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FRANK RIMI DEMEYER

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F. R. DEMEYER

A Galois theory for separable projective algebras over a class of commutative rings more general than Von Neumann regular rings with includes the full group of algebra automorphisms is presented. Fundamental facts concerning the group of units and the automorphisms of these algebras are also given.

A Galois theory for simple algebras over fields which included the full group of automorphisms was presented by G. Hochschile [5]. After the work of M. Auslander and O. Goldman [1] and S. Chase, D. K. Harrison, and A. Rosenberg [3] it was possible to generalize Hochschild's theory to semi-local rings with no idempotents other than 0 and 1 [4]. More recent results of O. Villamayor and D. Zelinsky [14], and H. Kreimer [8] permit the extension of Hochschild's theory to a much broader class of rings than in [4]. The context includes separable projective algebras over commutative regular rings and is stated explicitly at the beginning of § 3. Theorem 3 and Theorem 5 of § 2 have also been obtained by A. Magid and will appear in the Illinois Journal of Mathematics as part of an article entitled *Pierce's representation and separable algebras.*

We first state some of the results of [14] without proof. The Galois theory in [14] is then improved in a special case. Next, in § 2, several basic properties of separable algebras are developed. The Galois theory which generalizes [4] is presented in § 3. Henceforth R always is a commutative ring with identity. All rings have identity, all subrings share the common identity, all modules are unitary, and all algebra homomorphisms carry identity to identity. Also $\otimes$ will mean $\otimes_R$. A familiarity with the basic properties of separable algebras is assumed. In particular, we employ a result of Villamayor [10] in the sequel which asserts that any separable projective R-algebra is finitely generated as an R-module.

1. We first enumerate, with corresponding nomenclature, several results in [14]. Let $B(R)$ be the Boolean algebra of idempotents of the commutative ring R. If $e, f \in B(R)$ the Boolean operations are given by $e \oplus f = e + f - ef$ and $e * f = ef$. Let spec $B(R)$ be the space of maximal (= prime) ideals in $B(R)$ and call the subsets U_e of spec $B(R)$ defined for each $e \in B(R)$ by

$$U_e = \{p \in \operatorname{spec} B(R) \mid e \in p\}$$

a base for the open sets of spec $B(R)$. In this topology spec $B(R)$ is a totally disconnected compact Hausdorff space. If $p \in \operatorname{spec} B(R)$ we let $R_p = R/p \cdot R$, then R_p is a homomorphic image of R. If M is an R-module then $R_p \otimes M = M/pM$ is denoted M_p and the natural image of $a \in M$ in M_p is denoted a_p. In most of what follows we will use the compactness of spec $B(R)$ to recover information true at R_p for each $p \in \operatorname{spec} B(R)$ to information about R.

The next seven results are proved in [14].

(2.7) If M and N are finitely generated projective R-modules the natural map from $\operatorname{Hom}_R(M, N)$ to $\operatorname{Hom}_{R_p}(M_p, N_p)$ denoted $f \to f_p$ is an isomorphism.

(2.9) If a and b are elements of the finitely generated R-module M and $a_p = b_p$ for one $p \in \operatorname{spec} B(R)$ then $a_q = b_q$ for all q in some neighborhood U_e of p.

(2.11) If N is a submodule of the finitely generated R-module M and $N_p = M_p$ for all $p \in \operatorname{spec} B(R)$ (R_p is flat over R so $N_p c M_p$) then $N = M$.

(2.12) If S is a finitely generated R-algebra and u is an idempotent in S_p for some $p \in \operatorname{spec} B(R)$, then there is an idempotent v in S such that $v_p = u$.

(2.13) R_p has no idempotents other than 0 and 1 for each $p \in \operatorname{spec} B(R)$.

(2.14) Suppose S is an R-algebra that is finitely presented as an R-module, let F be a finite subset of S and let g be an R_p-algebra homomorphism of S_p which is the identity on F_p. Then there is an R-algebra homomorphism h of S such that h is the identity on F and $h_p = g$.

