DOMAIN-PERTURBED PROBLEMS FOR ORDINARY LINEAR DIFFERENTIAL OPERATORS

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The variation of the eigenvalues and eigenfunctions of an ordinary linear self-adjoint differential operator $L$ is considered under perturbations of the domain of $L$. The basic problem is defined as a suitable singular eigenvalue problem for $L$ on the open interval $\omega_- < s < \omega_+$ and is assumed to have at least one real eigenvalue $\lambda$ of multiplicity $k$. The perturbed problem is a regular self-adjoint problem defined for $L$ on a closed subinterval $[a, b]$ of $(\omega_-, \omega_+)$. It is proved under suitable conditions on the boundary operators of the perturbed problem that exactly $k$ perturbed eigenvalues $\mu_{ab}^i \to \lambda$ as $a, b \to \omega_-, \omega_+$. Further, asymptotic estimates are obtained for $\mu_{ab}^i - \lambda$ as $a, b \to \omega_-, \omega_+$. The other results are refinements which lead to asymptotic estimates for the eigenfunctions and variational formulæ for the eigenvalues.

Let $L$ be the $n$-th order ordinary linear differential operator defined by

$$(1.1) \quad Lx = \frac{1}{k(s)} \sum_{i=0}^{n} p_i(s)x^{(n-i)}(s)$$

on the open interval $\omega_- < s < \omega_+$, where $k$ and $p_i$, $i = 0, 1, \ldots, n$ are real-valued functions on this interval with the properties that

(i) $p_i \in C^{n-i}(\omega_-, \omega_+)$, $i = 0, 1, \ldots, n$;
(ii) $k$ is piecewise continuous on $(\omega_-, \omega_+)$; and
(iii) $p_0$ and $k$ are positive-valued. Furthermore the operator $k \cdot L$ is assumed to be formally self-adjoint, i.e. $k \cdot L$ coincides with its Lagrangian adjoint $[k \cdot L]^+$ where

$$(1.2) \quad [k \cdot L]^+ x = \sum_{i=0}^{n} (-1)^{n-i} [p_i x]^{(n-i)} .$$

The points $\omega_+$ and $\omega_-$ are in general singularities for $L$; the possibility that they are $\pm \infty$ is not excluded.

It will be convenient to use the following notations:

$$(1.3) \quad (x, y)_s^t = \int_{s}^{t} x(u)\overline{y(u)}k(u)du, \omega_- \leq s < t \leq \omega_+ ;$$

$$(1.4) \quad (x, y)_a^b = (x, y)_a^{\omega_+} ; (x, y)_b^c = (x, y)_b^{\omega_-} ;$$

$$(1.5) \quad (x, y) = (x, y)^{\omega_+}_{\omega_-} ;$$

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\( [xy](s) = \sum_{m=1}^{n} \sum_{j+k=m-1, j \geq 0, k \geq 0} (-1)^j x^{(j)}(s) [p_{n-m}(s) \overline{y(s)}]^{(k)}; \)

\( [xy](\pm) = \lim_{s \to \omega \pm} [xy](s). \)

Since the operator \( k \cdot L \) is formally self-adjoint Green's symmetric formula has the form

\( (Lx, y)_{\omega} = (x, Ly)_{\omega} = [xy](t) - [xy](s). \)

Let \( H, H[a, b] \) denote the Hilbert spaces which are the Lebesgue spaces with respective inner products \((x, y), (x, y)_{a}^{b}\) and norms \( \|x\| = (x, x)^{1/2}, \|x\|_{a} = [(x, x)_{a}^{b}]^{1/2}, \omega_{-} \leq a < b \leq \omega_{+}. \)

For \( c \) any intermediate point, \( \omega_{-} < c < \omega_{+} \), the symbols \( H(\omega_{-}, c), H(c, \omega_{+}) \) will similarly denote the Lebesgue spaces with respective inner products \((x, y)^{c}, (x, y)_{c}^{a}\) and norms \( \|x\|^{c} = [(x, x)^{c}]^{1/2}, \|x\|_{c} = [(x, x)_{c}^{a}]^{1/2}. \) From (1.8) it is clear that \([xy](+)\) (or \([xy](-)\)) exists provided \( x, y, Lx, Ly \) are in \( H(\omega_{-}, c) \) (or \( x, y, Lx, Ly \) are in \( H(\omega_{+}, c) \)).

