

# Pacific Journal of Mathematics

**DYNAMICAL SYSTEMS OF CHARACTERISTIC  $0^+$**

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## DYNAMICAL SYSTEMS OF CHARACTERISTIC $0^+$

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The main purpose of this paper is to classify the dynamical systems on the plane which satisfy a certain type of stability criterion. Such flows are referred to as dynamical systems of characteristic  $0^+$ . The classification is based on the consideration of three mutually exclusive and exhaustive cases: Dynamical systems of characteristic  $0^+$  which have no critical points. Those whose critical points form nonempty compact sets, and those whose critical points do not form compact sets.

Dynamical systems of characteristic  $0^+$  are those dynamical systems in which all closed positively invariant sets are positively  $D$ -stable, i.e., stable in Ura's sense (see [11]). If the phase space of a flow is regular, then a closed positively invariant set, which is positively stable in Liapunov's sense, is also positively  $D$ -stable. Thus, some simple examples of flows of characteristic  $0^+$  are those where the phase spaces are regular and all closed invariant sets are positively stable in Liapunov's sense.

In § 2 we give some of the basic definitions and notations that are used throughout the paper. In § 3 we prove some results of a more general nature which are later applied to flows of characteristic  $0^+$  on the plane. It is proved that if the phase space  $X$  of a flow is normal and connected and a closed invariant set  $F$  is globally + asymptotically stable, then  $F$  is connected. Further, if the phase space  $X$  of a flow of characteristic  $0^+$  is connected and locally compact, then a compact subset  $M$  of  $X$  is a positive attractor implies that  $M$  is globally + asymptotically stable.

In § 4 we discuss flows of characteristic  $0^+$  on the plane. It is shown that if the set of critical points  $S$  of such a flow is empty, then the flow is parallelizable. If  $S$  is compact, then it either consists of a single point which is a Poincaré center, or it is globally + asymptotically stable. If  $S$  is not compact, then either  $R^2 = S$ , or  $S$  is + asymptotically stable;  $S$  and the region of positive attraction  $A^+(S)$  of  $S$  each has a countable number of components. Further, each component of  $A^+(S)$  is homeomorphic to  $R^2$ . At the end of this section, we summarize all the results of this section in the form of a complete classification of such flows.

In § 5 we discuss flows of characteristic  $0^\pm$  on the plane, i.e., those in which every closed invariant set is positively and negatively stable in Ura's sense. We prove that such a flow is either parallelizable, or it has a single critical point which is a global Poincaré center, or all

points are critical points.

2. **Notations and definitions.** Let  $R, R^+$ , and  $R^-$  denote the sets of real numbers, nonnegative, and nonpositive real numbers, respectively. Given a topological space  $X$  and a mapping  $\pi$  of the product space  $X \times R$  into  $X$ , we say  $(X, \pi)$  defines a *dynamical system* or *flow* on the phase space  $X$  if the following conditions are satisfied.

1. *Identity axiom:*  $\pi(x, 0) = x$ .
2. *Homomorphism axiom:*  $\pi(\pi(x, t), s) = \pi(x, s + t)$ .
3. *Continuity axiom:*  $\pi$  is continuous on  $X \times R$ .

For brevity, we denote  $\pi(x, t)$  by  $xt$ . For each  $x \in X$ , we let  $C(x)$  denote the *trajectory* or *orbit* through  $x$ , i.e.,  $C(x) = xR$ . Similarly, the *positive* and *negative semi-trajectories* through  $x$  are represented by  $C^+(x)$  and  $C^-(x)$ , respectively, i.e.,  $C^+(x) = xR^+$  and  $C^-(x) = xR^-$ . We let  $L^+(x)$  denote the *positive* (or  $\omega$ -) *limit set* of  $x$ , i.e.,  $L^+(x) = \bigcap \{ \overline{C^+(xt)} : t \in R \}$ . Similarly,  $L^-(x)$  denotes the *negative* (or  $\alpha$ -) *limit set* of  $x$ . A point  $x$  is called a *critical* or *rest point* if  $xR = x$ . A subset  $M$  of  $X$  is said to be *invariant* if  $C(M) = M$ , and *positively* (negatively) *invariant* if  $C^+(M) = M$  ( $C^-(M) = M$ ). A closed invariant set  $M$  is *minimal* if it has no proper subset which is closed and invariant.

Throughout this paper, we use  $\partial M$  and  $\bar{M}$  to represent the boundary and closure of  $M$ . Given a Jordan curve  $C$  on the plane  $R^2$ , we let  $\text{int}(C)$  denote the bounded component of  $R^2 - C$ . Let  $(R^2)^* = R^2 \cup \{\omega\}$  be the one point compactification of the plane.

A closed positively invariant set  $M$  is said to be *positively Liapunov stable*, or more simply, *positively stable*, if for every neighborhood  $U$  of  $M$ , there exists a neighborhood  $V$  of  $M$  such that  $C^+(V) \subset U$ .  $M$  is said to be a *positive attractor* if there exists a neighborhood  $U$  of  $M$  such that  $\varphi \neq L(x) \subset M$  for all  $x$  in  $U$ . The largest such neighborhood  $U$  is called the *region of positive attraction* of  $M$  and will be denoted by  $A^+(M)$ .  $M$  is said to be *asymptotically stable* if it is both positively stable and a positive attractor. It is said to be globally *asymptotically stable* if it is *asymptotically stable* and  $A^+(M) = X$ .

For each  $x \in X$ , the (first) *positive* (*negative*) *prolongation*  $D^+(x)$  ( $D^-(x)$ ) of  $x$  is given by

$$D^+(x) = \bigcap_{N \in \eta(x)} \{ \overline{C^+(N)} \} \quad (D^-(x) = \bigcap_{N \in \eta(x)} \{ \overline{C^-(N)} \} ),$$

where  $\eta(x)$  is the neighborhood filter of  $x$ .

