DYNAMICAL SYSTEMS OF CHARACTERISTIC $0^+$

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The main purpose of this paper is to classify the dynamical systems on the plane which satisfy a certain type of stability criterion. Such flows are referred to as dynamical systems of characteristic 0+. The classification is based on the consideration of three mutually exclusive and exhaustive cases: Dynamical systems of characteristic 0+ which have no critical points, those whose critical points form nonempty compact sets, and those whose critical points do not form compact sets.

Dynamical systems of characteristic 0+ are those dynamical systems in which all closed positively invariant sets are positively D-stable, i.e., stable in Ura’s sense (see [11]). If the phase space of a flow is regular, then a closed positively invariant set, which is positively stable in Liapunov’s sense, is also positively D-stable. Thus, some simple examples of flows of characteristic 0+ are those where the phase spaces are regular and all closed invariant sets are positively stable in Liapunov’s sense.

In § 2 we give some of the basic definitions and notations that are used throughout the paper. In § 3 we prove some results of a more general nature which are later applied to flows of characteristic 0+ on the plane. It is proved that if the phase space $X$ of a flow is normal and connected and a closed invariant set $F$ is globally + asymptotically stable, then $F$ is connected. Further, if the phase space $X$ of a flow of characteristic 0+ is connected and locally compact, then a compact subset $M$ of $X$ is a positive attractor implies that $M$ is globally + asymptotically stable.

In § 4 we discuss flows of characteristic 0+ on the plane. It is shown that if the set of critical points $S$ of such a flow is empty, then the flow is parallelizable. If $S$ is compact, then it either consists of a single point which is a Poincaré center, or it is globally + asymptotically stable. If $S$ is not compact, then either $R^2 = S$, or $S$ is + asymptotically stable; $S$ and the region of positive attraction $A^+(S)$ of $S$ each has a countable number of components. Further, each component of $A^+(S)$ is homeomorphic to $R^2$. At the end of this section, we summarize all the results of this section in the form of a complete classification of such flows.

In § 5 we discuss flows of characteristic 0± on the plane, i.e., those in which every closed invariant set is positively and negatively stable in Ura’s sense. We prove that such a flow is either parallelizable, or it has a single critical point which is a global Poincaré center, or all
points are critical points.

2. Notations and definitions. Let \( R, R^+, \) and \( R^- \) denote the sets of real numbers, nonnegative, and nonpositive real numbers, respectively. Given a topological space \( X \) and a mapping \( \pi \) of the product space \( X \times R \) into \( X \), we say \((X, \pi)\) defines a dynamical system or flow on the phase space \( X \) if the following conditions are satisfied.

1. Identity axiom: \( \pi(x, 0) = x \).
2. Homomorphism axiom: \( \pi(\pi(x, t), s) = \pi(x, s + t) \).
3. Continuity axiom: \( \pi \) is continuous on \( X \times R \).

For brevity, we denote \( \pi(x, t) \) by \( xt \). For each \( x \in X \), we let \( C(x) \) denote the trajectory or orbit through \( x \), i.e., \( C(x) = xR \). Similarly, the positive and negative semi-trajectories through \( x \) are represented by \( C^+(x) \) and \( C^-(x) \), respectively, i.e., \( C^+(x) = xR^+ \) and \( C^-(x) = xR^- \).

We let \( L^+(x) \) denote the positive (or \( \omega \)-) limit set of \( x \), i.e., \( L^+(x) = \bigcap \{ C^+(xt) : t \in \mathbb{R} \} \). Similarly, \( L^-(x) \) denotes the negative (or \( \alpha \)-) limit set of \( x \). A point \( x \) is called a critical or rest point if \( xR = x \). A subset \( M \) of \( X \) is said to be invariant if \( C(M) = M \), and positively (negatively) invariant if \( C^+(M) = M \) (\( C^-(M) = M \)). A closed invariant set \( M \) is minimal if it has no proper subset which is closed and invariant.

Throughout this paper, we use \( \partial M \) and \( \bar{M} \) to represent the boundary and closure of \( M \). Given a Jordan curve \( C \) on the plane \( \mathbb{R}^2 \), we let \( \text{int}(C) \) denote the bounded component of \( \mathbb{R}^2 - C \). Let \( (\mathbb{R}^2)^* = \mathbb{R}^2 \cup \{ \omega \} \) be the one point compactification of the plane.

A closed positively invariant set \( M \) is said to be positively Liapunov stable, or more simply, positively stable, if for every neighborhood \( U \) of \( M \), there exists a neighborhood \( V \) of \( M \) such that \( C^+(V) \subset U \). \( M \) is said to be a positive attractor if there exists a neighborhood \( U \) of \( M \) such that \( \phi \neq L(x) \subset M \) for all \( x \) in \( U \). The largest such neighborhood \( U \) is called the region of positive attraction of \( M \) and will be denoted by \( \text{A}^+(M) \). \( M \) is said to be + asymptotically stable if it is both positively stable and a positive attractor. It is said to be globally + asymptotically stable if it is + asymptotically stable and \( \text{A}^+(M) = X \).

For each \( x \in X \), the (first) positive (negative) prolongation \( D^+(x) \) \( (D^-(x)) \) of \( x \) is given by

\[
D^+(x) = \bigcap_{N \in \gamma(x)} \{ C^+(N) \} \quad (D^-(x) = \bigcap_{N \in \gamma(x)} \{ C^-(N) \}),
\]

where \( \gamma(x) \) is the neighborhood filter of \( x \).

