & \longrightarrow & A^*/F^nA
\end{array}
$$

commutes and it follows from 2.3.2 that the induced map on the
quotients is $\Theta: E(k, n + 1) \rightarrow E(k, n)$. Unfortunately there does not seem to be a coherent way to reverse the vertical arrows in order to obtain the inclusion $E(k, n) \subset E(k, n + 1)$.

**Proposition 2.6.** Let $K \supset k$ be a field and

$$\Psi: E(k) \longrightarrow E(K)$$

$G \longmapsto G_K$

the natural map.

(i) If $K$ is purely inseparable over $k$, then $\Psi$ is surjective.

(ii) If $k$ is algebraically closed in $K$ and $K$ is separable over $k$,
then $\Psi$ is injective.

**Proof.** (i) Let $G = (F^n, \tau) \in E(K)$, $\tau = 1 + a_iF + \cdots + a_mF^m$. There is an integer $r \geq 0$ such that $a_i^{p^r} = a_i \in k$, $i = 1, \ldots, m$. Let

$$\tau' = 1 + a_iF + \cdots + a_mF^m$$

and $G' = (F^{n+r}, \tau') \in E(k)$. Then $\tau' = \tau^{(r)}$
over $K$ and 2.3 implies $G_K' = (F^{n+r}, \tau^{(r)}) \cong (F^n, \tau) = G$.

(ii) Let $G = (F^n, \tau), \tau = 1 + \sum a_iF^i \in A, \rho = \sum x_iF^i \in A_k^*$ with $x_i = 0$ for $i \geq n$, and $\sigma = \sum y_iF^i \in A_k^*$. Suppose $(\rho^{(n)}\tau + F^n\sigma)x_0^{-p^n} = 1 + \sum b_iF^i = \tau' \in A$, that is,

$$(*) \quad \left(\sum_{j=0}^{t-1} x_j^{p^n}a_{i-j}^{p^n} + x_i^{p^n} + y_i^{p^n}\right)x_0^{-p^{n+i}} = b_i \in k$$
for $i \geq 1$. (Set $y_i = 0$ for $i < 0$). We have to show that the same can be done with $x_i, y_i \in k$. We may clearly assume $G \not\cong G_a$. Then not all $a_i \in k^{p^n}$ and there is an $r \geq 1$ such that $a_i, \ldots, a_{r-1} \in k^{p^n}$ but $a_r \notin k^{p^n}$. If $r > 1$, we can replace $\tau$ by $(1 - a_r F)\tau$ (since $a_i \in k^{p^n}$) which has a zero linear term. By an obvious induction argument, we can assume $a_1 = \cdots = a_{r-1} = 0$. Then (*) gives for $i = r$

\[
\begin{equation}
\begin{aligned}
\alpha_r x_0^{p^n - r + r} + x_r^{p^n} x_0^{-p^n + r} + y_r^{p^n} x_0^{-p^n + r} = b_r.
\end{aligned}
\end{equation}
\]

Put $u = x_r^{-1}$, $v = x_r x_0^{-p^n}$ if $r < n$ (and so $y_{r-n} = 0$), and $v = y_{n-r} x_r^{p^n}$ if $r \geq n$ (and so $x_r = 0$). In both cases $a_r u^{(p^n-1)p^n} + v^{p^n} = b_r$. Extracting $p$-th roots in $k$ from $a_r$ and $b_r$ as far as possible, we can write

\[
\begin{equation}
\begin{aligned}
\alpha_r \xi(\psi) \tau: G \to \beta
\end{aligned}
\end{equation}
\]

$\psi \in \text{Hom}_k(G, G_1)$ is nonzero. Then $\Theta^n \psi: G_a \to \Theta^n G_1$ is nonzero, and since a nonzero homomorphic image of $G_a$ is isomorphic with $G_a$ (cf. [6], p. 101, lemma), we must have $n_2 = n - n_1 \geq 0$. Now $F'^n \xi \psi$ has inseparable degree $\geq p^n$ and therefore factors through $\xi$. This gives a commutative diagram

\[
G \xrightarrow{\xi} G_a \quad \quad \quad G_1 \xrightarrow{\xi_1} G_a
\]

\[
\eta \downarrow \quad \quad \quad \quad \quad \quad \eta_1 \downarrow
\]

\[
G_a \xrightarrow{F'^n} G_a \quad \quad \quad \quad \quad \quad G_a \xrightarrow{F'^{n_1}} G_a
\]
If $\psi$ is separable, so are $\tau_2$ and $\tau_2^{[n^2]}\tau_2$. This shows that one can use the big square to define $G$ as a fiber product, that is, $G \cong (F^*, \tau_2^{[n^2]}\tau_2)$. By 2.3 there exist $\rho \in A^*$ and $\sigma \in A$ such that

\[
\tau_2^{[n^2]}\tau_2 = \rho^{[n]}\tau + F^n\sigma.
\]

(No $c$ appears since $\xi$ is left unchanged.) Conversely, if $\tau_2$ satisfies $(\ast)$, there is a unique $\psi$ making the diagram commutative. So separable homomorphisms $\psi: G \to G_1$ are in one-to-one correspondence with those $\tau_2 \in A^*$ for which a solution to $(\ast)$ exists.