Let \( a_{0} \) and \( b_{0} \) be fixed numbers satisfying \( \omega_{-} < a_{0} < b_{0} < \omega_{+} \) and let \( R_{0} \) be the rectangle in the \( a - b \)-plane described by the inequalities \( \omega_{-} < a \leq a_{0}, b_{0} \leq b < \omega_{+}. \) Then every closed, bounded interval \([a, b] \), \( \omega_{-} < a \leq a_{0}, b_{0} \leq b < \omega_{+}, \) can be associated in a one-to-one manner with a point of \( R_{0}. \) For \( k = 0, 1, \ldots, n - 1, \) let \( \alpha_{ik}(a), i = 1, 2, \ldots, m, \) and \( \beta_{jk}(b), j = 1, 2, \ldots, n - m \) be real-valued functions defined on the respective intervals \( \omega_{-} < a \leq a_{0}, b_{0} \leq b < \omega_{+}, \) such that for every \( [a, b] \in R_{0} \) the boundary operators

\[
\begin{align*}
U_{a}^{i}y &= \sum_{k=0}^{n-1} \alpha_{ik}(a)y^{(k)}(a), \quad i = 1, 2, \ldots, m \\
U_{b}^{j}y &= \sum_{k=0}^{n-1} \beta_{jk}(b)y^{(k)}(b), \quad j = 1, 2, \ldots, n - m
\end{align*}
\]

yield a linearly independent self-adjoint set of boundary conditions

\( \begin{cases} 
U_{a}^{i}y = 0, & i = 1, 2, \ldots, m \\
U_{b}^{i}y = 0, & i = 1, 2, \ldots, n - m
\end{cases} \)

for \( L \) (see [3] Chapter 11). Also for each \( [a, b] \in R_{0} \) let \( D[a, b] \) denote the set of all \( y \in H[a, b] \) which have the properties that

(i) \( y \in C^{n-1}[a, b], y^{(n-1)} \) is absolutely continuous on \([a, b];\)

(ii) \( Ly \in H[a, b]; \) and

(iii) \( y \) satisfies (1.10).

Then the self-adjoint eigenvalue problem

\( Ly = \mu y, \quad y \in D[a, b] \)

is known to have a countable set of real eigenvalues with no finite
cluster point and a corresponding set of (real) eigenfunctions complete in $H[a, b]$. Our problem is to obtain estimates for each eigenvalue $\mu = \mu_{ab}$ of (1.11) for $a, b$ near $\omega_-, \omega_+$ under hypotheses that will ensure that the limits of $\mu_{ab}$ as $a, b \to \omega_-, \omega_+$ will exist. Accordingly, eigenvalues $\lambda$ of suitable singular eigenvalue problems for $L$ on $(\omega_-, \omega_+)$ will be assumed to exist. If the eigenspace of $\lambda$ is $k$-dimensional the first theorem shows in particular that at least $k$ eigenvalues of (1.11) converge to $\lambda$ as $a, b \to \omega_-, \omega_+$. The other results are refinements of this which lead to asymptotic estimates for eigenfunctions. The method of estimation used is due to H. F. Bohnenblust [1]. Results like these have been previously obtained for second order cases by C. A. Swanson [8], [9]. See also [10] where he considers the biharmonic operator.

