

# Pacific Journal of Mathematics

**SOME INCLUSIONS IN MULTIPLIERS**

RAOUF DOSS

## SOME INCLUSIONS IN MULTIPLIERS

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**$G$  is a compact abelian group. The main result of this paper is that if  $T$  is a  $(p, 1)$  multiplier,  $1 < p \leq 2$ , then  $T$  is a  $(p, s)$  multiplier for all  $s$  in the range  $1 \leq s < p$  and also an  $(r, r)$  multiplier for  $p < r < p'$  ( $p'$  conjugate of  $p$ ).**

An operator  $T$  defined on  $L^p(G)$ , whose range lies in the set of measurable functions on  $G$  is said to be of *weak type*  $(p, q)$  if there is a number  $A$  such that

$$m(\{x \in G: |Tf(x)| > t\}) \leq \left(\frac{A\|f\|_p}{t}\right)^q$$

for all  $f \in L^p$  and all  $t > 0$ . ( $m$  is Haar measure.)  $T$  need not be linear.

A linear operator, defined on  $L^p(G)$  is said to be of *strong type*  $(p, q)$  if there exists a number  $A$  such that

$$\|Tf\|_q \leq A\|f\|_p.$$

If  $T$  is of strong type  $(p, q)$  and commutes with translations (or equivalently with convolutions), then  $T$  is called a  $(p, q)$  multiplier. In this case we write  $T \in M_p^q$ . The Banach space of  $(p, p)$  multipliers is denoted  $M_p$ .

If  $T \in M_p^q$ , then there is a function  $\varphi$  on the dual of  $G$  such that  $(Tf)^\wedge = \varphi \hat{f}$ , for all  $f \in L^p$ , where  $\wedge$  denotes the Fourier transformation.  $T$  and  $\varphi$  are in one-to-one correspondence and we shall often write  $T_\varphi$  for  $T$ .

Using a deep theorem of E. M. Stein [3] on limits of sequences of operators we prove the following theorems:

**THEOREM 1.** *If  $T \in M_p^1$ ,  $1 \leq p \leq 2$  then  $T$  is of weak type  $(p, p)$ .*

**THEOREM 2.** *(converse.) Let  $T$  be a linear map of  $L^p$ ,  $1 < p \leq 2$  which commutes with translation and is of weak type  $(p, p)$ . Then  $T$  is of strong type  $(p, s)$  for all  $s$  in the range  $1 \leq s < p$ .*

These theorems imply the following corollaries.

**COROLLARY 1.** *If  $T \in M_p^1$ ,  $1 < p \leq 2$ , then  $T \in M_p^s$  for all  $s$  in the range  $1 \leq s < p$ .*

**COROLLARY 2.** *If  $T \in M_p^1$ ,  $1 \leq p \leq 2$ , then  $T \in M_r$  for all  $r$  in the*

range  $p < r < p'$  ( $p'$  conjugate of  $p$ ).

LEMMA 1. Let  $T = T_\varphi \in M_p^1$ ,  $1 < p \leq 2$  and let  $g \in L^1$ . Then  $T_{\hat{g}\varphi}$  is of weak type  $(p, p)$

*Proof.* The operator  $T_{\hat{g}\varphi}: L^p \rightarrow L^1$  is defined by  $(T_{\hat{g}\varphi}f)^\wedge = \hat{g}\varphi\hat{f}^\wedge = \hat{g}(Tf)^\wedge$ . Let  $g_n$  be the function  $g$  cut off at  $n$  and put  $T_n = T_{\hat{g}_n\varphi}: L^p \rightarrow L^1$ . If  $f \in L^p$ , then  $T_n f = g_n * Tf$  and since  $g_n$  is bounded, then  $T_n f \in L^\infty \subset L^p$ . Thus  $T_n$  is of strong type  $(p, p)$ , by the closed graph theorem.

We shall show that for any  $f \in L^p$  we have

$$(1) \quad \limsup |T_n f(x)| < \infty \quad \text{for almost every } x.$$

In fact, by Fubini's theorem

$$|g| * |Tf|(g) < \infty \quad \text{on a set } E \text{ of measure } 1.$$

By dominated convergence

$$T_n f(x) = g_n * Tf(x) \longrightarrow g * Tf(x) \quad x \in E \text{ which proves (1).}$$

A theorem of Stein [3] states that (1) implies that the operator  $T^*$  defined by

$$T^* f(x) = \sup |T_n f(x)|$$

is of weak type  $(p, p)$ . A fortiori the operator  $T_{\hat{g}\varphi}$  is of weak type  $(p, p)$  and the lemma is proved.