The (first) positive (negative) prolongational limit set of \( x \) is given by
\[ J^+(x) = \bigcap_{t \in R} \{ D^+(xt) \} \quad (J^-(x) = \bigcap_{t \in R} \{ D^-(xt) \}). \]

It is known and easy to verify that \( L^+(x) \subseteq J^+(x) \). Further, if \( X \) is a Hausdorff space, then \( D^+(x) = C^+(x) \cup J^+(x) \).

A closed positively invariant set \( M \) is said to be *positively D-stable* if \( D^+(M) = M \).

It is easy to verify that if \( X \) is regular and a closed positively invariant set \( M \) is positively stable (i.e., stable in Liapunov's sense as defined above), it is also positively \( D \)-stable. The converse is false.

The following theorem, which we use several times in this paper, is due to Ura [11].

**Theorem (Ura).** Let \((X, \pi)\) be a dynamical system on a locally compact space \( X \), and let \( M \) be a compact subset of \( X \). Then \( M \) is positively stable if and only if it is positively \( D \)-stable.

**Remark.** The statement "\( X \) is locally compact" is used in the Bourbaki sense throughout this paper, i.e., \( X \) is assumed to be a Hausdorff space.

3. Flows of characteristic \( 0^+ \). Before discussing flows of characteristic \( 0^+ \), we prove a lemma and a proposition concerning flows in general.

**Lemma 1.** Let \((X, \pi)\) be any dynamical system. If \( x \in X \) and \( y_1, y_2 \in L^+(x) \), then \( y_1 \in D^+(y_2) \) and \( y_2 \in D^+(y_1) \).

**Proof.** We note that
\[ D^+(y_1) = \bigcap_{N \in \eta(y_1)} \{ C^+(N) \}, \]

where \( \eta(y) \) denotes the neighborhood filter of \( y \). Since \( y_1, y_2 \in L^+(x) \), for each \( N \in \eta(y_1) \) and \( M \in \eta(y_2) \), there exist \( t_1, t_2 \in R^+ \) with \( xt_1 \in N \) and \( xt_1t_2 = x(t_1 + t_2) \in M \). Hence \( y_2 \in C^+(N) \), and consequently, \( y_2 \in D^+(y_1) \). Similarly, \( y_1 \in D^+(y_2) \).

**Proposition 3.1.** Let \((X, \pi)\) be a dynamical system on a normal (and Hausdorff) connected topological space \( X \). If a closed invariant subset \( F \) of \( X \) is globally + asymptotically stable, then \( F \) is connected.

**Proof.** Suppose \( F \) is not connected. Then there exist two non-
empty disjoint closed sets \( F_1 \) and \( F_2 \) such that \( F = F_1 \cup F_2 \). Since \( X \) is normal, there exist two disjoint open neighborhoods \( U_i \) and \( U_s \) of \( F_1 \) and \( F_2 \), respectively. On the other hand, since \( F \) is positively stable, corresponding to the neighborhood \( U = U_i \cup U_s \) of \( F \), there is an open neighborhood \( V \) of \( F \) such that \( C^+(V) \subset U \). Therefore, if we let \( V_i = V \cap U_i, i = 1, 2 \), then for each \( x \in V_i \), \( C^+(x) \subset U_i \) since \( C^+(x) \) is connected. Thus, \( L^+(x) \subset F_i \) i.e., \( V_i \subset A^+(F_i) \) since \( \overline{U_i} \cap F_j = \emptyset, i \neq j \). Hence, we have shown that \( F_1 \) and \( F_2 \) are positive attractors; consequently \( A^+(F_1) \) and \( A^+(F_2) \) are open, since the boundary of each is closed and invariant. But this contradicts the assumption that \( X \) is connected, since \( X = A^+(F) = A^+(F_1) \cup A^+(F_2) \), where \( A^+(F_1) \) and \( A^+(F_2) \) are clearly nonempty disjoint open sets. This completes the proof of Proposition 3.1.

**Definition 3.1.** A dynamical system \((X, \pi)\) is said to have characteristic \( 0^+ \) if and only if \( D^+(x) = C^+(x) \) for all \( x \in X \).

The above definition is equivalent to saying that \((X, \pi)\) has characteristic \( 0^+ \) if and only if every closed positively invariant subset of \( X \) is positively \( D \)-stable.

It follows that if the phase space \( X \) of a flow of characteristic \( 0^+ \) is a Hausdorff space, then \( D^+(x) = C^+(x) \cup L^+(x) \), for all \( x \in X \).

**Lemma 2.** Let \((X, \pi)\) be a flow of characteristic \( 0^+ \). If \( x \in X \) such that \( L^+(x) \neq \emptyset \), then \( x \in L^+(x) \).

**Proof.** Suppose \( L^+(x) \neq \emptyset \) and let \( y \in L^+(x) \). Then, \( y \in D^+(x) \), and hence \( x \in D^+(y) = C^+(y) \). On the other hand, \( y \in L^+(x) \) implies that \( C^+(y) \subset L^+(x) \), since \( L^+(x) \) is a closed invariant set. Therefore, \( x \in L^+(x) \).

**Proposition 3.2.** Let \((X, \pi)\) be a flow of characteristic \( 0^+ \) on a connected locally compact space \( X \). If \( M \) is a compact positively invariant subset of \( X \) and \( M \) is a positive attractor, then \( M \) is globally \( + \) asymptotically stable.