**Theorem 3.1.** Let $G$ be a form of $G_α$, $G \not\cong G_α$. Then $\text{End}_k G$ may be identified with a finite subfield of $k$. If $\text{End}_{k_s} G_{k_s} = F_q$ and $k \subset K \subset k_s$, then $\text{End}_K G_K = K \cap F_q$.

**Proof.** Let $G = (F^n, \tau)$, $n = n(G)$, and suppose $\psi: G \to G$ is non-zero. If $\psi$ is not separable, there is a nonzero homomorphism $\Theta G \to G$. Since $n(\Theta G) < n(G)$, this is impossible, as we have seen. So $\psi$ is separable and $\tau_2 = \Theta^{\ast}\psi$ satisfies a relation

\[
\tau_2^{[n^2]}\tau_2 = \rho^{[n]}\tau + F^n\sigma.
\]

We will assume, as we may, that $\deg \rho < p^n$. Since $\Theta^{\ast}, r \geq 0$, is a faithfully flat base change functor, $\Theta^{\ast}: \text{End}_k G_1 \to \text{End}_k \Theta^{\ast}G$ is injective and moreover $\Theta^{\ast}\psi$ is a monomorphism (epimorphism) if and only if $\psi$ is. Taking $r = n - 1$, we see that it is enough to prove the first statement in case $n = 1$. We can then choose $\rho = a \in k^*$ and $\tau = 1 + a_1F^{m_1} + \cdots + a_sF^{m_s}$ with $1 \leq m_1 < m_2 < \cdots < m_s$ and $a_i \notin k^p$. Let $\tau_2 = c_0 + c_1F + \cdots + c_sF^r$, $c_0 \neq 0$ and $c_r \neq 0$. Comparing coefficients in $(\ast)$, we get $a_sc_r^{m} \in k^p$ unless $r = 0$. Since $m_s \geq 1$ and $a_s \notin k^p$, we actually have $r = 0$ and $\tau_2 = c_0 = c \in k^*$. $(\ast)$ now reduces to $a^p\tau - \tau c \in F^A$, and this gives $a^p - c = 0$ and $(c - c^p)\alpha_i \in k^p$, $i = 1, \ldots, s$. Since $a_i \notin k^p$, this implies $c - c^p = 0$. Or, equivalently, $c - c^p = 0$ where $m$ is the greatest common divisor of $m_1, \ldots, m_s$. Conversely $\tau c = c\tau$ for such $c$ and if $c \neq 0$, it lifts to an automorphism of $G$. Hence $\text{End}_k G = k \cap F^*_p$ in this case.

Now let $n \geq 1$, $F_q = \text{End}_{k_s} G_{k_s}$, $k \subset K \subset k_s$ and $\tau_2 = c \in K \cap F^*_q$. To
show that $c \in \text{End}_K G_K$, we have to solve (*) with $\rho, \sigma \in A_K$. However there exists a solution over $k_s$, and applying to it a $K$-automorphism $\lambda$ of $k_s$, we get $\tau c = \lambda(\tau c) = \lambda(\rho^n) + F^n \lambda(\sigma)$ and $0 = (\rho^n - \lambda(\rho^n))\tau + F^n(\sigma - \lambda(\sigma))$. Multiplying by $\tau^{-1}$ (in $A_K$), we have $0 = (\rho^n - \lambda(\rho^n)) + F^n(\sigma - \lambda(\sigma))\tau^{-1}$, giving $\rho^n = \lambda(\rho^n)$ and $\sigma = \lambda(\sigma)$ since $\deg \rho < p^n$. Hence $\rho, \sigma \in A_K$.

The theorem states that the automorphism functor of $G$ coincides with the functor $\mu_r$ ($r$-th roots of unity, $r = q - 1$ prime to $p$) on separable algebraic extensions of $k$. Galois cohomology therefore gives (for details we refer to [8], in particular I, §5, II, §1 and III, §1):

**Corollary 3.1.1.** Let $E(k_s/k, G)$ be the set of $k_k$-forms of $G$. Then $E(k_s/k, G) = H^1(k, F_q^*) \cong k^*/k^{*q-1}$.