The (first) *positive* (*negative*) *prolongational limit set* of  $x$  is given by

$$J^+(x) = \bigcap_{t \in \mathbb{R}} \{D^+(xt)\} \quad (J^-(x) = \bigcap_{t \in \mathbb{R}} \{D^-(xt)\}) .$$

It is known and easy to verify that  $L^+(x) \subset J^+(x)$ . Further, if  $X$  is a Hausdorff space, then  $D^+(x) = C^+(x) \cup J^+(x)$ .

A closed positively invariant set  $M$  is said to be *positively  $D$ -stable* if  $D^+(M) = M$ .<sup>1</sup>

It is easy to verify that if  $X$  is regular and a closed positively invariant set  $M$  is positively stable (i.e., stable in Liapunov's sense as defined above), it is also positively  $D$ -stable. The converse is false.

The following theorem, which we use several times in this paper, is due to Ura [11].

**THEOREM (Ura).** *Let  $(X, \pi)$  be a dynamical system on a locally compact space  $X$ , and let  $M$  be a compact subset of  $X$ . Then  $M$  is positively stable if and only if it is positively  $D$ -stable.*

**REMARK.** The statement “ $X$  is locally compact” is used in the Bourbaki sense throughout this paper, i.e.,  $X$  is assumed to be a Hausdorff space.

**3. Flows of characteristic  $0^+$ .** Before discussing flows of characteristic  $0^+$ , we prove a lemma and a proposition concerning flows in general.

**LEMMA 1.** *Let  $(X, \pi)$  be any dynamical system. If  $x \in X$  and  $y_1, y_2 \in L^+(x)$ , then  $y_1 \in D^+(y_2)$  and  $y_2 \in D^+(y_1)$ .*

*Proof.* We note that

$$D^+(y_1) = \bigcap_{N \in \eta(y_1)} \{\overline{C^+(N)}\} ,$$

where  $\eta(y_1)$  denotes the neighborhood filter of  $y_1$ . Since  $y_1, y_2 \in L^+(x)$ , for each  $N \in \eta(y_1)$  and  $M \in \eta(y_2)$ , there exist  $t_1, t_2 \in \mathbb{R}^+$  with  $xt_1 \in N$  and  $(xt_1)t_2 = x(t_1 + t_2) \in M$ . Hence  $y_2 \in \overline{C^+(N)}$ , and consequently,  $y_2 \in D^+(y_1)$ . Similarly,  $y_1 \in D^+(y_2)$ .

**PROPOSITION 3.1.** *Let  $(X, \pi)$  be a dynamical system on a normal (and Hausdorff) connected topological space  $X$ . If a closed invariant subset  $F$  of  $X$  is globally + asymptotically stable, then  $F$  is connected.*

*Proof.* Suppose  $F$  is not connected. Then there exist two non-

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<sup>1</sup> The theory of prolongation and  $D$ -stability is due to Ura (see [11], [12], and [13]). Ura [11] refers to  $D$ -stability as *stability* and to Liapunov stability as *L-stability*.

empty disjoint closed sets  $F_1$  and  $F_2$  such that  $F = F_1 \cup F_2$ . Since  $X$  is normal, there exist two disjoint open neighborhoods  $U_1$  and  $U_2$  of  $F_1$  and  $F_2$ , respectively. On the other hand, since  $F$  is positively stable, corresponding to the neighborhood  $U = U_1 \cup U_2$  of  $F$ , there is an open neighborhood  $V$  of  $F$  such that  $C^+(V) \subset U$ . Therefore, if we let  $V_i = V \cap U_i$ ,  $i = 1, 2$ , then for each  $x \in V_i$ ,  $C^+(x) \subset U_i$  since  $C^+(x)$  is connected. Thus,  $L^+(x) \subset F_i$  i.e.,  $V_i \subset A^+(F_i)$  since  $\overline{U_i} \cap F_j = \emptyset$ ,  $i \neq j$ . Hence, we have shown that  $F_1$  and  $F_2$  are positive attractors; consequently  $A^+(F_1)$  and  $A^+(F_2)$  are open, since the boundary of each is closed and invariant. But this contradicts the assumption that  $X$  is connected, since  $X = A^+(F) = A^+(F_1) \cup A^+(F_2)$ , where  $A^+(F_1)$  and  $A^+(F_2)$  are clearly nonempty disjoint open sets. This completes the proof of Proposition 3.1.

**DEFINITION 3.1.** A dynamical system  $(X, \pi)$  is said to have characteristic  $0^+$  if and only if  $D^+(x) = \overline{C^+(x)}$  for all  $x \in X$ .

The above definition is equivalent to saying that  $(X, \pi)$  has characteristic  $0^+$  if and only if every closed positively invariant subset of  $X$  is positively  $D$ -stable.

It follows that if the phase space  $X$  of a flow of characteristic  $0^+$  is a Hausdorff space, then  $D^+(x) = C^+(x) \cup L^+(x)$ , for all  $x \in X$ .

**LEMMA 2.** Let  $(X, \pi)$  be a flow of characteristic  $0^+$ . If  $x \in X$  such that  $L^-(x) \neq \emptyset$ , then  $x \in L^-(x)$ .

*Proof.* Suppose  $L^-(x) \neq \emptyset$  and let  $y \in L^-(x)$ . Then,  $y \in D^-(x)$ , and hence  $x \in D^+(y) = \overline{C^+(y)}$ . On the other hand,  $y \in L^-(x)$  implies that  $\overline{C^+(y)} \subset L^-(x)$ , since  $L^-(x)$  is a closed invariant set. Therefore,  $x \in L^-(x)$ .