**Proof.** Since \( M \) is a closed positively invariant set, we have \( D^+(M) = M \). Therefore, \( M \) is positively stable by Ura's theorem. It is sufficient to show that \( \partial A^+(M) = \emptyset \). Suppose that \( \partial A^+(M) \neq \emptyset \), and let \( x \in \partial A^+(M) \). Let \( \gamma_{A}(x) \) be the trace of the neighborhood filter \( \gamma(x) \) of \( x \) on \( A = A^+(M) \). Then, for each \( N_{A} \in \gamma_{A}(x), \emptyset \neq L^+(N_{A}) \subset M \). Since \( M \) is compact, the cluster set of the filter base \( \{L^+(N_{A}) \mid N_{A} \in \gamma_{A}(x)\} \) is a nonempty subset of \( M \); hence \( J^+(x) \cap M \neq \emptyset \). However, this
contradicts the assumption that $(X, \pi)$ has characteristic $0^+$, since
$\partial A^+(M)$ is a closed invariant set disjoint with $M$. Therefore, $\partial A^+(M) = \emptyset$
and the proof of Proposition 3.2 is complete.

4. Flows of characteristic $0^+$ on the plane. Throughout this
section, we assume the phase space to be the plane $\mathbb{R}^2$ and $(\mathbb{R}^2, \pi)$ to
be a fixed flow of characteristic $0^+$. We let $S$ denote the set of rest
points of this flow.

Lemma 3. For each $x \in X$, if $L^+(x) \neq \emptyset$, then $L^+(x)$ is either a
periodic orbit or it consists of a single rest point.

Proof. If $L^+(x)$ contains a rest point $s_0$, then $L^+(x) = \{s_0\}$. For,
y $y \in L^+(x)$ implies that $y \in D^+(s_0) = \{s_0\}$, by Lemma 1. Suppose that
$L^+(x)$ consists of regular points only. Then, to complete the proof of
the lemma, it is sufficient to prove that $L^+(x)$ is compact. We note
that if $y \in L^+(x)$, then $C^+(y) = L^+(x)$. For, $z \in D^+(y) = C^+(y)$.
Also, $C^+(y) \subset L^+(x)$ since $L^+(x)$ is a closed invariant
set, and hence $C^+(y) = L^+(x)$. Since $C^+(y) \subset C(y) \subset L^+(x)$, we have
$C(y) = L^+(x)$. Therefore, $L^+(x)$ is a minimal set. We recall that if $M$
is a minimal subset of $\mathbb{R}^2$ which is not compact, then for each
$m \in M$, $L^+(m) = \emptyset$ (c.f. p. 37 of [6]). Suppose that $L^+(x)$ is not
compact, and let $y_1$ and $y_2$ be two distinct points in $L^+(x)$. Then,
y $y_1 \in D^+(y_1) = C^+(y_1)$ and $y_2 \in D^+(y_2) = C^+(y_2)$. But, if $t_1$ and $t_2$
are positive numbers such that $y_1 = y_1t_1$ and $y_2 = y_2t_2$, then $y_1 = y_1(t_1 + t_2)$;
showing that $C^+(y_1)$ is a periodic orbit. Hence, $L^+(x)$ is a periodic orbit,
since $L^+(x) = C^+(y_1)$, as it is a minimal set; thus contradicting the as-
sumption that $L^+(x)$ is not compact.

For a proof of the following theorem see [5].

Theorem (Bhatia). A flow $F$ on a metric space $X$ is dispersive
if and only if for each $x \in X$, $D^+(x) = C^+(x)$ and there are no rest
points or periodic orbits.

Theorem 4.1. If $S = \emptyset$, then the flow $(\mathbb{R}^2, \pi)$ is parallelizable.

Proof. We note that for each $x \in \mathbb{R}^2$, $L^+(x) = \emptyset$, and hence
$D^+(x) = C^+(x) = C^+(x)$. For, if $L^+(x) \neq \emptyset$, then by Lemma 3, it must
be a periodic orbit since it consists of regular points only. But this
is impossible since the bounded component of a periodic orbit contains
a rest point. Thus, the proof of our assertion follows from Bhatia’s
Theorem, stated above (c.f. Auslander [2]) and the fact that the notions
of parallelizability and dispersiveness are equivalent for a flow on the plane (see Antosiewicz and Dugundji [1]).

**Theorem 4.2.** If \( R^2 \) contains a periodic point, then \( S \) is a singleton. Further, if \( S = \{ s_0 \} \), then one of the following holds.

1. \( s_0 \) is a global Poincaré center.\(^2\)
2. \( s_0 \) is a local Poincaré center. The neighborhood \( N \) of \( s_0 \), consisting of \( s_0 \) and periodic orbits surrounding \( s_0 \), is a globally asymptotically stable simply connected continuum. Further, if \( x \in N \), then \( L^*(x) = \delta N \).