Let $l_0$ be any fixed complex number, $\text{Im} l_0 \neq 0$, and let $\psi_i, i = 1, 2, \cdots, n$, denote linearly independent solutions (hereafter to be referred to as basic solutions) of $L_0x = 0$ where $L_0 = L - l_0$. If basic solutions $\psi_i, i = 1, 2, \cdots, n$ exist such that the limit $|\psi_i/\psi_j|$ is either 0 or $\infty$ as $s \to \omega_+$ for each pair $\psi_i, \psi_j, i, j = 1, 2, \cdots, n, i \neq j$, then $\omega_+$ will be referred to as a class 1 singularity. On the other hand, $\omega_+$ will be called a class 2 singularity when the behaviour of the basic solutions is essentially arbitrary as $s \to \omega_+$. Similar definitions also apply to the singularity $\omega_-$. The singularity $\omega_+$ (or $\omega_-$) is further characterized by the number of basic solutions in $H[\omega_-, c]$ and in $H[c, \omega_+]$ where $c$ is any number satisfying $\omega_- < c < \omega_+$. For $n = 2$ this reduces to Weyl’s well-known limit circle, limit point classification of singular points [3, p. 225].

For the present perturbation problems will be considered for which both $\omega_-$ and $\omega_+$ are both class 2 singularities and all basic solutions are in $H(\omega_-, c)$ and in $H[c, \omega_+]$. In another paper class 1 singularities (and mixed cases) as well as examples will be considered.

2. Basic and perturbed problems. Rather than general spectral theory, one is interested in cases that the limits of $\mu_{ab}$ as $a, b \to \omega_-, \omega_+$ exist in an elementary sense. Thus, eigenvalues of suitable singular eigenvalue problems for $L$ on $(\omega_-, \omega_+)$ are supposed to exist. Such eigenvalue problems may be established by following basically the methods suggested by Kodaira [5] and Coddington [2]. Note that for the particular case $n = 2$, a theorem of Weyl [7] leads to singular “limit circle” problems which possess eigenvalues.

Let $D$ be the set of all $x \in H$ such that $x \in C^{n-1}(\omega_-, \omega_+)$ and $x^{(n-1)}$ is absolutely continuous on every closed bounded sub-interval of $(\omega_-, \omega_+)$. Let $\chi_i, i = 1, 2, \cdots, n$ be functions (to remain fixed) such that

( i) $L\chi_i \in H, i = 1, 2, \cdots, n$;

(ii) the end conditions $[x\chi_i](-) = 0, i = 1, 2, \cdots, m$ are linearly
independent; and

(iii) the end conditions \([x\chi_i](\pm) = 0, i = m + 1, m + 2, \cdots, n,\) are linearly independent.

Then the basic problem is the singular eigenvalue problem

\[ Lx = \lambda x, \quad x \in D_o \]

where \(D_o\) is the set of all \(x \in D\) such that

\[
\begin{align*}
[x\chi_i](-) &= 0, i = 1, 2, \cdots, m \\
[x\chi_i](+) &= 0, i = m + 1, \cdots, n.
\end{align*}
\]

Again (2.1) is to be a reasonable eigenvalue problem, i.e., at least one eigenvalue \(\lambda\) is supposed to exist which is assumed to be real. Note that the methods used by Coddington [2] and Kodaira [5] ensure that all eigenvalues are real. The eigenvalue problem (1.11) is to be regarded as a perturbation of (2.1) and hence will be referred to as the perturbed problem.

For the class of perturbation problems to be considered, the basic solutions are not necessarily ordered according to their asymptotic behaviour at \(\omega_+\) or at \(\omega_-\). Consequently strong conditions have to be imposed on the limiting behaviour of the boundary operators \(U_i, U_b^i\) as \(a, b \to \omega_-, \omega_+\). In particular every \(n - 1\) times differentiable function \(y\) shall satisfy

\[
\begin{align*}
[U_i^i y] = [y\chi_i]^m j [1 + o(1)] & \text{ as } a \to \omega_-, \quad i = 1, 2, \cdots, m \\
[U_i^b y] = [y\chi_i]^m j [1 + o(1)] & \text{ as } b \to \omega_+, \quad i = 1, 2, \cdots, n - m.
\end{align*}
\]