**PROPOSITION 3.2.** Let  $(X, \pi)$  be a flow of characteristic  $0^+$  on a connected locally compact space  $X$ . If  $M$  is a compact positively invariant subset of  $X$  and  $M$  is a positive attractor, then  $M$  is globally + asymptotically stable.

*Proof.* Since  $M$  is a closed positively invariant set, we have  $D^+(M) = M$ . Therefore,  $M$  is positively stable by Ura's theorem. It is sufficient to show that  $\partial A^+(M) = \emptyset$ . Suppose that  $\partial A^+(M) \neq \emptyset$ , and let  $x \in \partial A^+(M)$ . Let  $\eta_A(x)$  be the trace of the neighborhood filter  $\eta(x)$  of  $x$  on  $A \equiv A^+(M)$ . Then, for each  $N_A \in \eta_A(x)$ ,  $\emptyset \neq L^+(N_A) \subset M$ . Since  $M$  is compact, the cluster set of the filter base  $\{L^+(N_A) \mid N_A \in \eta_A(x)\}$  is a nonempty subset of  $M$ ; hence  $J^+(x) \cap M \neq \emptyset$ . However, this

contradicts the assumption that  $(X, \pi)$  has characteristic  $0^+$ , since  $\partial A^+(M)$  is a closed invariant set disjoint with  $M$ . Therefore,  $\partial A^+(M) = \emptyset$  and the proof of Proposition 3.2 is complete.

4. **Flows of characteristic  $0^+$  on the plane.** Throughout this section, we assume the phase space to be the plane  $R^2$  and  $(R^2, \pi)$  to be a fixed flow of characteristic  $0^+$ . We let  $S$  denote the set of rest points of this flow.

LEMMA 3. *For each  $x \in X$ , if  $L^+(x) \neq \emptyset$ , then  $L^+(x)$  is either a periodic orbit or it consists of a single rest point.*

*Proof.* If  $L^+(x)$  contains a rest point  $s_0$ , then  $L^+(x) = \{s_0\}$ . For,  $y \in L^+(x)$  implies that  $y \in D^+(s_0) = \{s_0\}$ , by Lemma 1. Suppose that  $L^+(x)$  consists of regular points only. Then, to complete the proof of the lemma, it is sufficient to prove that  $L^+(x)$  is compact. We note that if  $y \in L^+(x)$ , then  $\overline{C^+(y)} = L^+(x)$ . For,  $z \in L^+(x)$  implies that  $z \in D^+(y) = \overline{C^+(y)}$ . Also,  $\overline{C^+(y)} \subset L^+(x)$  since  $L^+(x)$  is a closed invariant set, and hence  $\overline{C^+(y)} = L^+(x)$ . Since  $\overline{C^+(y)} \subset \overline{C(y)} \subset L^+(x)$ , we have  $\overline{C(y)} = L^+(x)$ . Therefore,  $L^+(x)$  is a minimal set. We recall that if  $M$  is a minimal subset of  $R^2$  which is not compact, then for each  $m \in M$ ,  $L^+(m) = \emptyset$  (c.f. p. 37 of [6]). Suppose that  $L^+(x)$  is not compact, and let  $y_1$  and  $y_2$  be two distinct points in  $L^+(x)$ . Then,  $y_1 \in D^+(y_2) = C^+(y_2)$  and  $y_2 \in D^+(y_1) = C^+(y_1)$ . But, if  $t_1$  and  $t_2$  are positive numbers such that  $y_1 = y_2 t_1$  and  $y_2 = y_1 t_2$ , then  $y_1 = y_1(t_1 + t_2)$ ; showing that  $C^+(y_1)$  is a periodic orbit. Hence,  $L^+(x)$  is a periodic orbit, since  $L^+(x) = C^+(y_1)$ , as it is a minimal set; thus contradicting the assumption that  $L^+(x)$  is not compact.

For a proof of the following theorem see [5].

THEOREM (Bhatia). *A flow  $F$  on a metric space  $X$  is dispersive if and only if for each  $x \in X$ ,  $D^+(x) = C^+(x)$  and there are no rest points or periodic orbits.*

THEOREM 4.1. *If  $S = \emptyset$ , then the flow  $(R^2, \pi)$  is parallelizable.*

*Proof.* We note that for each  $x \in R^2$ ,  $L^+(x) = \emptyset$ , and hence  $D^+(x) = \overline{C^+(x)} = C^+(x)$ . For, if  $L^+(x) \neq \emptyset$ , then by Lemma 3, it must be a periodic orbit since it consists of regular points only. But this is impossible since the bounded component of a periodic orbit contains a rest point. Thus, the proof of our assertion follows from Bhatia's Theorem, stated above (c.f. Auslander [2]) and the fact that the notions

of parallelizability and dispersiveness are equivalent for a flow on the plane (see Antosiewicz and Dugundji [1]).

**THEOREM 4.2.** *If  $R^2$  contains a periodic point, then  $S$  is a singleton. Further, if  $S = \{s_0\}$ , then one of the following holds.*

1.  $s_0$  is a global Poincaré center.<sup>2</sup>

2.  $s_0$  is a local Poincaré center. The neighborhood  $N$  of  $s_0$ , consisting of  $s_0$  and periodic orbits surrounding  $s_0$ , is a globally + asymptotically stable simply connected continuum. Further, if  $x \in N$ , then  $L^+(x) = \partial N$ .