**Proof.** Let \( x_0 \) be any periodic point, and let \( S_0 = \text{int} (C^+(x_0)) \cap S \). We note that \( \text{int} (C^+(x_0)) \neq S_0 \) since \( S \) is closed; and for each regular point \( x \) in \( \text{int} (C^+(x_0)) \), \( C^+(x) \) is a periodic orbit, by virtue of Lemma 2.\(^3\) Let \( (B_a)_{\alpha \in I} \) be the family of all periodic orbits such that for each \( \alpha \in I \), \( \text{int} (B_\alpha) \cap S = S_0 \). Let \( B = \bigcup_{\alpha \in I} \text{int} (B_\alpha) \). If \( \partial B = \emptyset \), then \( B = R^2 \). Suppose that \( \partial B \neq \emptyset \). Then \( \partial B \) is a closed invariant set since \( B \) is invariant. Further, \( \partial B \cap S = \emptyset \). For, if \( b_0 \in \partial B \cap S \), then one can choose a simple closed curve \( C \) such that \( \text{int}(C) \cap S_0 = \emptyset \), since \( S_0 \subset \text{int} (C^+(x_0)) \subset B \) and \( S_0 \) is closed. Clearly, there is no neighborhood \( W \) of \( b_0 \) with \( C^+(W) \subset \text{int}(C) \), since \( x \in W \cap B - S_0 \) would imply that \( x \) is a periodic point, by Lemma 2, and \( \text{int} (C^+(x)) \cap S_0 \neq \emptyset \). But this contradicts the fact that \( \{ b_0 \} \) is positively stable, as \( D^+(b_0) = \{ b_0 \} \); thus showing that \( \partial B \cap S = \emptyset \). This also shows that \( \partial B \) is not a singleton since it is invariant and consists of regular points.

We note that if \( x \in B \) and \( x \in S_0 \), then \( x \) is a periodic point, by Lemma 2, with \( C^+(x) \subset B \) and \( \text{int} (C^+(x)) \cap S_0 \neq \emptyset \). For, \( x \) belongs to \( \text{int} (B_\alpha) \) for some \( \alpha \in I \). Thus, \( x \in S \) since \( \text{int} (B_\alpha) \cap S = S_0 \). Further \( L^*(x) \neq \emptyset \) and \( C^+(x) \subset B \) since \( x \) is surrounded by the periodic orbit \( B_\alpha \). Thus, \( x \) is a periodic point with \( \text{int} (C^+(x)) \cap S_0 \neq \emptyset \) since \( C^+(x) \subset \text{int} (B_\alpha) \) and \( \text{int} (B_\alpha) \cap S = S_0 \). Now we wish to show that \( \partial B \) is a periodic orbit. In order to accomplish this, we consider two cases.

**Case 1.** Suppose \( \partial B \cap C^+(x_0) \neq \emptyset \). Then, since \( \partial B \) is invariant, we must have \( C^+(x_0) \subset \partial B \). On the other hand, \( \partial B \subset C^+(x_0) \). For, assume \( \partial B \not\subset C^+(x_0) \), and let \( b \in \partial B - C^+(x_0) \). Then, \( b \in \text{int} (C^+(x_0)) \) since \( \text{int} (C^+(x_0)) \subset B \). Thus, one can choose a neighborhood \( U \) of \( b \) such that \( U \cap \text{int} (C^+(x_0)) = \emptyset \) since \( b \in \text{int} (C^+(x_0)) \), as \( b \in C^+(x_0) \) and

\(^2\) \( s_0 \) is a global Poincaré center if for each \( x \neq s_0 \), \( C(x) \) is a periodic orbit surrounding \( s_0 \). It is a local Poincaré center if it has a neighborhood \( M \) such that for each \( x \in M - \{ s_0 \} \), \( C(x) \) is a periodic orbit surrounding \( s_0 \).

\(^3\) It is a known fact about flows on the plane that a point is positively (or negatively) Poisson stable if and only if it is either a rest point or a periodic point (see [10]).
b \in \text{int}(C^+(x_0))$. Let $x \in U \cap B$. Then, $x \in S_0$ since $S_0 \subset \text{int}(C^+(x_0))$. Thus $C^+(x)$ is a periodic orbit. Since $\text{int}(C^+(x_0))$ is connected, $\text{int}(C^+(x)) \cap \text{int}(C^+(x_0)) \neq \emptyset$, as $\text{int}(C^+(x)) \cap S_0 \neq \emptyset$ and

$$\partial \text{int}(C^+(x)) \cap \text{int}(C^+(x_0)) = C^+(x) \cap \text{int}(C^+(x_0)) = \emptyset,$$

it follows that $\text{int}(C^+(x_0)) \subset \text{int}(C^+(x))$. But, $C^+(x_0) \subset \text{int}(C^+(x)) \subset B$ contradicts the assumption that $\partial B \cap C^+(x_0) \neq \emptyset$, as $B$ is open; hence $\partial B = C^+(x_0)$.