(2.17) If S is an R-algebra and J is a finite group of R-algebra automorphisms of S; then for all $p \in \operatorname{spec} B(R)$, $(S^J)_p = (S_p)_p^J$ where S^J denotes the fixed ring under J.

Let K be a commutative ring containing R as a subring.

If K is a projective separable R-algebra and if $K^G = R$ where G is the group of all R-automorphisms of K, then K is called a normal separable extension of R with group G. If H is a subset of G, the closure $\bar{H}$ of H is the set of all automorphisms g of K such that for each $p \in \operatorname{spec} B(R)$ and each minimal idempotent f_p in K_p, $f_p \cdot g_p = f_p \cdot h_p$

for some $h \in H$. A closed subgroup of G is one equal to its own closure.

Another result from [14] is

LEMMA 1. *Let K be a normal separable extension of R with group G, then the following statements are equivalent.*

(1) *For each $p \in \operatorname{spec} B(R)$, K_p is a normal separable extension of R_p with group G_p.*

(2) $\operatorname{Hom}_R(K, K)$ *is generated as a K-module by R-algebra automorphism of K.*

(3) *R is the fixed ring of some finite set (equivalently, finite group) of automorphisms of S.*

An example in [14] shows that when K is a normal separable R-algebra the equivalent conditions of Lemma 1 need not always hold. It is only in the situation where these conditions hold that the techniques we employ are effective. As a consequence of Lemma 1 we can restate Theorem 3.8 of [14].

THEOREM 3.8. *Let K be a normal separable R extension of R with group G, if there is a finite subset J of G with $K^J = R$ then there is a one to one correspondence between the set of all separable subalgebras of K and the set of all subgroups H of G which are the closure of some finite subgroup of G by $H \to K^H$.*

The result we require is in the situation where every idempotent in K belongs to R. In this case Theorem 3.8 can be improved.

COROLLARY 2. *Let K be a normal separable extension of R with group G. Assume R is the fixed ring of a finite subset of G and assume every idempotent in K belongs to R. Then there is a one-to-one correspondence between the separable subalgebras of K containing R and the closed subgroups H of G by $H \to K^H$. The closed subgroup H of G is a normal subgroup of G if and only if the corresponding subalgebra is a normal extension of R with group G/H.*

Proof. We apply Theorem 3.8 to give the subgroup-subring correspondence. Assume H is a closed normal subgroup of G, then G/H acts as a group of automorphisms of K^H in the usual way and $(K^H)^{G/H} = R$. Let σ be an R-automorphism of K^H. By hypothesis on the idempotents in K and Theorem 3.5 of [3] we have for each $p \in \operatorname{spec} B(R)$ an automorphism τ_p of K_p which extends σ_p on $K_p^{H_p}$. Use (2.14) to lift τ_p to an automorphism τ on K. Since K^H is finitely generated and projective over R (Proposition 1.5 of [4]), $\operatorname{Hom}_R(K^H, K)$ is finitely generated over R so there is by (2.9) and (2.7) a neighborhood

U of p such that for each $q \in U$, τ_q restricted to $(N^H)_q$ is σ_q. Cover spec $B(R)$ with such neighborhoods and use compactness of spec $B(R)$ to obtain a decomposition $R = Re_1 \oplus \cdots \oplus Re_n$ and automorphisms τ_i of Ke_i with τ_i restricted to $(K^H)e_i$ equal to σ on $(K^H)e_i$ (since $e_i \in R$, $\sigma(e_i) = \tau(e_i) = e_i$). The automorphism τ on K constructed from the τ_i extends σ and is in G so K^H is a normal separable extension of R with group G/H. Conversely, assume H is a closed subgroup of G and K^H is a normal separable extension of R. Then by Theorem 2.3 of [3] and (2.17) we know that for each $p \in$ spec $B(R)$ and each $\sigma \in G$

$$\sigma_p(K^H)_p = \sigma_p(K_p^{Hp}) = K_p^{Hp} = (K^H)_p .$$

Since K^H is finitely generated over R, (2.7) implies $\sigma(K^H) = K^H$. This is equivalent to the statement that H is a normal subgroup of G and completely proves the corollary.