*Proof.* Let  $x_0$  be any periodic point, and let  $S_0 = \text{int}(C^+(x_0)) \cap S$ . We note that  $\text{int}(C^+(x_0)) \neq S_0$  since  $S$  is closed; and for each regular point  $x$  in  $\text{int}(C^+(x_0))$ ,  $C^+(x)$  is a periodic orbit, by virtue of Lemma 2.<sup>3</sup> Let  $(B_\alpha)_{\alpha \in I}$  be the family of all periodic orbits such that for each  $\alpha \in I$ ,  $\text{int}(B_\alpha) \cap S = S_0$ . Let  $B = \bigcup_{\alpha \in I} \text{int}(B_\alpha)$ . If  $\partial B = \emptyset$ , then  $B = R^2$ . Suppose that  $\partial B \neq \emptyset$ . Then  $\partial B$  is a closed invariant set since  $B$  is invariant. Further,  $\partial B \cap S = \emptyset$ . For, if  $b_0 \in \partial B \cap S$ , then one can choose a simple closed curve  $C$  such that  $\text{int}(C) \cap S_0 = \emptyset$ , since  $S_0 \subset \text{int}(C^+(x_0)) \subset B$  and  $S_0$  is closed. Clearly, there is no neighborhood  $W$  of  $b_0$  with  $C^+(W) \subset \text{int}(C)$ , since  $x \in W \cap B - S_0$  would imply that  $x$  is a periodic point, by Lemma 2, and  $\text{int}(C^+(x)) \cap S_0 \neq \emptyset$ . But this contradicts the fact that  $\{b_0\}$  is positively stable, as  $D^+(b_0) = \{b_0\}$ ; thus showing that  $\partial B \cap S = \emptyset$ . This also shows that  $\partial B$  is not a singleton since it is invariant and consists of regular points.

We note that if  $x \in B$  and  $x \notin S_0$ , then  $x$  is a periodic point, by Lemma 2, with  $C^+(x) \subset B$  and  $\text{int}(C^+(x)) \cap S_0 \neq \emptyset$ . For,  $x$  belongs to  $\text{int}(B_\alpha)$  for some  $\alpha \in I$ . Thus,  $x \in S$  since  $\text{int}(B_\alpha) \cap S = S_0$ . Further  $L^-(x) \neq \emptyset$  and  $C^+(x) \subset B$  since  $x$  is surrounded by the periodic orbit  $B_\alpha$ . Thus,  $x$  is a periodic point with  $\text{int}(C^+(x)) \cap S_0 \neq \emptyset$  since  $C^+(x) \subset \text{int}(B_\alpha)$  and  $\text{int}(B_\alpha) \cap S = S_0$ . Now we wish to show that  $\partial B$  is a periodic orbit. In order to accomplish this, we consider two cases.

*Case 1.* Suppose  $\partial B \cap C^+(x_0) \neq \emptyset$ . Then, since  $\partial B$  is invariant, we must have  $C^+(x_0) \subset \partial B$ . On the other hand,  $\partial B \subset C^+(x_0)$ . For, assume  $\partial B \not\subset C^+(x_0)$ , and let  $b \in \partial B - C^+(x_0)$ . Then,  $b \notin \text{int}(C^+(x_0))$  since  $\text{int}(C^+(x_0)) \subset B$ . Thus, one can choose a neighborhood  $U$  of  $b$  such that  $U \cap \overline{\text{int}(C^+(x_0))} = \emptyset$  since  $b \in \overline{\text{int}(C^+(x_0))}$ , as  $b \in C^+(x_0)$  and

<sup>2</sup>  $s_0$  is a global Poincaré center if for each  $x \neq s_0$ ,  $C(x)$  is a periodic orbit surrounding  $s_0$ . It is a local Poincaré center if it has a neighborhood  $M$  such that for each  $x \in M - \{s_0\}$ ,  $C(x)$  is a periodic orbit surrounding  $s_0$ .

<sup>3</sup> It is a known fact about flows on the plane that a point is positively (or negatively) Poisson stable if and only if it is either a rest point or a periodic point (see [10]).

$b \in \text{int}(C^+(x_0))$ . Let  $x \in U \cap B$ . Then,  $x \notin S_0$  since  $S_0 \subset \text{int}(C^+(x_0))$ . Thus  $C^+(x)$  is a periodic orbit. Since  $\overline{\text{int}(C^+(x_0))}$  is connected,  $\text{int}(C^+(x)) \cap \overline{\text{int}(C^+(x_0))} \neq \emptyset$ , as  $\text{int}(C^+(x)) \cap S_0 \neq \emptyset$  and

$$\partial \text{int}(C^+(x)) \cap \overline{\text{int}(C^+(x_0))} = C^+(x) \cap \overline{\text{int}(C^+(x_0))} = \emptyset,$$

it follows that  $\overline{\text{int}(C^+(x_0))} \subset \text{int}(C^+(x))$ . But,  $C^+(x_0) \subset \text{int}(C^+(x)) \subset B$  contradicts the assumption that  $\partial B \cap C^+(x_0) \neq \emptyset$ , as  $B$  is open; hence  $\partial B = C^+(x_0)$ .

