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FACTORIZATION OF A SPECIAL POLYNOMIAL OVER A FINITE FIELD

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Let $q = p^z$, where p is a prime and $z \geq 1$, and put $r = q^n$, $n \geq 1$. Consider the polynomial

$$F(x) = x^{2r+1} + x^{r-1} + 1.$$

Mills and Zierler proved that, for $q = 2$, the degree of every irreducible factor of $F(x)$ over $GF(2)$ divides either $2n$ or $3n$. We shall show that, for arbitrary q , the degree of every irreducible factor of $F(x)$ over $GF(q)$ divides either $2n$ or $3n$.

We shall follow the notation of Mills and Zierler [1]. Put

$$(1.1) \quad K = GF(r), \quad L = GF(r^2), \quad M = GF(r^3).$$

The identity

$$\begin{aligned} (x^{(2r+1)r} + x^{(r-1)r} + 1) - x^{r^2-r}(x^{2r+1} + x^{r-1} + 1) \\ = (x^{r^2-1} - 1)(x^{r^2+r+1} - 1) \end{aligned}$$

is easily verified. Since

$$(x^{2r+1} + x^{r-1} + 1)^r = x^{(2r+1)r} + x^{(r-1)r} + 1,$$

it is clear that

$$(1.2) \quad F^r(x) - x^{r^2-r}F(x) = (x^{r^2-1} - 1)(x^{r^2+r+1} - 1).$$

Let $F(\alpha) = 0$, where α lies in some finite extension of $GF(q)$. Then by (1.2)

$$(\alpha^{r^2-1} - 1)(\alpha^{r^2+r+1} - 1),$$

so that either

$$(1.3) \quad \alpha^{r^2-1} - 1 = 0$$

or

$$(1.4) \quad \alpha^{r^2+r+1} - 1 = 0.$$

Clearly (1.4) implies

$$\alpha^{r^3-1} - 1 = 0.$$

Hence α lies in either L or M .

Assume $\alpha \in K$. Then $\alpha^r = \alpha$, so that $F(\alpha) = 0$ reduces to

$$(1.5) \quad \alpha^3 + 2 = 0 .$$

There are now several possibilities. First the case $p = 2$ can be ruled out since $\alpha \neq 0$. Next if $p = 3$, (1.5) reduces to $\alpha^3 = 1$, so that $\alpha = 1$. If $p > 3$ and $r \equiv 2 \pmod{3}$ then again α is uniquely determined by (1.5) and is in $GF(p)$. If $p > 3$, $p \equiv 2 \pmod{3}$ but $r \equiv 1 \pmod{3}$, then $\alpha \in K$ if and only if

$$(1.6) \quad (-2)^{(r-1)/3} \equiv 1 \pmod{p} .$$

Since $p^2 - 1 \mid r - 1$, it is clear that this condition is satisfied; hence there are three distinct values of $\alpha \in K$ that satisfy (1.5). Finally if $p \equiv 1 \pmod{3}$, (1.5) will be satisfied with $\alpha \in K$ if and only if (1.6) holds and again there are three distinct values of α .

There is also a possibility that $F(x)$ has multiple roots when $p > 2$. Since

$$F'(x) = (2r + 1)x^{2r} + (r - 1)x^{r-2} = x^{2r} - x^{r-2} ,$$

it follows that a multiple root must satisfy

$$(1.7) \quad \alpha^{r+2} = 1 .$$

Then

$$0 = \alpha^3 F(\alpha) = \alpha^{2r+4} + \alpha^{r+2} + \alpha^3 ,$$

so that $\alpha^3 + 2 = 0$. On the other hand, combining (1.7) with either (1.3) or (1.4) gives $\alpha^3 = 1$. Hence $p = 3$, $\alpha = 1$. Since $F'''(1) = 2$ the multiplicity is 2.

To sum up we state the following two theorems.

THEOREM 1. *The degree of every irreducible factor of*

$$F(x) = x^{2r+1} + x^{r-1} + 1$$

over $GF(q)$ divides either $2n$ or $3n$.