With the hypothesis of Corollary 2, if N is a normal separable extension of R in K which corresponds to the closed normal subgroup H of G observe that there is a finite subset of G/H on N leaving exactly R fixed. Therefore our Galois theory also applies to the extension N of R with group G/H.

2. We now derive some properties of separable algebras. The group of isomorphism classes of rank $= 1$ projective R-modules forms a group called the class group (Picard group) of R. Generalizing Theorem 3.6 of [1] we have

THEOREM 3. *Let A be a central separable R-algebra. Assume the class group of R_p is trivial for each $p \in$ spec $B(R)$, then every R-algebra automorphism of A is an inner automorphism.*

Proof. Let σ be an algebra automorphism of A. By Theorem 3.6 of [1] we know σ_p induces an inner automorphism of A_p for each $p \in$ spec $B(R)$. Let $\sigma_p = i_{\bar{x}}$ on A_p, then there exists a $\bar{y} \in A_p$ with $\bar{x}\bar{y} = 1$. Lift $\bar{x}$ and $\bar{y}$ to elements x' and y' in A. By (2.9) there is a neighborhood U_e of p so that for all $q \in U_e$ we have $x'_q y'_q = 1$. Therefore $(x'e)(y'e) = e$ and $(x'e)_p = \bar{x}$, $(y'e)_p = \bar{y}$. Let $x = (1 - e) + x'e$ and $y = (1 - e) + y'e$. Then $x, y \in A$ and $xy = 1$. Moreover i_x is an inner automorphism of A with $(i_x)_p = i_{\bar{x}}$. Now $\mathrm{Hom}_R(A, A)$ is a finitely generated projective R-module so by (2.9) there is a neighborhood U of p in spec $B(R)$ with $(i_x)_q = \sigma_q$ for all $q \in U$. Cover spec $B(R)$ with such neighborhoods, since spec $B(R)$ is compact one can find a finite subcover. Thus $R = Re_1 \oplus \cdots \oplus Re_n$ with $e_i^2 = e_i \in R$ and there are elements $x_j \in Ae_j (j = 1, \cdots n)$ such that x_j is a unit in Ae_j and σ on Ae_j is inner automorphism by x_j. Let $x = x_1 + \cdots + x_n$, then σ is inner automorphism by x on A.

A corollory of this result shows that we are in the context of [9].

COROLLARY 4. *Let A be a separable projective R-algebra with center K, and assume the class group of K_p is trivial for each $p \in \operatorname{spec} B(R)$, then every R-algebra automorphism of A which is not inner is represented nontrivially on K.*

Proof. For $p \in \operatorname{spec} B(R)$, K_p is a finitely generated projective separable R_p-algebra and R_p has no idempotents other than 0 and 1. Thus C_p is a finite direct sum of finitely generated projective separable R_p-algebras whose only idempotents are 0 and 1. The class group of K_p is the product of the class groups of the summands and the summands correspond to the elements of spec $B(K)$ so Theorem 3 applies.

Our inability to extend the next result restricts the generality of the Galois theory in § 3. See the examples following Theorem 1.1 and Theorem 1.2 of [4].

THEOREM 5. *Let R be a commutative ring, assume for each $p \in \operatorname{spec} B(R)$ that R_p is a semi-local ring, (finite number of maximal ideals). Let A be a separable projective R-algebra with center K and let B be a separable R-subalgebra of A with center C containing K. Assume every idempotent in C belongs to R. Let σ be an R-algebra monomorphism from B into A leaving K fixed, then σ can be extended to an inner automorphism of A.*