Let \(A\) denote the matrix \((A_{ij})\) where

\[ A_{ij} = \begin{bmatrix}
[y\chi_i](\pm), i = 1, 2, \cdots, n; j = 1, 2, \cdots, m
\end{bmatrix}
\]

and let \(Q = \det A\). Then since \(Q = \det A^t\), where \(A^t\) is the transpose of \(A\), and since \(l_0\) is nonreal it follows immediately that \(Q \neq 0\) (otherwise \(l_0\) would be an eigenvalue of (2.1)). Also for each \(j, j = 1, 2, \cdots, n,\) \(y_j, L_p y_j, \chi_j, L\chi_j\) are in \(H\); hence (1.8) implies that each limit \([y\chi_i](\pm)\) exists (finite) for \(i, j = 1, 2, \cdots, n,\). This implies that \(Q\) is equal to some nonzero constant.

Let \(A(a, b)\) denote the matrix \((A_{ij}(a, b))\) where

\[ A_{ij}(a, b) = \begin{bmatrix}
U_j^i y_j, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n
\end{bmatrix}
\]

and let \(Q(a, b) = \det A(a, b)\). Since \([y\chi_i](a)\) and \([y\chi_i](b)\) are finite as \(a \to \omega_-\) and \(b \to \omega_+\), for \(i, j = 1, 2, \cdots, n,\) it follows from (2.3) that numbers \(a_o, b_o,\) can be selected (which may be pre-supposed to be the original
choices) and a constant $C$ such that
\begin{equation}
(2.4) \quad |U_{\alpha}^i \psi_j| \leq C, \quad |U_{\beta}^j \psi_j| \leq C, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,
\end{equation}
whenever $\omega_ - < a \leq \alpha_0, b_0 \leq b < \omega_ +$. Also by (2.3) the element in the $i$-th row and $j$-th column in $A(a, b)$ approaches the element in the $i$-th row and $j$-th column in $A^i$ as $a, b \to \omega_-, \omega_+$. This implies that
\begin{equation}
(2.5) \quad \Omega(a, b) \to \Omega \neq 0
\end{equation}
as $a, b \to \omega_-, \omega_+$ and hence by (2.4) and (2.5) the numbers $\alpha_0, b_0$ previously chosen can be assumed to be such that $\Omega(a, b)$ is bounded above and away from zero whenever $\omega_ - < a \leq \alpha_0, b_0 \leq b < \omega_ +$.

3. Comparison of the basic and perturbed problems. The two problems (1.11) and (2.1) will be compared, with (1.11) regarded as a perturbation of (2.1). An estimate will be obtained for the variation of the eigenvalues and eigenfunctions under the perturbation $D_0 \to D[a, b]$. In particular it will be shown that this variation has the limit 0 as $a, b \to \omega_-, \omega_+$. Let $\lambda$ be an eigenvalue of (2.1) and let $A_\lambda$ denote the eigenspace of dimension $k$ corresponding to $\lambda$. Let $x_1, j = 1, 2, \ldots, k$ be an orthonormal basis for $A_\lambda$ and define $\tau_a^i(x), \tau_b^i(x), \Gamma_a(x)$ and $\Gamma_b(x)$ by
\begin{align}
(3.1) \quad & \tau_a^i(x) = \sum_{j=1}^k |U_{\alpha}^i x_j|; \quad \tau_b^i(x) = \sum_{j=1}^k |U_{\beta}^i x_j| ; \\
(3.2) \quad & \Gamma_a(x) = \sum_{i=1}^m \tau_a^i(x); \quad \Gamma_b(x) = \sum_{i=1}^{n-m} \tau_b^i(x).
\end{align}
Then (2.2) and (2.3) clearly imply that $\tau_a^i(x) = o(1), i = 1, 2, \ldots, m$ and $\tau_b^i(x) = o(1), i = 1, 2, \ldots, n - m$ and hence
\begin{equation}
(3.3) \quad \Gamma_a(x) = o(1), \quad \Gamma_b(x) = o(1)
\end{equation}
as $a \to \omega_-, b \to \omega_+$. The following theorem proves the convergence of the eigenvalues of (1.11) to those of (2.1).