THEOREM 2. *The only possible irreducible factors of $F(x)$ of degree dividing n are determined as follows:*

- (i) $p = 3$, $x - 1$,
- (ii) $p > 3$, $r \equiv 2 \pmod{3}$, *linear factor*,
- (iii) $p > 3$, $p \equiv 2 \pmod{3}$, $r \equiv 1 \pmod{3}$, $x^3 + 2$,
- (iv) $p \equiv 1 \pmod{3}$, $(-2)^{(r-1)/3} \equiv 1 \pmod{p}$, $x^3 + 2$,
- (v) $p \equiv 1 \pmod{3}$, $(-2)^{(r-1)/3} \not\equiv 1 \pmod{p}$, 1 .

$F(x)$ has multiple roots if and only if $p = 3$; when $p = 3$, $\alpha = 1$ is a root of multiplicity 2.

Let $F_0(x)$ denote the product of the irreducible divisors of $F(x)$ over $GF(q)$ of degree dividing n and put $f_0 = \deg F_0(x)$. Then Theorem 2 implies

THEOREM 3. *We have*

(i) $f_0 = 2,$

(ii) $f_0 = 1,$

(iii) $f_0 = 3,$

(iv) $f_0 = 3,$

(v) $f_0 = 0,$

where the cases (i), \dots , (v) have the same meaning as in Theorem 2. When $p = 2, f_0 = 0.$

2. If α denotes a root of $F(x)$, put

$$(2.1) \quad \beta = \alpha^{2r+1} .$$

Thus

$$\beta + \alpha^{r-1} + 1 = 0 ,$$

so that

$$(2.2) \quad (\beta + 1)^{2r+1} + \beta^{r-1} = 0 .$$

Expanding the left member of (2.2) we get

$$\beta^{2r+1} + \beta^{2r} + 2\beta^{r+1} + 2\beta^r + \beta^{r-1} + \beta + 1 = 0 ;$$

this is the same as

$$(2.3) \quad (\beta^r + \beta^{r-1} + 1)(\beta^{r+1} + \beta + 1) = 0 .$$

Now define

$$G(x) = (x^r + x^{r-1} + 1)(x^{r+1} + x + 1) .$$

It follows that if α is a root of $F(x)$, then α^{2r+1} is a root of $G(x)$.

As in [1], put

$$G_1(x) = x^r + x^{r-1} + 1 , \quad G_2(x) = x^{r+1} + x + 1 ,$$

so that

$$G(x) = G_1(x)G_2(x) .$$

Also it is convenient to put

$$H(x) = x^r + x + 1 .$$

The roots of $H(x)$ are the inverse of the roots of $G_1(x)$.

If $H(\beta) = 0$ then

$$\beta^r = -\beta - 1, \quad \beta^{r^2} = -\beta^r - 1 = \beta,$$

so that $\beta \in L$. If we assume $\beta \in K$, so that $\beta^r = \beta$, it follows that $2\beta + 1 = 0$. Thus for $p > 2$, $H(x)$ has a unique root in K (indeed in $GF(p)$). Since $H'(\beta) = 1$ it is clear that $H(x)$ has no multiple root. Thus, except for the root -2 , all the roots of $G_1(x)$ lie in L and not in K .

Next if $G_2(\beta) = 0$ we have

$$\beta^{r+1} = -\beta - 1,$$

so that

$$\beta^{r^2+r+1} = -\beta(\beta + 1) = -\beta^{r+1} - \beta = 1.$$

Hence $\beta^{r^3-1} = 1$, so that $\beta \in M$. If we assume $\beta \in K$ we get

$$(2.4) \quad \beta^2 + \beta + 1 = 0.$$

This equation is solvable in K if and only if $p = 3$ or $r \equiv 1 \pmod{3}$. Thus, except for these cases, the roots of $G_2(x)$ lie in M and not in K . Since

$$G_2'(x) = x^r + 1 = (x + 1)^r,$$

it follows that $G_2(x)$ has no multiple roots.