Proof. We know for any $p \in \operatorname{spec} B(R)$ that A_p is a projective separable R_p algebra with center K_p and B_p is a separable subalgebra with center C_p containing K_p. By hypothesis on the idempotents in C it follows that the only idempotents in C_p are 0 and 1. By the lemma on page 25 of [4], B is projective over R so $\operatorname{Hom}_R(B, A)$ is a finitely generated projective R-module and (2.7) implies $\operatorname{Hom}_R(B, A)_p = \operatorname{Hom}_{R_p}(B_p, A_p)$. With this identification, Theorem 1.2 of [4] gives an inner automorophism $i_{\bar{x}}$ on A_p which extends σ_p on B_p. As in the proof of Theorem 3 lift $\bar{x}$ to a unit x in A so that i_x is an inner automorphism of A with $(i_x)_p = i_{\bar{x}}$. Apply (2.9) to get a neighborhood U_e of p so that for all $q \in U_e$, $\sigma_q = (i_x)_q$ on B_q. Applying the same compactness argument as used in Theorem 3 we construct an inner automorphism i_y on A which extends σ.

If R is a regular ring (in the sense of Von Neumann) then R satisfies the hypothesis of Theorem 5 since every principal ideal in a regular ring is generated by an idempotent. Thus if R is regular, R_p is a field for each $p \in \operatorname{spec} B(R)$. The next result improves Theorem

1.5 of [4].

THEOREM 6. *Let A be a separable projective algebra over K and let J be a finite group of ring automorphisms of A with the property that J restricted to K is a group of automorphisms of K isomorphic to J. Let $R = K^J$ and assume K is a separable finitely generated R-algebra with every idempotent in K belonging to R. If $B = A^J$, then B is a separable projective R-algebra and $A = B \otimes K$.*

Proof. The hypothesis of Theorem 1.5 in [4] is satisfied at each $p \in \operatorname{spec} B(R)$. Therefore the natural homomorphism from $B \otimes K$ to A is an isomorphism at each $p \in \operatorname{spec} B(R)$. Applying (2.7) and (2.11) we conclude $B \otimes K = A$. Now K is a finitely generated, projective, separable R-algebra so $R \cdot 1$ is an R-direct summand of K (see Proposition A.4 of [1]). By ([2], IX, 9.1) B is separable over R. By the lemma on page 2.5 of [4] B is projective over R.

The next result includes a correction of Lemma 1.8 in [4]. Let Z be the ring of integers.

LEMMA 7. *Let A be a separable projective R-algebra, then A is generated as an algebra by its units if and only if A_p is generated as an R_p algebra by its units for all $p \in \operatorname{spec} B(R)$. Moreover, if R_p is semi-local then A_p is not generated as an algebra by its units if and only if $R_p/\operatorname{Rad}(R_p) = Z/(2) \oplus R'$ and the central component of $A_p/\operatorname{Rad}(A_p)$ over $Z/(2)$ contains a direct factor isomorphic to $Z/(2) \oplus Z/(2)$.*

Proof. Assume that A_p is generated by its units for all $p \in \operatorname{spec} B(R)$. As in the proof of Theorem 3, each unit in A_p lifts to a unit in A. Thus if N is the R-subalgebra of A generated by the units of A we know $N_p = A_p$ for all $p \in \operatorname{spec} B(R)$. By (2.11) this implies $N = A$. The converse is clear.

For the last statement observe that the proof of Lemma 1.8 in [4] implies A_p is generated by its units over R_p if and only if $A_p/\operatorname{Rad}(A_p)$ is generated by its units over $R_p/\operatorname{Rad}(R_p)$. Now $R_p/\operatorname{Rad}(R_p)$ is a finite direct sum of fields and $A_p/\operatorname{Rad}(A_p)$ is a finite direct sum of *semi*-simple algebras over fields. It is an elementary exercise to show that the only semi-simple algebras over a field not generated by units occurs when the algebra contains $Z/(2) \oplus Z/(2)$ as a direct summand over the field $Z/(2)$. This proves the lemma.