*Case 2.* Suppose  $\partial B \cap C^+(x_0) = \emptyset$ , and let  $b_1, b_2 \in \partial B$ . First we show that  $b_2 \in D^+(b_1)$  and  $b_1 \in D^+(b_2)$ . In order to show that  $b_2 \in D^+(b_1)$ , it is sufficient to show that if  $C_1$  and  $C_2$  are any simple closed curves with  $b_1 \in \text{int}(C_1)$  and  $b_2 \in \text{int}(C_2)$ , then there exist  $x_1 \in \text{int}(C_1)$  and  $t_1 \in R^+$  such that  $x_1 t_1 \in \text{int}(C_2)$ . Let  $y_1 \in \text{int}(C_1) \cap B - \text{int}(C^+(x_0))$ , so that  $y_1$  is a periodic point with  $\text{int}(C^+(y_1)) \cap S = S_0$ . Since  $B$  is open and  $b_1, b_2 \in \partial B$ , there exists a point  $y_2 \in \text{int}(C_2) \cap B \cap (R^2 - \overline{\text{int}(C^+(y_1))})$ . Then,  $y_2$  is a periodic point with  $C^+(y_2) \subset R^2 - \overline{\text{int}(C^+(y_1))}$  and  $\text{int}(C^+(y_2)) \cap S_0 \neq \emptyset$ . Since  $\text{int}(C^+(y_2)) \cap \overline{\text{int}(C^+(y_1))} \neq \emptyset$ ,  $\overline{\text{int}(C^+(y_1))}$  is connected and  $\partial \text{int}(C^+(y_2)) \cap \overline{\text{int}(C^+(y_1))} = \emptyset$ , we must have  $\overline{\text{int}(C^+(y_1))} \subset \text{int}(C^+(y_2))$ . This implies that  $\text{int}(C_1) \cap \text{int}(C^+(y_2)) \neq \emptyset$ . It is also clear that  $\text{int}(C_1) \cap (R^2 - \overline{\text{int}(C^+(y_2))}) \neq \emptyset$  since  $b_1 \in \partial B$  and  $B$  is open. Therefore,  $C^+(y_2) \cap \text{int}(C_1) \neq \emptyset$  since  $\text{int}(C_1)$  is connected. Certainly, for each  $x_1 \in C^+(y_2) \cap \text{int}(C_1)$ , there exists  $t_1 \in R^+$  such that  $x_1 t_1 \in \text{int}(C_2)$  since  $C^+(x_1) = C^+(y_2)$  and  $y_2$  is a periodic point. This shows that  $b_2 \in D^+(b_1)$ . Similarly,  $b_1 \in D^+(b_2)$ . If  $L^+(b_1) \neq \emptyset$ , then it is a periodic orbit, by Lemma 3, since  $\partial B \cap S = \emptyset$  and  $L^+(b_1) \subset \partial B$ . That  $L^+(b_1) \subset \partial B$  follows from the fact that  $\partial B$  is a closed invariant set, as  $B$  is invariant. Further,  $\partial B \subset L^+(b_1)$ , since  $b \in \partial B$  and  $y \in L^+(b_1)$  implies  $b \in D^+(y) = \overline{C^+(y)} = L^+(b_1)$ , as  $L^+(b_1)$  is a periodic orbit contained in  $\partial B$ . Therefore  $\partial B = L^+(b_1)$  is a periodic orbit. Similarly, if  $L^+(b_2) \neq \emptyset$ , then  $\partial B$  is a periodic orbit. Suppose  $L^+(b_1) = L^+(b_2) = \emptyset$ . Then we must have  $b_1 \in C^+(b_2)$  and  $b_2 \in C^+(b_1)$ , which again implies that  $C^+(b_1)$  is a periodic orbit containing  $b_2$  (see proof of Lemma 3). Thus, we conclude that  $\partial B$  is a periodic orbit.

Let  $N = \partial B \cup \text{int}(\partial B)$ . We wish to show that  $N = \bar{B}$ . Since  $S$  is closed, one can choose a simple closed curve  $C$  such that  $N \subset \text{int}(C)$  and  $(\text{int}(C) - N) \cap S = \emptyset$ . We note the  $N$  is positively stable since  $D^+(N) = N$ . Thus, there exists a neighborhood  $V$  of  $N$  such that  $C^+(V) \subset \text{int}(C)$ . It follows that  $(V - N) \cap B = \emptyset$ . For, if  $x \in (V - N) \cap B$ , then  $x$  is a periodic point, by Lemma 2, since  $x$  is surrounded by some periodic orbit  $B_\alpha$ . Therefore, we must have  $\partial B \subset \text{int}(C^+(x))$ , since  $C^+(x) \subset \text{int}(C)$  and  $(\text{int}(C) - N) \cap S = \emptyset$ . But, it is impossible to have

$\partial B \subset \text{int}(C^+(x))$  since  $\text{int}(C^+(x)) \subset B$ . Thus, we have established that  $(V - N) \cap B = \emptyset$ , and hence  $\text{int}(\partial B) \cap B \neq \emptyset$ , since  $\partial B \cap B = \emptyset$ , as  $B$  is open. We note that  $B$  is connected since it is the union of the family of connected sets  $(\text{int}(B_\alpha))_{\alpha \in I}$  with  $\emptyset \neq S_0 \subset \bigcap_{\alpha \in I} \text{int}(B_\alpha)$ . Therefore,  $B \subset \text{int}(\partial B)$  since  $B \cap \partial(\text{int}(\partial B)) = B \cap \partial B = \emptyset$ . Now, suppose  $\text{int}(\partial B) \neq B$ . Then, clearly,  $\text{int}(\partial B) \cap B$  is a nonempty open set. Also,  $\text{int}(\partial B) - B$  is a nonempty open set. For,  $x \in \text{int}(\partial B) - B$  implies that  $x \in \partial B$  and  $x \notin B$ ; hence  $x \notin \bar{B}$ . Let  $V$  be a neighborhood of  $x$  such that  $V \cap \bar{B} = \emptyset$ . Then  $U = V \cap \text{int}(\partial B)$  is a neighborhood of  $x$  and  $U \subset \text{int}(\partial B) - B$ . Hence,  $\text{int}(\partial B)$  is disconnected; a contradiction to the Jordan Curve Theorem. We have thus shown that  $N = \partial B \cup B$ .