This proves

LEMMA 1. *Except for the root -2 when $p > 2$, all the roots of $G_1(x)$ lie in L and not in K . Except for the root 1 when $p = 3$, all the roots of $G_2(x)$ lie in M and not in K .*

We shall now prove

LEMMA 2. *Let α be a root of $F(x)$ and put $\beta = \alpha^{2r+1}$, so that β is a root of $G(x)$. If β is a root of $G_1(x)$, then $\alpha \in L$; if β is a root of $G_2(x)$, then $\alpha^{r^2+r+1} = 1$ so that $\alpha \in M$.*

Proof. By hypothesis

$$0 = F(\alpha) = \beta + \alpha^{r-1} + 1,$$

so that

$$(2.5) \quad \beta = -\alpha^{r-1} - 1.$$

Assume first that $G_1(\beta) = 0$. Then

$$1 = -\beta^{r-1}(\beta + 1) = \alpha^{(2r+1)(r-1)} \cdot \alpha^{r-1} = \alpha^{2r^2-2},$$

so that

$$\alpha^{2(r^2-1)} = 1$$

and $\alpha^2 \in L$. But since either $\alpha \in L$ or $\alpha \in M$ it follows that $\alpha \in L$.

Next let $G_2(\beta) = 0$. Then by (2.5)

$$\alpha^{r-1} = -\beta - 1 = \beta^{r+1} = \beta^{(r+1)(2r+1)},$$

which gives

$$(2.6) \quad \alpha^{2(r^2+r+1)} = 1.$$

This implies $\alpha^2 \in M$. If $\alpha \in L$, (2.6) reduces to $\alpha^{2r+4} = 1$; this in turn gives

$$\beta^2 = \alpha^{4r+2} = 1,$$

so that $B = \pm 1$. Since $G_2(\beta) = 0$ we must have $p = 3, \beta = 1$.

3. By Theorem 1 we have

$$(3.1) \quad F(x) = F_1(x)F_2(x)/F_0(x)$$

where every root of $F_1(x)$ is in L , every root of $F_2(x)$ is in M , every root of $F_0(x)$ is in K .

We shall now prove

LEMMA 3. *A number $\alpha \in L$ is a root of $F_1(x)$ if and only if $\beta = \alpha^{2r+1}$ is a root of $G_1(x)$.*

Proof. By Lemma 2, if α is a root of $F_1(x)$, then β is a root of $G_1(x)$. Let $\alpha \in L, \beta = \alpha^{2r+1}, G_1(\beta) = 0$. Then since $\alpha^{r^2-1} = 1$ it follows that

$$(\alpha\beta)^{r-1} = (\alpha^{2r+2})^{r-1} = \alpha^{2(r^2-1)} = 1.$$

Consequently

$$\begin{aligned} \beta^{r-1}F(\alpha) &= \beta^{r-1}(\beta + \alpha^{r-1} + 1) \\ &= \beta^r + \beta^{r-1} + (\alpha\beta)^{r-1} \\ &= \beta^r + \beta^{r-1} + 1 \\ &= G_1(\beta) = 0, \end{aligned}$$

so that $F(\alpha) = 0$.

LEMMA 4. *Let α be an element of M such that $\alpha^{r^2+r+1} = 1$. Then α is a root of $F_2(x)$ if and only if $\beta = \alpha^{2r+1}$ is a root of $G_2(x)$.*

Proof. By Lemma 2, if α is a root of $F_2(x)$, then β is a root of $G_2(x)$. Let $\alpha \in M$, $\alpha^{r^2+r+1} = 1$, $\beta = \alpha^{2r+1}$, $G_2(\beta) = 0$. Since

$$\beta^{r+1} = \alpha^{(r+1)(2r+1)} = \alpha^{2r^2+3r+1} = \alpha^{r-1},$$

we get

$$0 = G_2(\beta) = \beta^{r+1} + \beta + 1 = \alpha^{2r+1} + \alpha^{r-1} + 1 = F(\alpha),$$

so that $F(\alpha) = 0$.

LEMMA 5. *Let β be a nonzero element of L and let $R(\beta)$ denote the number of elements α in L such that $\alpha^{2r+1} = \beta$. Then*

$$(3.2) \quad R(\beta) = \begin{cases} 1 & (r \equiv 0, 2 \pmod{3}) \\ 3 & (r \equiv 1 \pmod{3}, \beta = \gamma^3, \gamma \in L) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Any common divisor of $2r + 1$ and $r^2 - 1$ must divide

$$(2r - 1)(2r + 1) - 4(r^2 - 1) = 3.$$

If $r \equiv 0, 2 \pmod{3}$ then $2r + 1 \equiv 1, 2 \pmod{3}$, so that $(2r + 1, r^2 - 1) = 1$. It follows that the equation $\alpha^{2r+1} = \beta$ has a unique solution $\alpha \in L$. If $r \equiv 1 \pmod{3}$ we have $(2r + 1, r^2 - 1) = 3$; thus the equation $\alpha^{2r+1} = \beta$ is insolvable in L if and only if $\beta = \gamma^3, \gamma \in L$. If $\beta = \gamma^3, \gamma \in L$, there are exactly three solutions; otherwise there are none.