4. We turn now to forms of $A^1$ that fail to be groups by just the absence of a rational point.

**Proposition 4.1.** Let $X$ be a form of $A^1$ and suppose that $X_{k_s}$ admits a group structure. Then $X$ is a principal homogeneous space for a form $G$ of $G_a$ determined uniquely by $X$. Moreover, $X = \text{Spec } k[x, y]/I, G = \text{Spec } k[u, v]/J$ where $I$ and $J$ are generated respectively by $y^{p^n} - b - f(x)$ and $v^{p^n} - f(u)$ with $b \in k$ and $f$ a separable $p$-polynomial. Conversely, if $X$ and $G$ are defined as above, then $X$ is a principal homogeneous space for $G$.

**Proof.** Let $X = \text{Spec } R$. As in the proof of 2.1, we have $(k, \varphi^n) \otimes_k R \cong k[t]$ for some $n, t = \sum a_i \otimes y_i$ with $a_i \in k$ linearly independent over $k^{p^n}$, and $y_i^{p^n} = g_i(x) \in k[x]$ with $x = F_k^n(t)$. Let $q \in X_{k_s}$ be rational over $k_s$ and let $c_i \in k_s$ be the residue of $y_i$ at $q$. Put $y_i' = y_i - c_i, t' = t - \sum a_i c_i^{p^n} = t - c$ and $x' = x - c$. Then $t' = \sum a_i \otimes y_i'$, $q$ lies above the point $t' = 0$ of $A_1^{k_s} \cong \Theta^* X_{k_s}$ and we can choose $q$ as the origin of the group structure supposed to exist on $X_{k_s}$. The $a_i$ remain linearly independent over $k^{p^n}$ and we have $y_i^{p^n} = f_i(x')$ with $f_i$ a $p$-polynomial as in the proof of 2.1. Hence $g_i(x) = y_i^{p^n} = b_i + f_i(x)$ with $b_i = c_i^{p^n} - f_i(c)$, and $g_i(x) \in k[x]$ implies $b_i \in k$ and $f_i(x) \in k[x]$. If $y$ is a separating variable for $\kappa(X)$ picked from the $y_i$, we get $y^{p^n} = b + f(x)$ where $f$ has nonzero linear term. As before, this implies $R = k[x, y]$. Let $G = \text{Spec } S, S = k[u, v]$ with $v^{p^n} = f(u)$. Then $\alpha: R \rightarrow R \otimes_k S, \alpha(x) = x \otimes 1 + 1 \otimes u$ and $\alpha(y) = y \otimes 1 + 1 \otimes v$, defines an action of $G$ on $X$. $\bar{\alpha}: R \otimes_k R \rightarrow R \otimes_k S$ defined by $\bar{\alpha}(w \otimes z) = (w \otimes 1)\alpha(z)$ is an isomorphism and gives an isomorphism (over $X$) $G \times_k X \cong X \times_k X$. Hence $X$ is a principal homogeneous space for $G$. If this is also true for $G_1$, we get an isomorphism (over $X$) $G \times_k X \cong G_1 \times_k X$. Applying 2.6 (ii) to the fiber over the generic
point of $X$, we see that $G \cong G_x$.
Principal homogeneous spaces for $G$ are classified by $H^1(k, G)$ (cf. [8], I, Proposition 33). Let $G = (F^n, \tau)$. Then there is a commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & \ker \gamma & \rightarrow & G & \rightarrow & G_a & \rightarrow & 0 \\
& & \downarrow \xi & & \downarrow \eta & & \downarrow \tau & & \\
0 & \rightarrow & \ker \tau & \rightarrow & G_a & \rightarrow & G_a & \rightarrow & 0.
\end{array}
$$

The exact cohomology sequence and $H^1(k, G_a) = 0$ give $H^1(k, G) = k/f(k) + k^n$, where $f$ is the $p$-polynomial corresponding to $\tau$. The Galois group of the splitting field of $0 = b + f(x) = b + a_0x + \cdots + a_m x^m$, $a_0 \neq 0$, is isomorphic to a subgroup of $f^{-1}(0) \subset k$. Hence $f(k) = k$ if $k$ has no normal extension of degree $p$, and $H^1(k, G) = 0$ for all forms $G$ of $G_a$ in that case. The author does not know whether the converse of this statement is true if $k$ is not perfect.