$N$  is a simply connected continuum, by Schoenflie's Theorem. We wish to show that  $N$  is globally + asymptotically stable. In view of Proposition 3.2, it is sufficient to show that  $N$  is a positive attractor. Since  $N$  is compact and  $S$  is closed, we can choose a compact neighborhood  $U_0$  of  $N$  such that  $U_0 \cap (S - S_0) = \emptyset$ . Then, there exists a neighborhood  $V_0$  of  $N$  such that  $C^+(V_0) \subset U_0$ . For each  $x \in V_0 - N$ ,  $L^+(x) \neq \emptyset$  and  $L^+(x) \cap S = \emptyset$ . Hence,  $L^+(x)$  is a periodic orbit and  $S_0 \subset \text{int}(L^+(x))$ . Similarly, if  $y \in \text{int}(L^+(x)) - N$ , then  $S_0 \subset \text{int}(L^+(y))$ . It follows from the way  $N$  was constructed that  $L^+(x) = \partial N$ .

We note that if  $B = R^2$ , then  $S = S_0$ . Also, if  $B \neq R^2$ , then  $S = S_0$  since  $N \cap (S - S_0) = \emptyset$  and  $N$  is a globally + asymptotically stable neighborhood of  $S_0$ . In particular, since  $x_0$  was an arbitrary periodic point, it follows that  $S$  is contained in the interior of every periodic orbit. Now, we wish to show that  $S$  is a singleton. This will complete the proof of the theorem, since  $B = R^2$  will then imply the first and  $B \neq R^2$  the second assertion of the theorem. Let  $D = \bigcap_{\alpha \in I} \text{int}(B_\alpha)$ . Then, we have  $S \subset D$ . Suppose that  $D$  contains a regular point  $d$ . Then,  $L^-(d) \neq \emptyset$  since  $d$  is surrounded by periodic orbits, and hence  $C^+(d)$  is a periodic orbit (see footnote 3). But this would imply that  $d \in \text{int}(C^+(d))$ , which is impossible. For, as we pointed out above,  $S = S_0$  and  $S_0$  is contained in the interior of every periodic orbit. Hence every periodic orbit belongs to the family  $(B_\alpha)_{\alpha \in I}$  and, consequently,  $D$  is contained in the interior of every periodic orbit. Therefore,  $D = S$ . Let  $d_1 \in \partial D$ , and suppose that  $D$  contains a point  $d_2$  distinct from  $d_1$ . Let  $C_1$  be a simple closed curve such that  $d_1 \in \text{int}(C_1)$  and  $d_2 \notin \text{int}(C_1)$ . Since  $\{d_1\}$  is positively stable, there exists a neighborhood  $W_1$  of  $d_1$  with  $C^+(W_1) \subset \text{int}(C_1)$ . But, if  $x$  is a regular point in  $W_1 \cap B$ , then we must have  $D \subset \text{int}(C^+(x))$ , and in particular,  $d_2 \in \text{int}(C^+(x))$ , which is impossible. This completes the proof of Theorem 4.2.

For flows of characteristic  $0^+$ , the following theorem is a rather strong generalization of Bendixson's theorem (see [4]), which states that for every isolated critical point  $s$  on the plane, either there exists

a point  $y \neq s$  such that  $L^+(y) = \{s\}$  or  $L^-(y) = \{s\}$ , or every neighborhood of  $s$  contains a periodic orbit surrounding  $s$ .

**THEOREM 4.3.** *If  $S$  has a compact component  $S_0$  which is isolated from  $S - S_0$ , then one of the following holds.<sup>4</sup>*

(1)  $S$  is a singleton and one of the two assertions of Theorem 4.2 holds.

(2)  $S_0$  is globally + asymptotically stable, and consequently,  $S_0 = S$ .

*Proof.* Let  $V$  be a compact neighborhood of  $S_0$  such that  $V \cap (S - S_0) = \emptyset$ . Since  $D^+(S_0) = S_0$ ,  $S_0$  is positively stable. Let  $U$  be a neighborhood of  $S_0$  such that  $C^+(U) \subset V$ . Then, for each  $x \in U$ ,  $L^+(x) \neq \emptyset$ . If a periodic orbit exists, then the proof follows from Theorem 4.2. If there are no periodic orbits, then for each  $x \in U$ ,  $L^+(x)$  consists of a single rest point, by Lemma 3. Further,  $L^+(x) \subset S_0$  since  $L^+(x) \subset V$ . Therefore,  $S_0$  is globally + asymptotically stable, by Proposition 3.2, and hence  $S_0 = S$ .

**COROLLARY.** *If  $S$  contains a point  $s_0$  which is isolated from  $S - \{s_0\}$ , then  $S = \{s_0\}$ .*

**THEOREM 4.4.** *If  $S$  is compact, then either  $S$  is a singleton and one of the two assertions of Theorem 4.2 holds, or  $S$  is globally + asymptotically stable.*

*Proof.* Let  $C$  be a simple closed curve such that  $S \subset \text{int}(C)$ . Since  $S$  is positively stable, as  $D^+(S) = S$ , there exists a neighborhood  $V$  of  $S$  such that  $C^+(V) \subset \text{int}(C)$ . Therefore, for each  $x \in V$ ,  $L^+(x) \neq \emptyset$ . If a periodic orbit exists, then the proof follows from Theorem 4.2. If there are no periodic orbits, then  $L^+(x)$  consists of a single rest point, by Lemma 3. Hence,  $S$  is globally + asymptotically stable, by Proposition 3.2.