**Case 2.** Suppose $\partial B \cap C^+(x_0) = \emptyset$, and let $b_1, b_2 \in \partial B$. First we show that $b_2 \in D^+(b_1)$ and $b_1 \in D^+(b_2)$. In order to show that $b_2 \in D^+(b_1)$, it is sufficient to show that if $C_1$ and $C_2$ are any simple closed curves with $b_1 \in \text{int}(C_1)$ and $b_2 \in \text{int}(C_2)$, then there exist $x_1 \in \text{int}(C_1)$ and $t_1 \in R^+$ such that $x_1 t_1 \in \text{int}(C_2)$. Let $y_1 \in \text{int}(C_1) \cap B - \text{int}(C^+(x_0))$, so that $y_1$ is a periodic point with $\text{int}(C^+(y_1)) \cap S = S_0$. Since $B$ is open and $b_1, b_2 \in \partial B$, there exists a point $y_2 \in \text{int}(C_1) \cap B \cap (R^2 - \text{int}(C^+(y_1)))$. Then, $y_2$ is a periodic point with $C^+(y_2) \subset R^2 - \text{int}(C^+(y_1))$ and $\text{int}(C^+(y_2)) \cap S = S_0$. Since $\text{int}(C^+(y_1)) \cap \text{int}(C^+(y_2)) \neq \emptyset$, $\text{int}(C^+(y_1))$ is connected and $\partial \text{int}(C^+(y_2)) \cap \text{int}(C^+(y_1)) = \emptyset$, we must have $\text{int}(C^+(y_2)) \subset \text{int}(C^+(y_1))$. This implies that $\text{int}(C_1) \cap \text{int}(C^+(y_2)) \neq \emptyset$. It is also clear that $\text{int}(C_1) \cap (R^2 - \text{int}(C^+(y_2)) = \emptyset$ since $b_1 \in \partial B$ and $B$ is open. Therefore, $C^+(y_2) \cap \text{int}(C_1) \neq \emptyset$ since $\text{int}(C_1)$ is connected. Certainly, for each $x_1 \in C^+(y_2) \cap \text{int}(C_1)$, there exists $t_1 \in R^+$ such that $x_1 t_1 \in \text{int}(C_2)$ since $C^+(x_1) = C^+(y_2)$ and $y_2$ is a periodic point. This shows that $b_2 \in D^+(b_1)$. Similarly, $b_1 \in D^+(b_2)$. If $L^+(b_1) \neq \emptyset$, then it is a periodic orbit, by Lemma 3, since $\partial B \cap S = \emptyset$ and $L^+(b_1) \subset \partial B$. That $L^+(b_1) \subset \partial B$ follows from the fact that $\partial B$ is a closed invariant set, as $B$ is invariant. Further, $\partial B \subset L^+(b_2)$, since $b \in \partial B$ and $y \in L^+(b_2)$ implies $b \in D^+(y) = C^+(y) = L^+(b_1)$, as $L^+(b_1)$ is a periodic orbit contained in $\partial B$. Therefore $\partial B = L^+(b_1)$ is a periodic orbit. Similarly, if $L^+(b_2) \neq \emptyset$, then $\partial B$ is a periodic orbit. Suppose $L^+(b_1) = L^+(b_2) = \emptyset$. Then we must have $b_1 \in C^+(b_2)$ and $b_2 \in C^+(b_1)$, which again implies that $C^+(b_1)$ is a periodic orbit containing $b_1$ (see proof of Lemma 3). Thus, we conclude that $\partial B$ is a periodic orbit.

Let $N = \partial B \cup \text{int}(\partial B)$. We wish to show that $N = \bar{B}$. Since $S$ is closed, one can choose a simple closed curve $C$ such that $N \subset \text{int}(C)$ and $(\text{int}(C) - N) \cap S = \emptyset$. We note the $N$ is positively stable since $D^+(N) = N$. Thus, there exists a neighborhood $V$ of $N$ such that $C^+(V) \subset \text{int}(C)$. It follows that $(V - N) \subset B$. For, if $x \in (V - N) \cap B$, then $x$ is a periodic point, by Lemma 2, since $x$ is surrounded by some periodic orbit $B_x$. Therefore, we must have $\partial B \cap \text{int}(C^+(x))$, since $C^+(x) \subset \text{int}(C)$ and $(\text{int}(C) - N) \cap S = \emptyset$. But, it is impossible to have
$\partial B \subset \text{int} (C^+(x))$ since $\text{int} (C^+(x)) \subset B$. Thus, we have established that 

$$(V - N) \cap B = \emptyset,$$

and hence $\text{int}(\partial B) \cap B \neq \emptyset$, since $\partial B \cap B = \emptyset$, as $B$ is open. We note that $B$ is connected since it is the union of the family of connected sets $(\text{int}(B_a))_{a \in I}$ with $\emptyset \neq S_0 \subset \bigcap_{a \in I} \text{int}(B_a)$. Therefore, $B \subset \text{int}(\partial B)$ since $B \cap \partial(\text{int}(\partial B)) = B \cap \partial B = \emptyset$. Now, suppose $\text{int}(\partial B) \neq \emptyset$. Then, clearly, $\text{int}(\partial B) \cap B$ is a nonempty open set. Also, $\text{int}(\partial B) = B$ is a nonempty open set. For, $x \in \text{int}(\partial B) - B$ implies that $x \notin \partial B$ and $x \in B$; hence $x \in B$. Let $V$ be a neighborhood of $x$ such that $V \cap B = \emptyset$. Then $U = V \cap \text{int}(\partial B)$ is a neighborhood of $x$ and $U \subset \text{int}(\partial B) - B$. Hence, $\text{int}(\partial B)$ is disconnected; a contradiction to the Jordan Curve Theorem. We have thus shown that $N = \partial B \cup B$.

$N$ is a simply connected continuum, by Schoenflie’s Theorem. We wish to show that $N$ is globally $+$ asymptotically stable. In view of Proposition 3.2, it is sufficient to show that $N$ is a positive attractor. Since $N$ is compact and $S$ is closed, we can choose a compact neighborhood $U_0$ of $N$ such that $U_0 \cap (S - S_0) = \emptyset$. Then, there exists a neighborhood $V_0$ of $N$ such that $C^+(V_0) \subset U_0$. For each $x \in V_0 - N$, $L^+(x) \neq \emptyset$ and $L^+(x) \cap S = \emptyset$. Hence, $L^+(x)$ is a periodic orbit and $S_0 \subset \text{int}(L^+(x))$. Similarly, if $y \in \text{int}(L^+(y)) - N$, then $S_0 \subset \text{int}(L^+(y))$. It follows from the way $N$ was constructed that $L^+(x) = \partial N$.