LEMMA 2. (Varopoulos-Johnson-Rieffel.) Let  $f_n \in L^p$ ,  $f_n \rightarrow 0$  in  $L^p$ . Then there are  $g \in L^1$  and  $g_n \in L^p$  such that  $f_n = g * g_n$  and  $g_n \rightarrow 0$  in  $L^p$ .

For a proof see Rieffel [2].

THEOREM 1. If  $T_\varphi \in M_p^1$ ,  $1 \leq p \leq 2$  then  $T$  is of weak type  $(p, p)$ .

*Proof.* Assume  $T_\varphi$  is not of weak type  $(p, p)$ . Then, to every positive integer  $n$  there corresponds a real number  $t_n$  and  $f_n \in L^p$  such that

$$(1) \quad m(\{x: |T_\varphi f_n(x)| > t_n\}) > \left(\frac{n \|f_n\|_p}{t_n}\right)^p.$$

Multiplying  $t_n$  and  $f_n$  by an appropriate constant we may suppose that  $\|f_n\|_p = n^{-1/2}$ . We now apply Lemma 2, writing

$$f_n = g * g_n, \quad g \in L^1$$

$$(2) \quad g_n \longrightarrow 0 \quad \text{in } L^p.$$

By Lemma 1, there is a constant  $A = A_g$  such that

$$m(\{x: |T_{\hat{g}_\varphi} g_n(x)| > t_n\}) \leq \left(\frac{A \|g_n\|_p}{t_n}\right)^p.$$

This is

$$m(\{x: |T_\varphi f_n(x)| > t_n\}) \leq \left(\frac{A \|g_n\|_p}{t_n}\right)^p.$$

Hence, by (1)

$$\frac{n \cdot n^{-1/2}}{t_n} = \frac{n \|f_n\|_p}{t_n} \leq \frac{A \|g_n\|_p}{t_n}$$

$$n^{1/2} \leq A \|g_n\|_p.$$

This contradicts (2). Therefore  $T_\varphi$  is of weak type  $(p, p)$  and the theorem is proved

**THEOREM 2.** (converse.) *Let  $T$  be a linear map of  $L^p, 1 < p \leq 2$ , which is of weak type  $(p, p)$  and which commutes with translation. Then  $T$  is of strong type  $(p, s)$  i.e.,  $T \in M_p^s$  for all  $s$  in the interval  $1 \leq s < p$ .*

*Proof.* We first show that if  $f \in L^p$ , then  $Tf \in L^s$ . For, there is a constant  $A$  such that for every  $f \in L^p$  and every positive  $t$  we have

$$(1) \quad m(\{x: |Tf(x)| > t\}) \leq \left(\frac{A \|f\|_p}{t}\right)^p.$$

Now it is well known (see, e.g., [1] 13.7.3) that for any nonnegative measurable function  $g$  we have

$$\int_G g^s = \int_0^\infty st^{s-1} m(\{x \in G: g(x) > t\}) dt.$$

Then, by (1), for  $1 \leq s < p$

$$\int_G |Tf|^s \leq \int_0^1 sm(G) dt + \int_1^\infty st^{s-1} \left(\frac{A \|f\|_p}{t}\right)^p dt < \infty.$$

This means  $Tf \in L^s$ . By the closed graph theorem we deduce easily  $T \in M_p^s$ .

**COROLLARY 1.** *If  $1 < p \leq 2, 1 \leq s < p$ , then  $M_p^1 = M_p^s$ .*

This is an immediate consequence of Theorems 1 and 2 and of the trivial inclusion  $M_p^s \subset M_p^1$ .

**COROLLARY 2.** *If  $1 \leq p < 2$ , then  $M_p^1 \subset \bigcap_{p < r \leq 2} M_r$ .*

*Proof.* Let  $T \in M_p^1$ . By Theorem 1,  $T$  is of weak type  $(p, p)$ .  $T$  is also strong type  $(2, 2)$ . The Marcinkiewicz interpolation theorem shows that  $T$  is of strong type  $(r, r)$  for each  $r$  satisfying  $p < r \leq 2$ . Since  $T$  commutes with translation, then  $T \in M_r$ , for  $p < r \leq 2$ .

**COROLLARY 3.** *If  $1 \leq p \leq 2$  then  $\bigcap_{p < r \leq 2} M_r = \bigcap_{p < r \leq 2} M_r^1$ .*

*Proof.* One part of the inclusion is due to  $M_r \subset M_r^1$ . The other part is a consequence of Corollary 2.

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