If $r \equiv 0, 2 \pmod{3}$ it follows at once from Lemmas 3 and 5 that there is a one-to-one correspondence between the roots of $F_1(x)$ and of $G_1(x)$. We may therefore state the following.

THEOREM 4. *Let $r \equiv 0, 2 \pmod{3}$. Then the degree of $F_1(x)$ is equal to r .*

If we put $f_1 = \deg F_1(x), f_2 = \deg F_2(x), f_0 = \deg F_0(x)$, then by (3.1) we have

$$(3.3) \quad f_0 + 2r + 1 = f_1 + f_2.$$

Thus for $r \equiv 0, 2 \pmod{3}$, f_2 can be computed by means of (3.3) and Theorem 3.

4. We shall now determine f_1 when $r \equiv 1 \pmod{3}$. By Lemmas 3 and 5, f_1 is three times the number of roots of $G_1(x)$ that are cubes

in L . Then, if as above

$$H(x) = x^r + x + 1,$$

f_1 is three times the number of roots of $H(x)$ that are cubes in L .

Put $\lambda = \beta^{r+1}$, where $H(\beta) = 0$. Since $\beta^{r^2} = \beta$, it follows that $\lambda^r = \beta^{r^2+r} = \lambda$, so that $\lambda \in K$. In the next place λ is a cube in K if and only if β is a cube in L . To see this let γ denote a primitive root of L . Then $\beta = \gamma^t$, where t is some integer. If β is a cube in L then $t = 3u$, where u is an integer. Thus

$$\lambda = \beta^{r+1} = \gamma^{3u(r+1)}.$$

Since $\gamma^{r+1} \in K$, it follows that λ is a cube in K . To prove the converse, it is clear first that $\lambda = \gamma^{a(r+1)}$, where a is an integer. If λ is a cube in K it follows that $a = 3b$, where b is an integer. Thus $\lambda = \beta^{r+1}$ becomes

$$\gamma^{3b(r+1)} = \gamma^{t(r+1)},$$

so that

$$3b(r + 1) \equiv t(r + 1) \pmod{r^2 - 1}.$$

This implies

$$3b \equiv t \pmod{r - 1}.$$

Since $r \equiv 1 \pmod{3}$ we conclude that $3/t$.

The relation $\lambda = \beta^{r+1}$, where $H(\beta) = 0$, is equivalent to

$$(4.1) \quad \beta^2 + \beta + \lambda = 0.$$

We have seen above that, except for $\beta = -1/2$, all the roots of $H(\beta) = 0$, are in L and not in K (of course this case occurs only when $p > 2$). Moreover $\beta = -1/2, \lambda = 1/4$ do indeed satisfy (4.1). Also 2 is a cube in L if and only if it is a cube in K , that is, if and only if

$$(4.2) \quad 2^{(r-1)/3} \equiv 1 \pmod{p}.$$

Thus aside from the exceptional case just described we must determine the number of cubes of K that are not of the form $\tau(\tau + 1)$ with τ in K (for convenience we replace λ in (4.1) by its negative). We denote this number by N . If N_0 denotes the number of nonzero cubes of K that are of the form $\tau(\tau + 1)$ with τ in K , it is clear that

$$(4.3) \quad N + N_0 = \frac{1}{3}(r - 1).$$

As for f_1 , we have

$$(4.4) \quad f_1 = 6N + 3E,$$

where $E = 1$ when (4.2) is satisfied and $E = 0$ otherwise. The coefficient 6 occurs because for given $\lambda \neq 1/4$ there are two distinct values of β ; however when $\lambda = 1/4$ there is a single value of β and hence the coefficient 3.