More information about automorphisms of separable algebras is contained in

LEMMA 8. *Let A be a projective separable R-algebra with center K and assume every idempotent in K belongs to R, then any R-homomorphism σ from A to A is an automorphism of A.*

Proof. Observe that σ is an R-homomorphism from K to K. Consider the exact sequence

$$0 \longrightarrow \ker(\sigma) \longrightarrow K \xrightarrow{\sigma} \sigma(K) \longrightarrow 0 .$$

Now $\sigma(K)$ is a separable R-subalgebra of K so by the lemma on page 25 of [4] $\sigma(K)$ is projective over R. Therefore $\ker(\sigma)$ is an ideal direct summand and is generated by an idempotent in K. But every idempotent in K belongs to R and σ is one-to-one on R so σ is one-to-one on K. For each $p \in \operatorname{spec} B(R)$ we know $\sigma_p: K_p \to K_p$ is one-to-one. K_p has no idempotents other than 0 and 1 and K_p can be imbedded in a Galois (in the sense of [3]) extension of R_p by [6] so Theorem 2.3 of [3] and the fact that $\operatorname{Rank}_{R_p}(K_p) = \operatorname{Rank}_{R_p}(\sigma_p(K_p))$ implies σ_p is also onto K_p. By (2.11) and (2.7) this implies $\sigma(K) = K$. Since there is a one-to-one correspondence between the two sided ideals of A and those of K by $a \, cK$ corresponds to Aa we know $\ker(\sigma) = 0$ on A and σ is one to one on A.

Now $\sigma(A)$ is a central separable K-algebra (since $\sigma(K) = K$) and $\operatorname{Rank}_K(A) = \operatorname{Rank}_K(\sigma(A))$ since $\sigma(A)$ is one to one. By Theorem 3.3 of [1], $A = \sigma(A) \otimes_K A'$ where A' is a central separable K-algebra. By a rank argument, $A' = K$ and σ maps A onto A. This proves the lemma.

LEMMA 9. *Let K and L be commutative separable R-subalgebras of the finitely generated projective R-algebra A, then $K \cap L$ and $K \cdot L$ are separable projective R-subalgebras of A.*

Proof. First, $K \cdot L$ is a homomorphic image of $K \otimes L$ and therefore is separable over R. By the lemma on page 25 of [4] $K \cdot L$ is projective over R. To prove $K \cap L$ is separable over R we employ the construction in § 3 of [12] to find a commutative projective separable R-algebra N containing $K \cdot L$ such that there is a finite group J of automorphisms of N with $N^J = R$. By Theorem 3.8 there are finite sets H, H' of R-automorphisms of N with $N^H = K$ and $N^{H'} = L$. Observe that $N^{H \cdot H'} = K \cap L$ which by Theorem 3.8 implies $K \cap L$ is projective and separable over R.

3. Throughout this section we shall adhere to the following notations and assumptions. R is a commutative ring with R_p with semi-local for each $p \in \operatorname{spec} B(R)$. A is a separable projective R-algebra with

center K, and every idempotent of K belongs to R. Let G be the group of all R-algebra automorphisms of A and assume $A^G = R$. Also we assume there is a finite subset of G whose restrictions give a finite set J of automorphisms of K with $K^J = R$.

If H is a subgroup of G we let H_0 be the normal subgroup of H consisting of those elements of H which are inner automorphisms of A. We let $R(H)$ be the subring of A generated as a K-algebra by all the units giving the inner automorphisms in H. If B is an R-subalgebra of A we let G_B be the group of all elements in G leaving B elementwise fixed.

Call a subgroup H of G complete if every inner automorphism of A by an element of $R(H)$ is in H and if when restricted to K, every element of G/H_0 which leaves K^H fixed is in H/H_0. We show later that this last condition implies that on K we have H a closed group of automorphisms in the sense defined in § 1. Call a subgroup H of G regular if H is complete, $R(H)$ is a separable K-algebra, and every central idempotent in $R(H)$ belongs to K.

