THE NUMERICAL RANGE OF AN OPERATOR

MARY RODRIGUEZ EMBRY
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Let $A$ be a continuous linear operator on a complex Hilbert space $X$ with inner product $\langle , \rangle$ and associated norm $\| \cdot \|$. Let $W(A) = \{ \langle Ax, x \rangle : \| x \| = 1 \}$ be the numerical range of $A$ and for each complex number $z$ let $M_z = \{ x : \langle Ax, x \rangle = z \| x \|^2 \}$. Let $\gamma M_z$ be the linear span of $M_z$ and $M_z \oplus M_z = \{ x + y : x \in M_z \text{ and } y \in M_z \}$. An element $z$ of $W(A)$ is characterized in terms of the set $M_z$ as follows:

**Theorem 1.** If $z \in W(A)$, then $\gamma M_z = M_z \oplus M_z$ and

(i) $z$ is an extreme point of $W(A)$ if and only if $M_z$ is linear;

(ii) if $z$ is a nonextreme boundary point of $W(A)$, then $\gamma M_z$ is a closed linear subspace of $X$ and $\gamma M_z = \bigcup \{ M_w : w \in L \}$, where $L$ is the line of support of $W(A)$, passing through $z$. In this case $\gamma M_z = X$ if and only if $W(A) \subset L$.

(iii) if $W(A)$ is a convex body, then $x$ is an interior point of $W(A)$ if and only if $\gamma M_z = X$.

It is well-known that $W(A)$ is a convex subset of the complex plane. Thus if $z \in W(A)$, either $z$ is an extreme point (not in the interior of any line segment with endpoints in $W(A)$), a nonextreme boundary point, or an interior point (with respect to the usual plane topology) of $W(A)$. Thus Theorem 1 characterizes every point of $W(A)$.

The following additional notation and terminology are used. If $K \subset X$, then $K^\perp$ denotes the orthogonal complement of $K$. An operator $A$ is **normal** if and only if $AA^* = A^*A$ and **hyponormal** only if $AA^* \ll A^*A$. A line $L$ is a **line of support** for $W(A)$ if and only if $W(A)$ lies in one of the closed half-planes determined by $L$ and $L \cap W(A) \neq \emptyset$.

In the last section of the paper consideration is given to $\bigcap \{ \text{maximal linear subspaces of } M_z \}$. One result is that if $A$ is hyponormal and $z$ a boundary point of $W(A)$, then $\bigcap \{ \text{maximal linear subspaces of } M_z \} = \{ x : Ax = zx \text{ and } A^*x = z^*x \}$. This generalizes Stampfli’s result in [3]: if $A$ is hyponormal and $z$ is an extreme point of $W(A)$, then $z$ is an eigenvalue of $A$. In [2] MacCluer proved this theorem for $A$ normal.

2. A proof of Theorem 1. Lemmas 1 and 2 provide the core of the proof of Theorem 1.

**Lemma 1.** Let $z$ be in the interior of a line segment with endpoints $a$ and $b$ in $W(A)$, $x \in M_a$, $y \in M_b$, $\| x \| = \| y \| = 1$. There exist
real numbers $s$ and $t$ in $(0, 1)$ and a complex number $\lambda$, $|\lambda| = 1$, such that $tx + (1 - t)\lambda y \in M_z$ and $sx - (1 - s)\lambda y \in M_z$. Consequently,

$$M_a \subset M_z \oplus M_z.$$ 

Proof. In proof of the convexity of $W(A)$ given in [1], pp. 317-318, it is shown that $tx + (1 - t)\lambda y \in M_z$ for some real number $t$ in $(0, 1)$ and some complex $\lambda$, $|\lambda| = 1$. A slight modification of the argument shows that $sx - (1 - s)\lambda y \in M_z$ for some real number $s$ in $(0, 1)$. Therefore, since $M_z$ is homogeneous and $s, t \in (0, 1)$, $x \in M_z \oplus M_z$, proving the last assertion.

Lemma 2. Let $L$ be a line of support of $W(A)$ and $N = \bigcup \{M_w \mid w \in L\}$.  

(i) There exists a real number $\theta$ such that $N = \{x \mid e^{i\theta}(A - z)x = e^{-i\theta}(A^* - z^*)x\}$ for all $z$ in $L$.

(ii) $N$ is a closed linear subspace of $X$.

(iii) $N = X$ if and only if $W(A) \subset L$.

Proof. (i) Let $\theta$ be such that $e^{i\theta}(w - z)$ is real for all $w$ and $z$ in $L$. Then $N = \{x \mid < e^{i\theta}(A - z)x, x > \text{ is real}\}$. Therefore since $L$ is a line of support of $W(A)$, Im $e^{i\theta}(A - z) = 0$ or $< 0$ and thus $N = \{x \mid e^{i\theta}(A - z)x = e^{-i\theta}(A^* - z^*)x\}$. Conclusion (ii) follows immediately from (i), and (iii) follows from the definition of $N$.