**Theorem 1.** Let $\omega_-$ and $\omega_+$ be singularities for $L$ as described in §1. Let $\lambda$ be an eigenvalue of (2.1) possessing $k$ orthonormal eigenfunctions. Then under assumption (2.3) there exists a rectangle $R_0$, and a constant $C$ on $R_0$, such that at least $k$ perturbed eigenvalues $\mu_{ab}^i$ of (1.11) satisfy
\begin{equation}
(3.4) \quad |\mu_{ab}^i - \lambda| \leq C[\Gamma_a(x) + \Gamma_b(x)]
\end{equation}
whenever $[a, b] \in R_0$.

**Proof.** Let $G_{ab}(s, t)$ be the Green’s function for the operator $k \cdot L_0$. ...
associated with (1.10) and let $G_{ab}$ be the linear transformation on $H[a, b]$ defined by

$$G_{ab}y = \int_a^b G_{ab}(s, t)y(t)k(t)dt, \quad y \in H[a, b].$$

It is well-known [3, Chapter 7], that for any function $y \in H[a, b]$, the function $w = G_{ab}y$ is the unique solution in $D[a, b]$ of the differential equation $L_0w = y$. For $\lambda$ an eigenvalue and $x$ any corresponding normalized eigenfunction of (2.1), we define a function $f$ on $[a, b]$ by

$$f = x - \gamma G_{ab}x, \quad \gamma = \lambda - l_0.$$  

It is easily verified because of the linearity of all the operators involved that $f$ is a solution of the boundary problem

$$L_0f = 0, \quad U_i f = U_i x, \quad i = 1, 2, \ldots, m,$$

$$U_i f = U_i x, \quad i = 1, 2, \ldots, n - m.$$  

Let $K^j(a, b)$ denote the determinant of the matrix obtained from $A(a, b)$ by replacing the $j$-th column by

$$U_a x, U_a x, \ldots, U_b x, \ldots, U_{n-m} x.$$  

Then Cramer's rule yields the following representation of $f$ in terms of the basic solutions:

$$f(s) = \frac{1}{\Omega(a, b)} \sum_{j=1}^n K^j(a, b) \psi_j(s).$$  

The solution $f$ of (3.6) is unique for if $g$ is any solution of (3.6) then the function $h = g - f$ satisfies $L_0h = 0, U_i h = 0, \quad i = 1, 2, \ldots, m,$

$$U_i h = 0, \quad i = 1, 2, \ldots, n - m.$$  

This implies that $h$ is the zero function or $g = f$.

It follows from (2.4), (3.1) and (3.2) that there exists a constant $C$ such that

$$|K^j(a, b)| \leq C[\Gamma_a(x) + \Gamma_b(x)]$$

for each $j, j = 1, 2, \ldots, n$ whenever $[a, b] \in R_\delta$. This in addition to (2.5), (3.5), (3.7) and the fact that all the basic solutions are in $H$, enables one to deduce that there exists a constant $C$ such that

$$||x - \gamma G_{ab}x||_a \leq C(\Gamma_a(x) + \Gamma_b(x)) ||x||_a$$

whenever $[a, b] \in R_\delta$. The following fundamental lemma was obtained by H. F. Bohnenblust the proof of which is outlined in [8, p. 1554].