If B is an R-algebra of A we call B regular if B is separable as an R-algebra, if every idempotent in the center of $B \otimes_{B \cap K} K$ is in K, and if the commutator of $K \cdot B$ in A is generated as a K-algebra by its units.

If R is a semi-local ring whose only idempotents are 0 and 1 these definitions reduce to [4]. For the same reasons as in the Galois theory for simple algebras over fields, it is the regular subgroups and the regular subalgebras which the Galois theory relates.

THEOREM 10. *If H is a regular subgroup of G, then A^H is a regular R-subalgebra of A, and H is the group of all automorphisms of A leaving A^H fixed.*

Proof. The commutator of $R(H)$ in A is A^{H_0}; thus by Theorem 2 of [7], A^{H_0} is a separable R-algebra. Now H leaves $R(H)$ setwise invariant so H leaves A^{H_0} setwise invariant. Restricting H to A^{H_0} yields a group of automorphisms H' of A^{H_0} leaving A^H fixed which is isomorphic to H/H_0. Similarly H leaves K setwise invariant. Since all automorphisms of A leaving K fixed are inner by Theorem 3, H restricted to K is also isomorphic to H/H_0 and can be viewed as the restriction of H' to K. By our assumptions there is a finite subset J of H' with $K^J = K^{H'} = K \cap A^H$. By Corollary 2, $K \cap H^H$ is separable over R. The center of A^{H_0}, being the same as the center of $R(H)$, is projective and separable over K. Furthermore K is projective and separable over R so A^{H_0} is projective and separable over R. By Theorem 6 we know $(A^{H_0})^J = B$ is separable over K^J and $A^{H_0} = B \otimes K$

(tensor over K^J). Since K^H is separable over R, B is separable over R. Every idempotent in the center of A^{H_0} is in K by assumption on $R(H)$. Now $A^H cB$ but if $\sigma \in H$ and $\sigma|_B \neq e$ then $\sigma \otimes 1$ is an automorphism of $B \otimes K = A^{H_0}$ which is the identity on K so $\sigma \otimes 1$ extends by Theorem 3 to an element of H_0. This is impossible since $B c A^{H_0}$. Thus $A^H = B$ is a regular subalgebra of A.

To prove the last statement of the theorem observe that H_0 contains all the inner automorphisms leaving A^H fixed. Also $H' = H/H_0$ contains all the automorphisms of A^{H_0} leaving A^H fixed since $A^{H_0} = A^H \otimes K$ (tensor over $A^H \cap K$). Moreover H/H_0 is closed when viewed as a group of automorphisms of K.

THEOREM 11. *Let B be a regular subalgebra of A containing R, then the group G_B is a regular subgroup of B and B is the fixed ring of G_B.*

Proof. If C is the center of B, then $C \otimes_{B \cap K} K$ is the center of $B \otimes_{B \cap K} K$, so $C \cdot K$ is the center of $B \cdot K$ and is separable over R. By Lemma 9 $C \cap K$ is separable over R, therefore $B \cap K$ is separable over R. By our assumption there is a finite subset of G whose restrictions to K leave exactly R elementwise fixed so by Theorem 6 we can write $A = A^G \otimes K$. By Corollary 2 there is a finite subset J' of the R-automorphism group of K with $K^{J'} = B \cap K$. The representation of A as $A^G \otimes K$ insures that every element of J' extends to an element in G/G_0. Next we show the natural map $j: B \otimes_{B \cap K} K \to B \cdot K$ is an isomorphism. Now $\ker j = I \cdot (B \otimes_{B \cap K} K)$ where I is the kernel of the restriction of j to $C \otimes_{C \cap K} K$. Consider the exact sequence of $C \otimes_{C \cap K} K$-modules

$$0 \longrightarrow I \longrightarrow C \otimes_{C \cap K} K \longrightarrow C \cdot K \longrightarrow 0 .$$

Now $Z \cdot K$ is a projective R-module, so since $C \otimes_{C \cap K} K$ is a separable R-algebra $Z \cdot K$ is a projective $C \otimes_{C \cap K} K$-module. Hence the sequence splits and I is generated by an idempotent. Since every idempotent in $C \otimes_{C \cap K} K$ belongs to K, $I = 0$ and j is an isomorphism.