We note that if $B = R^2$, then $S = S_0$. Also, if $B \neq R^2$, then $S = S_0$ since $N \cap (S - S_0) = \emptyset$ and $N$ is a globally $+$ asymptotically stable neighborhood of $S_0$. In particular, since $x_0$ was an arbitrary critical point, it follows that $S$ is contained in the interior of every periodic orbit. Now, we wish to show that $S$ is a singleton. This will complete the proof of the theorem, since $B = R^2$ will then imply the first and $B \neq R^2$ the second assertion of the theorem. Let $D = \bigcap_{a \in I} \text{int}(B_a)$. Then, we have $S \subset D$. Suppose that $D$ contains a regular point $d$. Then, $L^-(d) \neq \emptyset$ since $d$ is surrounded by periodic orbits, and hence $C^+(d)$ is a periodic orbit (see footnote 3). But this would imply that $d \in \text{int}(C^+(d))$, which is impossible. For, as we pointed out above, $S = S_0$ and $S_0$ is contained in the interior of every periodic orbit. Hence every periodic orbit belongs to the family $(B_a)_{a \in I}$ and, consequently, $D$ is contained in the interior of every periodic orbit. Therefore, $D = S$. Let $d_1 \in \partial D$, and suppose that $D$ contains a point $d_2$ distinct from $d_1$. Let $C_1$ be a simple closed curve such that $d_1 \in \text{int}(C_1)$ and $d_2 \in \text{int}(C_1)$. Since $[d_1]$ is positively stable, there exists a neighborhood $W_1$ of $d_1$ with $C^+(W_1) \subset \text{int}(C_1)$. But, if $x$ is a regular point in $W_1 \cap B$, then we must have $D \subset \text{int}(C^+(x))$, and in particular, $d_2 \in \text{int}(C^+(x))$, which is impossible. This completes the proof of Theorem 4.2.

For flows of characteristic $0^+$, the following theorem is a rather strong generalization of Bendixson’s theorem (see [4]), which states that for every isolated critical point $s$ on the plane, either there exists
a point \( y \neq s \) such that \( L^+(y) = \{s\} \) or \( L^-(y) = \{s\} \), or every neighborhood of \( s \) contains a periodic orbit surrounding \( s \).

**Theorem 4.3.** If \( S \) has a compact component \( S_0 \) which is isolated from \( S - S_0 \), then one of the following holds.

1. \( S \) is a singleton and one of the two assertions of Theorem 4.2 holds.
2. \( S_0 \) is globally \( + \) asymptotically stable, and consequently, \( S_0 = S \).

**Proof.** Let \( V \) be a compact neighborhood of \( S_0 \) such that \( V \cap (S - S_0) = \emptyset \). Since \( D^+(S_0) = S_0, S_0 \) is positively stable. Let \( U \) be a neighborhood of \( S_0 \) such that \( C^+(U) \subset V \). Then, for each \( x \in U, L^+(x) \neq \emptyset \). If a periodic orbit exists, then the proof follows from Theorem 4.2. If there are no periodic orbits, then for each \( x \in U, L^+(x) \) consists of a single rest point, by Lemma 3. Further, \( L^+(x) \subset S_0 \) since \( L^+(x) \subset V \). Therefore, \( S_0 \) is globally \( + \) asymptotically stable, by Proposition 3.2, and hence \( S_0 = S \).

**Corollary.** If \( S \) contains a point \( s_0 \) which is isolated from \( S - \{s_0\} \), then \( S = \{s_0\} \).

**Theorem 4.4.** If \( S \) is compact, then either \( S \) is a singleton and one of the two assertions of Theorem 4.2 holds, or \( S \) is globally \( + \) asymptotically stable.

**Proof.** Let \( C \) be a simple closed curve such that \( S \subset \text{int}(C) \). Since \( S \) is positively stable, as \( D^+(S) = S \), there exists a neighborhood \( V \) of \( S \) such that \( C^+(V) \subset \text{int}(C) \). Therefore, for each \( x \in V, L^+(x) \neq \emptyset \). If a periodic orbit exists, then the proof follows from Theorem 4.2. If there are no periodic orbits, then \( L^+(x) \) consists of a single rest point, by Lemma 3. Hence, \( S \) is globally \( + \) asymptotically stable, by Proposition 3.2.

**Remark.** If \( S \) is \( + \) asymptotically stable, then for each \( s \in \partial S \), there is a regular point \( y \) with \( L^+(y) = \{s\} \). For, if \( x \) is a regular point, then it follows from Lemma 2 and Theorem 4.2 that \( C^-(x) \) is unbounded. Thus, if \( C \) is a simple closed curve surrounding \( s \), then one can choose sequences \( \{x_n\} \) and \( \{t_n\} \) in \( R^2 \) and \( R^- \), respectively, such that \( \{x_n\} \) converges to \( s \) and \( \{x_n t_n\} \) converges to some point \( x_0 \in C \). But this would imply that \( x_0 \in D^-(s) \) or \( s \in D^+(x_0) \), and hence \( L^+(x_0) = \{s\} \).