**Lemma 1.** Let $P(\delta)$ be the projection mapping from the Hilbert space $H[a, b]$ onto its subspace $H_\delta[a, b]$ ($\delta > 0$) spanned by all the
eigenfunctions $y_i$ of (1.11) such that the corresponding eigenvalues $\mu^i$ satisfy $|\mu^i - \lambda| \leq \delta$. Then for any $w \in H[a, b]$,

$$|| w - P(\delta)w ||_a^b \leq \left(1 + \frac{1}{\delta}\right)|| w - \gamma G_{ab}w ||_a^b .$$

It follows from (3.8) and Lemma 1 that there exists a constant $C$ on $R_o$ such that

$$(3.9) \quad || x - P(\delta)x ||_a^b \leq \frac{C}{2\delta}(\Gamma_a(x) + \Gamma_b(x)) || x ||_a^b .$$

With the choice $\delta = C[\Gamma_a(x) + \Gamma_b(x)]$, we obtain

$$(3.10) \quad || x - P(\delta)x ||_a^b \leq \frac{1}{2} || x ||_a^b$$

and conclude that $P(\delta)x = 0$ implies $x = 0$ on $[a, b]$. But $\dim A_i = k$; hence there exists at least $k$ perturbed eigenvalues $\mu^i_{ab}$ (counting multiplicities) of (1.11) such that

$$| \mu^i_{ab} - \lambda | \leq C[\Gamma_a(x) + \Gamma_b(x)]$$

for $[a, b] \in R_o$. This completes the proof of the theorem.

Theorem 1 and (3.3) show in particular that if $\lambda$ is a basic eigenvalue of multiplicity $k$ there exist at least $k$ perturbed eigenvalues $\mu^i_{ab}$ (counting multiplicities) such that $\mu^i_{ab} \to \lambda$ when $a, b \to \omega_-, \omega_+$. To obtain the stronger result that exactly $k$ perturbed eigenvalues $\mu^i_{ab}$ satisfy (3.4) in Theorem 1, we require the monotonicity property that the absolute value of the $n$-th eigenvalue of (2.1), $|\lambda_1| \leq |\lambda_2| \leq \cdots$, is not larger than the absolute value of the $n$-th eigenvalue of (1.11), $|\mu_1| \leq |\mu_2| \leq \cdots$. Then an inductive proof similar to that used in [8, p. 1554] yields the following result:

**Theorem 2.** If in addition to the hypotheses of Theorem 1 the above monotonicity property holds, then for every basic eigenvalue $\lambda$ of (2.1), of multiplicity $k$, there exists a rectangle $R_o$ and a constant $C$ on $R_o$, such that exactly $k$ eigenvalues $\mu^i_{ab}$ (counting multiplicities) of (1.11) satisfy (3.4) whenever $[a, b] \in R_o$.

**Theorem 3.** Let the hypotheses of Theorem 2 be satisfied. Then corresponding to the eigenvalues $\lambda$ and $\mu^i_{ab}$ of Theorem 2, there are orthogonal eigenfunctions $x^i$ on $[a, b]$ associated with $\lambda$ and $y^i$ associated with the $\mu^i_{ab}$ such that

$$|| y^i_{ab} - x^j ||_a^b \leq C[\Gamma_a(x) + \Gamma_b(x)] , \quad || x^i ||_a^b = || y^j ||_a^b = 1 ,$$

$\quad j = 1, 2, \cdots, k$, whenever $[a, b] \in R_o$. 

}\]
Proof. Let \( \{y^i\} \) be a set of orthonormal eigenfunctions on \([a, b]\) corresponding to the set of eigenvalues \( \{\mu^i_{\alpha}\} \) in Theorem 2. Then \( H_\delta[a, b] \) is \( k \)-dimensional by Theorem 2 and \( P(\delta) x = 0 \) implies \( x = 0 \) by (3.10). Hence there exist \( k \) unique linearly independent eigenfunctions \( z^i \) corresponding to \( \lambda \) which \( P(\delta) \) maps into the orthonormal eigenfunctions \( y^i \) and by (3.9)

\[
||z^i - y^i||_a^b = O(\Gamma_a(x) + \Gamma_b(x)), \quad [a, b] \in R_0.
\]