In [4] Rosenlicht characterized curves that are "exceptional" in the sense that the genus $g$ is $\geq 1$ and the group of automorphisms (leaving a point fixed if $g = 1$) is infinite. We give another characterization, already implicit in [4], p. 10, theorem, assuming the exceptional case over $k_s$ only.

**Theorem 4.2.** Let $P$ be a complete regular curve such that $P_{k_s}$ is exceptional. Then $P$ has exactly one singular point $q$, $q$ is purely inseparable over $k$, and $X = P - \{q\}$ is a principal homogeneous space for a form of $G_a$.

*Proof.* It is enough to prove the first statement in case $k = k_s$. It is then taken directly from [4], p. 5, lemma. It is also shown there that $P_{k_s}$ has genus zero. Hence $X = P - \{q\}$ is a form of $\mathbb{A}^1$ and we have $F^n: X \rightarrow \theta^n X = \mathbb{A}^1 = \text{Spec } k[t]$ for some $n$. This gives an injection $\theta^n: \text{Aut}_k X \rightarrow \text{Aut}_k \mathbb{A}^1$. Now let $k = k_s$. It then follows from [4], loc. cit., that $\text{Aut}_k X$ has an infinite subset of automorphisms operating without fixed point. Hence $\theta^n(\text{Aut}_k X)$ contains infinitely many translations $t \mapsto t + b$. With notations as in the proof of 4.1, write $t = \sum a_i \otimes y_i$, $1 \otimes y_i = f_i(t)$, with $t$ so chosen that the point $q_0 \in X$ above $t = 0$ is rational. If $c_i$ is the residue of $y_i$ at $q_0$, we have $f_i(0) = c_i^o \in k^{p^n}$. Since $0 = \sum a_i f_i(0)$, we get $f_i(0) = 0$. If $T_b$ is the automorphism of $X$ inducing $t \mapsto t + b$, we have $t + b = \sum a_i \otimes T_b^*(y_i)$. Let $b_i \in k$ be the residue of $y_i$ at $T_b(q_0)$. Then $b = \sum a_i b_i^o$ and $t + b = \sum a_i \otimes (y_i + b_i)$. Hence $T_b^*(y_i) = y_i + b_i$ and $f_i(t + b) = 1 \otimes T_b^*(y_i) = f_i(t) + b_i^o$. With $t = 0$, this shows $b_i^o = f_i(b)$. Since this holds for infinitely many $b$, each $f_i$ is a $p$-polynomial. Hene $X$ has a group
If $X$ is a principal homogeneous space for a form $G$ of $G_\alpha$ and $P \supset X$ a complete regular curve, then $G(k_s) \subset \text{Aut}_{k_s} P_{k_s}$ is infinite. So $P_{k_s}$ is exceptional if the genus $g$ of $P$ is positive. The cases $g = 0$ as well as $g = 1$ can be settled completely. Excluding the trivial case $X = A^1$, we have: If $g = 0$, then char $k = 2$. If $g = 1$, then char $k = 3$. Moreover, $X = \text{Spec } k[x, y]/I$ where $I$ is generated by $y^p - b - x - ax^p$ with $p = 2$ or $3$ respectively and $a, b \in k$.

It is enough to prove the corresponding statement for the groups $G$ that are involved, that is, we may assume $X = G$ has a rational point. Now, by a theorem of Tate ([9], Corollary 2), the genus changes by a multiple of $1/2(p - 1)$ on passage from $X$ to $\theta X$. On the other hand, if $\mathcal{O}$ is the local ring of $P - X$, the genus change is $\dim_k \mathcal{O}/\mathcal{O}'$ where $\mathcal{O}' = (k, \varphi) \otimes_k \mathcal{O}$ and $\mathcal{O}_1$ is the normalization of $\mathcal{O}'$ (cf. [7], p. 73, example). So a drop in genus occurs unless $\mathcal{O}$ is nonsingular. But then $P$ is nonsingular, so $g = 0$ and $P \cong P^1$. Excluding the case $G = G_\alpha$ we must have $P^1 - G$ of degree 2 (cf. [5], p. 35 or the remark in the introduction). Hence $p = 2$ and $n(G) = 1$. If $p > 2$, we see that $g \geq 1/2n(G)(p - 1)$. So $g = 1$ implies $n(G) = 1$ and $p = 3$. In both cases ($g = 0$ or 1) $G = \text{Spec } k[x, y]$ with $y^p = x + a_1x^p + \cdots + a_mx^{pm}$ and $a_m \in k^p$ (cf. 2.1). Using [9], proposition, one checks that then $g = 1/2(p - 1)(p^m - 2)$. So necessarily $m = 1$.

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