---

\(^4\) \( S_0 \) is isolated from \( S - S_0 \) if \( S_0 \) has a neighborhood disjoint from \( S - S_0 \).
**Lemma 4.** If $S$ is $+$ asymptotically stable, then $A^{-}(S)$ is an open set.

**Proof.** We note that $\partial A^{+}(S)$ is a closed invariant set, since $A^{+}(S)$ is invariant. Thus, for each $x \in \partial A^{+}(S)$, $L^{+}(x) \subset \partial A^{+}(S)$. But, $\partial A^{+}(S) \cap S = \emptyset$ since $S$ is $+$ asymptotically stable. Therefore, $\partial A^{+}(S) \cap A^{+}(S) = \emptyset$, and hence $A^{+}(S)$ is open.

**Theorem 4.5.** If $S$ is unbounded, then the following hold.

1. Either $S = \mathbb{R}^2$, or $\mathbb{R}^2 - S$ is unbounded.
2. If $S \neq \mathbb{R}^2$, then $S$ is $+$ asymptotically stable. Further, if $S$ is disconnected, then it is not globally $+$ asymptotically stable.
3. $x \in A^{+}(S)$ implies that $L^{+}(x) = \emptyset$.

**Proof.** The first assertion follows from the fact that there are no periodic orbits, and consequently, if $x$ is a regular point, then $C^{-}(x)$ is unbounded. To prove (2), let $s \in \partial S$ and let $C$ be a simple closed curve such that $s \in \text{int}(C)$. Since $\{s\}$ is positively stable, there exists a neighborhood $U$ of $s$ such that $C^{+}(U) \subset \text{int}(C)$. Therefore, for each $x \in U$, $L^{+}(x) \neq \emptyset$, and hence $L^{+}(x) \subset S$ since there are no periodic orbits. The last assertion of (2) follows from Proposition 3.1. Statement (3) follows from Lemma 4 and the fact that $\partial A^{+}(S)$ is positively invariant and there are no periodic orbits.

**Theorem 4.6.** If $S \neq \mathbb{R}^2$ and $S$ is unbounded, then $A^{+}(S)$ has a countable number of components. The boundary of each component is constituted by a countable number of orbits $C(x)$ such that $L^{+}(x) = \emptyset$.

**Proof.** Since by Lemma 4, $A^{+}(S)$ is open, the first statement follows immediately from the fact that the components of $A^{+}(S)$ form a collection of mutually disjoint open subsets of $\mathbb{R}^2$. To prove the second assertion, let $K$ be any component of $A^{+}(S)$. We note that $\partial K$ is invariant and is thus constituted by whole trajectories. For each $x \in \partial K$, $L^{+}(x) = \emptyset$, since $x$ cannot belong to any component of $A^{+}(S)$ and there are no periodic orbits. Thus, $C_x = C(x) \cup \{\omega\}$ constitutes a simple closed curve in $(\mathbb{R}^2)^{+}$ and $K$ is contained in one of the components of $(\mathbb{R}^2)^{+} - C_x$. Let $K_x$ denote the component of $(\mathbb{R}^2)^{+} - C_x$ which is disjoint from $K$, i.e., $K_x \cap K = \emptyset$. Then we must have $K_x \cap \partial K = \emptyset$. If $y \in \partial K - C_x$, then $K_x \cap C_y = \emptyset$. For, suppose $K_x \cap C_y \neq \emptyset$. Then, $K_x \cap \partial K_y = K_x \cap C_y = \emptyset$ since $y \in \partial K$, $\partial K \cap K_x = \emptyset$ and $\partial K$ is invariant. Hence, $K_x \subset K_y$. Similarly, $K_y \subset K_x$ and thus $K_x = K_y$. Now, $y \in C_x$ and $y \in K_x$ since $K_x \cap \partial K = \emptyset$. Therefore, the component $(\mathbb{R}^2)^{+} - (K_x \cup C_x)$
must be a neighborhood of \( y \). But this is a contradiction to \( y \in \partial K_y \) since \((\mathbb{R}^2)^* - (K_x \cup C_\epsilon)\) contains no point of \( K_x = K_y \). This shows that \( K_x \cap K_y = \emptyset \). The second assertion of Theorem 4.6 now follows from the fact that \((\mathbb{R}^2)^*\) is a Lindelöf of space, and hence the collection \((K_x)_{x \in \partial K}\) is countable.

**Theorem 4.7.** If \( S \neq \mathbb{R}^2 \) and \( S \) is unbounded, then every component of \( A^+(S) \) is homeomorphic to \( \mathbb{R}^2 \).

**Proof.** Let \( K_0 \) be any component of \( A^+(S) \). Since \( K_0 \) is an open subset of \( \mathbb{R}^2 \), it is sufficient to show that \( K_0 \) is simply connected. Let \( C_0 \) be any simple closed curve such that \( C_0 \subset K_0 \). If \( x \) is a regular point in \( \text{int}(C_0) \), then \( L^-(x) = \emptyset \) since there are no periodic orbits. Therefore, \( C^-(x) \cap C_0 \neq \emptyset \). But \( x_0 \in C^-(x) \cap C_0 \) implies that \( x_0 \in A^+(S) \), and hence \( x \in A^+(S) \) since \( x \in C^+(x_0) \). This shows that \( \text{int}(C_0) \subset A^+(S) \), since \( S \subset A^+(S) \). Since \( \text{int}(C_0) \) is connected, \( \text{int}(C_0) \subset K_0 \), i.e., \( C_0 \) is retractible.