It remains therefore to evaluate N_0 . Clearly $6N_0$ is equal to the number of pairs $x, y \in K$ such that

$$(4.5) \quad x^2 + x = y^3 \neq 0.$$

Assume first that $p > 2$. Then (4.5) is equivalent to

$$(4.6) \quad z^2 = 4y^3 + 1, \quad y \neq 0.$$

Let $\psi(a)$ denote the quadratic character for K , that is

$$\psi(a) = \begin{cases} +1 & (a = b^2 \neq 0, b \in K) \\ 0 & (a = 0) \\ -1 & (\text{otherwise}). \end{cases}$$

Then the number of solutions of (4.6) is equal to

$$\sum_{\substack{y \in K \\ y \neq 0}} \{1 + \psi(4y^3 + 1)\},$$

so that

$$(4.7) \quad 6N_0 = r - 2 + \sum_{y \in K} \psi(4y^3 + 1),$$

where now the summation is over all $y \in K$.

Put

$$J(a) = \sum_{x \in K} \psi(x^3 + a) \quad (a \in K).$$

Then clearly

$$J(ac^3) = \psi(c)J(a) \quad (c \neq 0),$$

so that

$$(4.8) \quad J^2(ac^3) = J^2(c) \quad (c \neq 0).$$

$$\begin{aligned} \sum_a J^2(a) &= \sum_{x, y} \sum_a \psi((x^3 + a)(y^3 + a)) \\ &= \sum_{x^2=y^3} (r-1) - \sum_{x^3 \neq y^3} 1 \\ &= r \sum_{x^3=y^3} 1 - \sum_{x, y} 1 \\ &= r(3r-2) - r^2 \\ &= 2r(r-1), \end{aligned}$$

so that

$$(4.9) \quad \sum_a J^2(a) = 2r(r - 1) .$$

Let γ denote a fixed primitive root of K . Then by (4.8) and (4.9), since $J(0) = 0$,

$$(4.10) \quad J^2(1) + J^2(\gamma^2) + J^2(\gamma^4) = 6r .$$

On the other hand, since

$$\sum_c J(c^2) = \sum_x \sum_c \psi(x^3 + c^2) = r - 1 - \sum_{x \neq 0} 1 = 0 ,$$

it follows that

$$(4.11) \quad J(1) + J(\gamma^2) + J(\gamma^4) = 0 .$$

Combining (4.11) with (4.10), we get

$$(4.12) \quad J^2(1) + J(1)J(\gamma^2) + J^2(\gamma^2) = 3r .$$

It is easily seen that $J(1)$ is an even integer while $J(\gamma^2), J(\gamma^4)$ are odd. Thus (4.12) implies

$$(4.13) \quad r = A^2 + 3B^2 ,$$

where A, B are integers defined by

$$(4.14) \quad A = \frac{1}{2}J(1) , \quad B = \frac{1}{6}[J(1) + 2J(\gamma^2)] .$$

It follows from the definition that

$$(4.15) \quad J(1) \equiv 1 \pmod{3} .$$

Hence, by (4.11) and (4.12),

$$(4.16) \quad J(1) \equiv J(\gamma^2) \equiv J(\gamma^4) \equiv 1 \pmod{3} .$$

If $p \equiv 2 \pmod{3}$ it is clear from (4.13) that $A = \pm r^{1/2}, B = 0$. Thus, by (4.11), (4.14) and (4.16),

$$(4.17) \quad J(1) = \pm 2r^{1/2} \equiv 1 \pmod{3}$$

and

$$(4.18) \quad J(\gamma^2) = J(\gamma^4) = -\frac{1}{2}J(1) .$$

For $p \equiv 1 \pmod{3}$, on the other hand, we have the congruence

$$J(1) \equiv -\left(\frac{3m}{2m}\right)^{nz} \pmod{p} ,$$

where $p = 6m + 1$. Thus $J(1) \not\equiv 0 \pmod{p}$. Hence A^2, B^2 in (4.13) are uniquely determined. Then making use of (4.16), $J(1), J(\gamma^2), J(\gamma^4)$ are uniquely determined.