**REMARK.** If  $S$  is + asymptotically stable, then for each  $s \in \partial S$ , there is a regular point  $y$  with  $L^+(y) = \{s\}$ . For, if  $x$  is a regular point, then it follows from Lemma 2 and Theorem 4.2 that  $C^-(x)$  is unbounded. Thus, if  $C$  is a simple closed curve surrounding  $s$ , then one can choose sequences  $\{x_n\}$  and  $\{t_n\}$  in  $R^2$  and  $R^-$ , respectively, such that  $\{x_n\}$  converges to  $s$  and  $\{x_n t_n\}$  converges to some point  $x_0 \in C$ . But this would imply that  $x_0 \in D^-(s)$  or  $s \in D^+(x_0)$ , and hence  $L^+(x_0) = \{s\}$ .

<sup>4</sup>  $S_0$  is isolated from  $S - S_0$  if  $S_0$  has a neighborhood disjoint from  $S - S_0$ .

LEMMA 4. *If  $S$  is + asymptotically stable, then  $A^+(S)$  is an open set.*

*Proof.* We note that  $\partial A^+(S)$  is a closed invariant set, since  $A^+(S)$  is invariant. Thus, for each  $x \in \partial A^+(S)$ ,  $L^+(x) \subset \partial A^+(S)$ . But,  $\partial A^+(S) \cap S = \emptyset$  since  $S$  is + asymptotically stable. Therefore,  $\partial A^+(S) \cap A^+(S) = \emptyset$ , and hence  $A^+(S)$  is open.

THEOREM 4.5. *If  $S$  is unbounded, then the following hold.*

(1) *Either  $S = R^2$ , or  $R^2 - S$  is unbounded.*

(2) *If  $S \neq R^2$ , then  $S$  is + asymptotically stable.*

*Further, if  $S$  is disconnected, then it is not globally + asymptotically stable.*

(3)  *$x \in A^+(S)$  implies that  $L^\pm(x) = \emptyset$ .*

*Proof.* The first assertion follows from the fact that there are no periodic orbits, and consequently, if  $x$  is a regular point, then  $C^-(x)$  is unbounded. To prove (2), let  $s \in \partial S$  and let  $C$  be a simple closed curve such that  $s \in \text{int}(C)$ . Since  $\{s\}$  is positively stable, there exists a neighborhood  $U$  of  $s$  such that  $C^+(U) \subset \text{int}(C)$ . Therefore, for each  $x \in U$ ,  $L^+(x) \neq \emptyset$ , and hence  $L^+(x) \subset S$  since there are no periodic orbits. The last assertion of (2) follows from Proposition 3.1. Statement (3) follows from Lemma 4 and the fact that  $\partial A^+(S)$  is positively invariant and there are no periodic orbits.

THEOREM 4.6. *If  $S \neq R^2$  and  $S$  is unbounded, then  $A^+(S)$  has a countable number of components. The boundary of each component is constituted by a countable number of orbits  $C(x)$  such that  $L^\pm(x) = \emptyset$ .*

*Proof.* Since by Lemma 4,  $A^+(S)$  is open, the first statement follows immediately from the fact that the components of  $A^+(S)$  form a collection of mutually disjoint open subsets of  $R^2$ . To prove the second assertion, let  $K$  be any component of  $A^+(S)$ . We note that  $\partial K$  is invariant and is thus constituted by whole trajectories. For each  $x \in \partial K$ ,  $L^\pm(x) = \emptyset$ , since  $x$  cannot belong to any component of  $A^+(S)$  and there are no periodic orbits. Thus,  $C_x = C(x) \cup \{\omega\}$  constitutes a simple closed curve in  $(R^2)^*$  and  $K$  is contained in one of the components of  $(R^2)^* - C_x$ . Let  $K_x$  denote the component of  $(R^2)^* - C_x$  which is disjoint from  $K$ , i.e.,  $K_x \cap K = \emptyset$ . Then we must have  $K_x \cap \partial K = \emptyset$ . If  $y \in \partial K - C_x$ , then  $K_x \cap K_y = \emptyset$ . For, suppose  $K_x \cap K_y \neq \emptyset$ . Then,  $K_x \cap \partial K_y = K_x \cap C_y = \emptyset$  since  $y \in \partial K$ ,  $\partial K \cap K_x = \emptyset$  and  $\partial K$  is invariant. Hence,  $K_x \subset K_y$ . Similarly,  $K_y \subset K_x$  and thus  $K_x = K_y$ . Now,  $y \notin C_x$  and  $y \in K_x$  since  $K_x \cap \partial K = \emptyset$ . Therefore, the component  $(R^2)^* - (K_x \cup C_x)$

must be a neighborhood of  $y$ . But this is a contradiction to  $y \in \partial K_y$  since  $(R^2)^* - (K_x \cup C_x)$  contains no point of  $K_x = K_y$ . This shows that  $K_x \cap K_y = \emptyset$ . The second assertion of Theorem 4.6 now follows from the fact that  $(R^2)^*$  is a Lindelöf of space, and hence the collection  $(K_x)_{C(x) \subset \partial K}$  is countable.

**THEOREM 4.7.** *If  $S \neq R^2$  and  $S$  is unbounded, then every component of  $A^+(S)$  is homeomorphic to  $R^2$ .*

*Proof.* Let  $K_0$  be any component of  $A^+(S)$ . Since  $K_0$  is an open subset of  $R^2$ , it is sufficient to show that  $K_0$  is simply connected. Let  $C_0$  be any simple closed curve such that  $C_0 \subset K_0$ . If  $x$  is a regular point in  $\text{int}(C_0)$ , then  $L^-(x) = \emptyset$  since there are no periodic orbits. Therefore,  $C^-(x) \cap C_0 \neq \emptyset$ . But  $x_0 \in C^-(x) \cap C_0$  implies that  $x_0 \in A^+(S)$ , and hence  $x \in A^+(S)$  since  $x \in C^+(x_0)$ . This shows that  $\text{int}(C_0) \subset A^+(S)$ , since  $S \subset A^+(S)$ . Since  $\text{int}(C_0)$  is connected,  $\text{int}(C_0) \subset K_0$ , i.e.,  $C_0$  is retractible.