**Theorem 4.8.** If \( S \neq \mathbb{R}^2 \) and \( S \) is unbounded, then \( S \) has a countable number of components, each being simply connected. Further, the set of critical points in each component of \( A^+(S) \) form a component of \( S \).

**Proof.** We note that \( S \subset A^+(S) \), and by Theorem 4.6, \( A^+(S) \) is partitioned into a countable number of components. Therefore, in order to prove the first assertion, it is sufficient to show that if \( K_0 \) is any component of \( A^+(S) \) and \( S_0 = K_0 \cap S \), then \( S_0 \) is a component of \( S \). To show that \( S_0 \) is a component of \( S \), it is sufficient to show that \( S_0 \) is connected. For, it follows from the proof of Theorem 4.6 that \( \partial K_0 \cap S = \emptyset \), and consequently, the component of \( S \) containing \( S_0 \) is contained in \( K_x \). However, we note that \( S_0 \) is asymptotically stable, globally, in \( K_x \). Therefore, the fact that \( S_0 \) is connected follows from Proposition 3.1.

To prove that components of \( S \) are simply connected, let \( S_i \) be any component of \( S \) and let \( C_i \) be any simple closed curve such that \( C_i \subset S_i \). Suppose \( \text{int}(C_i) \) contains a regular point \( x \). Then \( L^-(x) \neq \emptyset \) since \( x \) is surrounded by the simple closed curve \( C_i \) consisting of rest points. But this implies that \( x \) is a periodic point (see footnote on page 10). Therefore, \( \text{int}(C_i) \) consists of rest points and is hence contained in \( S_i \), since \( S_i \) is a maximal connected subset of \( S \). This completes the proof.

It follows from Theorem 4.6 and the proof of Theorem 4.7 that
each component of $S$ is isolated from other points of $S$. Thus, using Theorem 4.3, we have the following sharpening of Theorem 4.3.

**Theorem 4.9.** If $S$ has a compact component, then one of the two possibilities stated in Theorem 4.3 holds.

We now summarize the results of this section.

**Case 1.** $S = \emptyset$ and $(\mathbb{R}^2, \pi)$ is parallelizable.

**Case 2.** $S$ is compact implies one of the following.
(a) $S = \{s_0\}$ is a singleton and $s_0$ is a global Poincaré center.
(b) $S = \{s_0\}$ is a singleton and $s_0$ is a local Poincaré center. Further, the set $N$ consisting of $s_0$ and periodic orbits surrounding $s_0$, is a globally + asymptotically stable simply connected continuum.
(c) $S$ is a globally + asymptotically simply connected continuum.

**Case 3.** If $S$ is unbounded, then either (A) $S = \mathbb{R}^2$ or (B) the following hold.
(a) $\mathbb{R}^2 - S$ is unbounded.
(b) $S$ is + asymptotically stable.
(c) $A^+(S)$ has a countable number of components each being homeomorphic to $\mathbb{R}^2$ and unbounded.
(d) $S$ has a countable number of components, each being non-compact and simply connected. For each $s \in \partial S$, there is a regular point $y$ with $L^+(y) = \{s\}$.
(e) $A^+(S)$ is a component of $A^+(S)$ if and only if $S_0$ is a component of $S$.
(f) For each $x \in \mathbb{R}^2$, $L^+(x)$ is either empty or consists of a single rest point. Further, $L^+(x) = \emptyset$ for all $x \in A^+(S)$ and $L^-(x) = \emptyset$ for all $x \in \mathbb{R}^2 - S$.

The above theorems indicate that imposing characteristic $0^+$ on a dynamical system on $\mathbb{R}^2$ is a fairly strong restriction. However, for more general phase spaces the situation is different. By way of illustration, we give the following example.

**Example 1.** Consider the subspace of $\mathbb{R}^3$ consisting of the $xy$-plane and the negative $z$-axis. Consider the flow in which the origin $0$ is a rest point, points on the $xy$-plane are periodic whose trajectories surround $0$ and points on the negative $z$-axis tend positively to $0$, i.e., $L^+(x) = 0$ for all $x$ on the negative $z$-axis.

We have clearly defined a flow of characteristic $0^+$ which has only
5. Flow of characteristic $0^\pm$ on the plane.

**DEFINITION 5.1.** A flow $(\mathbb{R}^2, \pi)$ on the plane is of characteristic $0^\pm$ if for each $x \in \mathbb{R}^2$, $D^+(x) = \overline{C}^+(x)$ and $D^-(x) = \overline{C}^-(x)$.

The above definition is equivalent to saying that a flow is of characteristic $0^\pm$ if and only if every closed invariant subset $M$ of $\mathbb{R}^2$ is positively and negatively $D$-stable (i.e., $D^+(M) = D^-(M) = M$). The following theorem completely classifies such flows. The proof of this theorem follows immediately from the previous section and is hence omitted.

**THEOREM 5.1.** Let $(\mathbb{R}^2, \pi)$ be a dynamical system of characteristic $0^\pm$ on the plane. Then one of the following holds.

1. $S = \emptyset$ and the flow is parallelizable.
2. $S = \mathbb{R}^2$.
3. $S = \{s_0\}$ is a singleton and $s_0$ is a global Poincaré center.

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