Returning to (4.7), we have

$$(4.19) \quad 6N_0 = r - 2 + \psi(2)J(2) .$$

Thus, by (4.3) and (4.4), we get

$$(4.20) \quad f_1 = r - \psi(2)J(2) - 3E .$$

We may state

THEOREM 5. *Let $p > 2, r \equiv 1 \pmod{3}$. Then the degree of $F_1(x)$ is determined by (4.20), where $J(2)$ is uniquely determined by (4.13), (4.16), (4.17) and (4.18); $E = 1$ when*

$$2^{(r-1)/3} \equiv 1 \pmod{p}$$

and $E = 0$ otherwise.

5. When $p = 2$ we have, as above, $f_1 = 6N$ and

$$N + N_0 = \frac{1}{3}(r - 1) ;$$

$6N_0$ is equal to the number of pairs $x, y \in K$ such that

$$(5.1) \quad x^2 + x = y^3 \neq 0 .$$

Now for $a \in K$ put

$$t(a) = a + a^2 + a^{2^2} + \dots + a^{2^{2s-1}}$$

and

$$e(a) = (-1)^{t(a)} .$$

Define

$$(5.2) \quad L(a) = \sum_{x \in K} e(ax^3) .$$

It follows from (5.2) that

$$(5.3) \quad L(ac^3) = L(a) \quad (c \neq 0) .$$

Since $e(a) = e(a^2)$ we have also

$$(5.4) \quad L(a) = L(a^2) = L(a^{-1}) \quad (a \neq 0) .$$

It is easy to show that

$$\sum_{x \in K} e(ax) = \begin{cases} r & (a = 0) \\ 0 & (a \neq 0) \end{cases} .$$

Then

$$\begin{aligned} \sum_{a \in K} L^2(a) &= \sum_{x,y} \sum_a e(a(x^3 + y^3)) \\ &= r \sum_{x^3=y^3} 1 \\ &= r[1 + 3(r - 1)] \\ &= r(3r - 2) . \end{aligned}$$

Since $L(a) = r$, it follows that

$$(5.5) \quad \sum_{a \neq 0} L^2(a) = 2r(r - 1) .$$

Let γ denote a fixed primitive root of K . Then, by (5.3) and (5.5),

$$L^2(1) + L^2(\gamma) + L^2(\gamma^2) = 6r .$$

In view of (5.4) this reduces to

$$(5.6) \quad L^2(1) + 2L^2(\gamma) = 6r .$$

In the next place

$$\sum_a L(a) = \sum_x \sum_a e(ax^3) = r ,$$

so that

$$\sum_{a \neq 0} L(a) = 0 .$$

By (5.3) and (5.4) this reduces to

$$(5.7) \quad L(1) + 2L(\gamma) = 0 .$$

Combining (5.7) with (5.6) we get

$$L^2(\gamma) = r , \quad L(\gamma) = \pm r^{1/2} .$$

But it is clear from the definition that

$$L(a) \equiv 1 \pmod{3}$$

for all $a \in K$. Therefore

$$(5.8) \quad L(\gamma) = L(\gamma^2) = (-2)^{nz/2}$$

and, by (5.7),

$$(5.9) \quad L(1) = (-2)^{(nz+2)/2} .$$

We now return to (5.1). For fixed y , the number of solutions of (5.1) is equal to

$$1 + e(y^3).$$

It follows that

$$\begin{aligned} 6N_0 &= \sum_{y \neq 0} \{1 + e(y^3)\} \\ &= r - 2 + L(1). \end{aligned}$$

Then

$$\begin{aligned} f_1 &= 6N = 6 \left[\frac{1}{3}(r - 1) - N_0 \right] \\ &= 2(r - 1) - [r - 2 + L(1)] \\ &= r - L(1). \end{aligned}$$

In view of (5.9) this becomes

$$f_1 = r - (-2)^{(nz+2)/2}.$$

This completes the proof of

THEOREM 6. *Let $p = 2$, $q = 2^z$, $r = q^n$. Then the degree of $F_1(x)$ is equal to*

$$2^{nz} - (-2)^{(nz+2)/2}.$$

The degree of $F_2(x)$ is determined by

$$f_0 + 2r + 1 = f_1 + f_2,$$

where $f_i = \deg F_i(x)$ and f_0 is given by Theorem 3.

We note that when $z = 1$, Theorem 6 reduces to Theorem 3 of [1].

REFERENCE

1. W. H. Mills and N. Zierler, *On a conjecture of Golomb*, Pacific J. Math. **28** (1969), 635-640.

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