**THEOREM 4.8.** *If  $S \neq R^2$  and  $S$  is unbounded, then  $S$  has a countable number of components, each being simply connected. Further, the set of critical points in each component of  $A^+(S)$  form a component of  $S$ .*

*Proof.* We note that  $S \subset A^+(S)$ , and by Theorem 4.6,  $A^+(S)$  is partitioned into a countable number of components. Therefore, in order to prove the first assertion, it is sufficient to show that if  $K_0$  is any component of  $A^+(S)$  and  $S_0 = K_0 \cap S$ , then  $S_0$  is a component of  $S$ . To show that  $S_0$  is a component of  $S$ , it is sufficient to show that  $S_0$  is connected. For, it follows from the proof of Theorem 4.6 that  $\partial K_0 \cap S = \emptyset$ , and consequently, the component of  $S$  containing  $S_0$  is contained in  $K_0$ . However, we note that  $S_0$  is +asymptotically stable, globally, in  $K_0$ . Therefore, the fact that  $S_0$  is connected follows from Proposition 3.1.

To prove that components of  $S$  are simply connected, let  $S_1$  be any component of  $S$  and let  $C_1$  be any simple closed curve such that  $C_1 \subset S_1$ . Suppose  $\text{int}(C_1)$  contains a regular point  $x$ . Then  $L^-(x) \neq \emptyset$  since  $x$  is surrounded by the simple closed curve  $C_1$  consisting of rest points. But this implies that  $x$  is a periodic point (see footnote on page 10). Therefore,  $\text{int}(C_1)$  consists of rest points and is hence contained in  $S_1$ , since  $S_1$  is a maximal connected subset of  $S$ . This completes the proof.

It follows from Theorem 4.6 and the proof of Theorem 4.7 that

each component of  $S$  is isolated from other points of  $S$ . Thus, using Theorem 4.3, we have the following sharpening of Theorem 4.3.

**THEOREM 4.9.** *If  $S$  has a compact component, then one of the two possibilities stated in Theorem 4.3 holds.*

We now summarize the results of this section.

*Case 1.*  $S = \emptyset$  and  $(R^2, \pi)$  is parallelizable.

*Case 2.*  $S$  is compact implies one of the following.

(a)  $S = \{s_0\}$  is a singleton and  $s_0$  is a global Poincaré center.

(b)  $S = \{s_0\}$  is a singleton and  $s_0$  is a local Poincaré center.

Further, the set  $N$  consisting of  $s_0$  and periodic orbits surrounding  $s_0$ , is a globally + asymptotically stable simply connected continuum.

(c)  $S$  is a globally + asymptotically simply connected continuum.

*Case 3.* If  $S$  is unbounded, then either (A)  $S = R^2$  or (B) the following hold.

(a)  $R^2 - S$  is unbounded.

(b)  $S$  is + asymptotically stable.

(c)  $A^+(S)$  has a countable number of components each being homeomorphic to  $R^2$  and unbounded.

(d)  $S$  has a countable number of components, each being non-compact and simply connected. For each  $s \in \partial S$ , there is a regular point  $y$  with  $L^+(y) = \{s\}$ .

(e)  $A^+(S_0)$  is a component of  $A^+(S)$  if and only if  $S_0$  is a component of  $S$ .

(f) For each  $x \in R^2$ ,  $L^+(x)$  is either empty or consists of a single rest point. Further,  $L^+(x) = \emptyset$  for all  $x \in A^+(S)$  and  $L^-(x) = \emptyset$  for all  $x \in R^2 - S$ .

The above theorems indicate that imposing characteristic  $0^+$  on a dynamical system on  $R^2$  is a fairly strong restriction. However, for more general phase spaces the situation is different. By way of illustration, we give the following example.

**EXAMPLE 1.** Consider the subspace of  $R^3$  consisting of the  $xy$ -plane and the negative  $z$ -axis. Consider the flow in which the origin  $0$  is a rest point, points on the  $xy$ -plane are periodic whose trajectories surround  $0$  and points on the negative  $z$ -axis tend positively to  $0$ , i.e.,  $L^+(x) = 0$  for all  $x$  on the negative  $z$ -axis.

We have clearly defined a flow of characteristic  $0^+$  which has only

one rest point, and yet none of the conditions of Theorems 4.2 or 4.3 hold.

### 5. Flow of characteristic $0^\pm$ on the plane.

DEFINITION 5.1. A flow  $(R^2, \pi)$  on the plane is of characteristic  $0^\pm$  if for each  $x \in R^2$ ,  $D^+(x) = \overline{C^+(x)}$  and  $D^-(x) = \overline{C^-(x)}$ .

The above definition is equivalent to saying that a flow is of characteristic  $0^\pm$  if and only if every closed invariant subset  $M$  of  $R^2$  is positively and negatively  $D$ -stable (i.e.,  $D^+(M) = D^-(M) = M$ ). The following theorem completely classifies such flows. The proof of this theorem follows immediately from the previous section and is hence omitted.

THEOREM 5.1. *Let  $(R^2, \pi)$  be a dynamical system of characteristic  $0^\pm$  on the plane. Then one of the following holds.*

- (1)  $S = \emptyset$  and the flow is parallelizable.
- (2)  $S = R^2$ .
- (3)  $S = \{s_0\}$  is a singleton and  $s_0$  is a global Poincaré center.

The author is grateful to Professor Taro Ura for suggesting this problem. His guidance and suggestions have been invaluable. The author also thanks the referee, particularly, for his suggestions concerning the exposition of this paper.

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Received January 5, 1969, and in revised form August 15, 1969.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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