Proof of Theorem 1. Let $z \in W(A)$. (i) In Lemma 2 of [3] it is proven that $M_z$ is linear if $z$ is an extreme point of $W(A)$. If $z$ is not an extreme point of $W(A)$, $z$ is in the interior of a line segment with end points $a$ and $b$ in $W(A)$. By Lemma 1, $M_a \subset M_z \oplus M_z$. Since $a \neq z$, $M_a \cap M_z = \{0\}$. Therefore $M_z$ cannot be linear. (ii) Assume now that $z$ is a nonextreme boundary point of $W(A)$. Let $L$ be the line of support of $W(A)$, passing through $z$, and let $N = \bigcup \{M_w \mid w \in L\}$. Lemma 1 implies that $M_w \subset M_z \oplus M_z$ whenever $w \in L$; consequently, $N \subset M_z \oplus M_z$. Lemma 2 (ii) implies that $\gamma M_z \subset N$. Therefore, $M_z \oplus M_z = \gamma M_z = N$ and thus by Lemma 2 (iii) $\gamma M_z = X$ if and only if $W(A) \subset L$. (iii) Assume now that $W(A)$ is a convex body. If $z$ is an interior point of $W(A)$, Lemma 1 implies that $M_a \subset M_z \oplus M_z$ for each $a$ in $W(A)$. Therefore

$$X = \bigcup \{M_a \mid a \in W(A)\} \subset M_z \oplus M_z \subset \gamma M_z = X.$$ 

On the other hand if $z$ is a boundary point of $W(A)$ either $\gamma M_z = M_z$ or $\gamma M_z = N$ and in either case $\gamma M_z \neq X$ since $W(A)$ is a convex body.

3. $\bigcap \{\text{Maximal linear subspaces of } M_z\}$. Although $M_z$ may
not be linear, it is homogeneous and closed. Therefore if \( M_z \neq \{0\} \) and \( x \in M_z \), there exists a nonzero maximal linear subspace of \( M_z \), containing \( x \). Consideration of the intersection of these maximal linear subspaces yields information about eigenvalues and eigenvectors of \( A \).

**Theorem 2.** Let \( z \in W(A) \) and \( K_z = \bigcap \{\text{maximal linear subspaces of } M_z\} \). If \( z \) is a boundary point of \( W(A) \), let \( N = \bigcup \{M_w \mid w \in L\} \), where \( L \) is a line of support for \( W(A) \), passing through \( z \).

(i) If \( z \) is a boundary point of \( W(A) \), \( x \in K_z \), and \( Ax \in N \), then \( Ax = zx \) and \( A^*x = z^*x \). Conversely, if \( Ax = zx \) and \( A^*x = z^*x \), then \( x \in K_z \).

(ii) If \( W(A) \) is a convex body and \( z \) is in the interior of \( W(A) \), \( K_z = \{x \mid Ax = zx \text{ and } A^*x = z^*x\} \).

**Proof.** By elementary techniques it can be shown that for each complex \( z \)

1. \( K_z = M_z \cap [(A - z)(\gamma M_z)]^{\perp} \cap [(A^* - z^*)(\gamma M_z)]^{\perp} \) and that if \( z \) is extreme,
2. \( M_z \subset [(A - z)N]^{\perp} \cap [(A^* - z^*)N]^{\perp} \).

(The proof of (2) depends upon the fact that \( M_z \) is linear if \( z \) is extreme.) (i) Let \( z \) be a boundary point of \( W(A) \). By Theorem 1, \( K_z = M_z \) if \( z \) is extreme and \( \gamma M_z = N \) if \( z \) is nonextreme. Moreover, if \( x \in K_z \) and \( Ax \in N \), Lemma 2 implies that

\[(A - z)x \in N \text{ and } (A^* - z^*)x \in N.\]

It now follows from (1) and (2) that \( Ax = zx \) and \( A^*x = z^*x \). The converse follows immediately from (1). (ii) If \( W(A) \) is a convex body and \( z \) is in the interior of \( W(A) \), \( \gamma M_z = X \) by Theorem 1 and (1) implies that \( K_z = \{x \mid Ax = zx \text{ and } A^*x = z^*x\} \).

**Corollary 1.** If \( A \) is hyponormal and \( z \) is a boundary point of \( W(A) \), \( \bigcap \{\text{maximal linear subspaces of } M_z\} = \{x \mid Ax = zx \text{ and } A^*x = z^*x\} \). In particular, if \( z \) is an extreme point of \( W(A) \), \( z \) is an eigenvalue of \( A \).

**Proof.** Again let \( N = \bigcup \{M_w \mid w \in L\} \), where \( L \) is a line of support for \( W(A) \), passing through \( z \). In Lemma 3 of [3] Stampfli proves that \( A(N) \subset N \). Thus by Theorem 2, (i) \( K_z = \{x \mid Ax = zx \text{ and } A^*x = z^*x\} \). Moreover, if \( z \) is extreme, \( K_z = M_z \neq \{0\} \).

One last remark about potential eigenvalues and eigenvectors: it is immediate from Lemma 2 (i) that if \( z \) is a boundary point of \( W(A) \), \( Ax = zx \) if and only if \( A^*x = z^*x \).
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