Extend each σ' in J' to $B \cdot K = B \otimes_{B \cap K} K$ by

$$\sigma'(bk) = b\sigma'(k) .$$

Lift σ', viewed as an element of G/G_0, to an element σ of G. Then σ restricted to K is σ' so $\sigma^{-1}\sigma': B \cdot K \to A$ leaves K fixed. Since every central idempotent in $B \cdot K$ belongs to K, $\sigma^{-1}\sigma'$ extends to an inner automorphism τ of A by Theorem 5. Now σ' is $\sigma\tau$ restricted to $B \cdot K$ so σ' extends to an automorphism of A. Each σ' in J' in this way extends to an element of G_B, and since J' defined on $B \cdot K$ leaves ex-

actly B fixed, G_B must leave exactly B fixed.

The commutator L in A of B is the commutator in A of $B \cdot K$, so is finitely generated and separable over K with every central idempotent of L in K. Clearly, any automorphism of G_B comes from a unit in L and every unit in L defines an inner automorphism in G_B. To show that $L = R(G_B)$ and consequently that G_B is regular, it suffices to show that L is generated as a K-algebra by its units. This follows from the definition of regularity for B.

Retain the notation of Theorem 11 in Theorem 12.

THEOREM 12. *If B is a regular subalgebra of A and G_B is a normal subgroup of G, then G/G_B is the group of all algebra automorphisms of B.*

Proof. Set $H = G_B$. The restriction map from G to the automorphism group of B has kernal H. Since H is normal in G, this implies every element in G/H can be viewed as an automorphism of B. The proof will be complete when we extend any R-automorphism σ' on B to an automorphism of A. Now $H' = H/H_0$ is a normal subgroup of $G' = G/G_0$ by the second isomorphism theorem so restricting our attention to K we know by Corollary 2 that $B \cap K$ is a normal separable extension of R with group G'/H' and G'/H' contains all automorphisms of $B \cap K$ leaving R fixed. View σ' as an element of G'/H' and extend to an element σ of G. Then $\tau' = \sigma^{-1}\sigma'$ is an automorphism of B leaving $B \cap K$ fixed. Define τ' on $B \otimes_{B \cap K} K = B \cdot K$ by $\tau'(bk) = \tau'(b)k$; then τ' is an automorphism of $B \cdot K$ leaving K fixed. By Theorem 5, τ' extends to an automorphism τ of G. For any $b \in B$, $\sigma\tau(b) = \sigma\tau'(b) = \sigma(\sigma^{-1}\sigma)(b) = \sigma'(b)$ so $\sigma\tau$ is the desired automorphism.

Following [5] we call a subalgebra B of A almost regular in case B is separable over R, every central idempotent in $B \cdot K$ belongs to K, and the commutator of $B \cdot K$ in A is generated by its units. Every regular algebra is almost regular and in this context the converse also holds.

THEOREM 14. *With A, K, R, and G as above, every almost regular R-subalgebra of A is regular.*

Proof. Let B be an almost regular R-subalgebra of A. Since every idempotent in $B \cdot K$ belongs to K it suffices to show that the natural map

$$j\colon B \otimes_{B \cap K} K \longrightarrow B \cdot K$$

is an isomorphism. As in the proof of Theorem 11 we can restrict our

attention to the natural map $j: C \otimes_{C \cap K} K \to C \cdot K$ where C is the center of B.

Since $C \cap K = L$ is separable over R by Lemma 9 and projective by the lemma on page 25 of [4] both C and K are projective when viewed as L-algebras. Thus $C \otimes_L K = C \cdot K \oplus \ker(j)$. By Theorem 2.3 of [4], $\ker(j) = 0$ for every $p \in \operatorname{spec} B(R)$ so $\ker(j) = 0$ and j is an isomorphism.

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