Since

\[
| (z^i, z^j)_a^b - (y^i, y^j)_a^b | \leq ||y^i||_a^b ||z^i - y^i||_a^b + ||z^i||_a^b ||z^j - y^j||_a^b
\]

by the Schwarz inequality

\[
(z^i, z^j)_a^b = \delta_{ij} + O(\Gamma_a(x) + \Gamma_b(x))
\]

for \( i, j = 1, 2, \cdots, k \) where \( \delta_{ij} \) denotes the Kronecker delta. Since the \( z^i \) are linearly independent, an orthonormal sequence \( x^i \) can be constructed by the Schmidt process as linear combinations of the \( z^i \) and it is easily verified that

\[
||x^i - z^i||_a^b = O(\Gamma_a(x) + \Gamma_b(x)).
\]

This combined with (3.11) gives the desired result.

4. Uniform estimate for eigenfunctions. For the class of singular problems under consideration, additional restrictions are needed on the basic solutions \( \psi_j, j = 1, 2, \cdots, n \), to obtain uniform estimates for \( y^i_a(s) - x^i(s), a \leq s \leq b, \) in Theorem 3. In particular the requirement will be that all basic solutions are bounded on \((\omega_-, \omega_+)\).

**Lemma 2.** Let \( G_{ab}(s, t) \) be the Green's function for \( k \cdot L_0 \) associated with (1.10). Then the positive function \( g_{ab}(s) \) defined by

\[
[g_{ab}(s)]^2 = \int_a^b |G_{ab}(s, t)|^2 k(t)dt
\]

is uniformly bounded on \( a \leq s \leq b \) provided \( a \leq a_0, b_0 \leq b \).

**Proof.** The Green's function \( G_{ab}(s, t) \) will be constructed first. From (1.6) it is clear that \([xy](s)\) may be written in the form

\[
[xy](s) = \sum_{i, j=0}^{n-1} B_{ij}(s)x^{(i)}(s)\overline{y^{(j)}}(s)
\]

with

\[
B_{ij}(s) = \begin{cases} (-1)^i P_\delta(s), & i + j = n - 1 \\
0, & i + j > n - 1. \end{cases}
\]
Let $B$ denote the $n$-by-$n$ matrix which has $B_{ij} = B_{ij}(s)$ in the $i + 1$-th row and $j + 1$-th column, $i, j = 0, 1, 2, \cdots, n - 1$. Then (4.2) implies that $B$ is nonsingular on $(\omega_- , \omega_+)$. Considering now the basic solutions one obtains from Green’s formula (1.8) that $[\psi_a \overrightarrow{\psi}_b](s)$ is a constant $[\psi_a \overrightarrow{\psi}_b]$ independent of $s, \alpha, \beta = 1, 2, \cdots, n$. With $S$ representing the matrix with element $[\psi_a \overrightarrow{\psi}_b]$ in the $\alpha$-th row and $\beta$-th column, it is easily verified that

(4.3) \[ S = Y^t B Y \]

where $Y$ denotes the Wronskian matrix $(\psi_{j-i}(s))$, $i, j = 1, 2, \cdots, n$ and $Y^t$ the transpose of the matrix $Y$. Since $B$, $Y$ (and hence $Y^t$) are nonsingular it follows that $S$ is a nonsingular constant matrix. Let $S^{-1} = (\gamma_{ab})$ denote the matrix inverse to $S$ and consider the function $K(s, t)$ defined by

(4.4) \[ K(s, t) = \sum_{\alpha, \beta=1}^{n} \gamma_{\alpha\beta} \psi_\alpha(t) \psi_\beta(s) . \]

Since $YS^{-1}Y^t = B^{-1}$ by (4.3) one obtains by inspection that

\[ K^{(i)}_s(s, t) = \begin{cases} 0, & i = 1, 2, \cdots, n - 2 \\ -1/p_\alpha(s), & i = n - 1. \end{cases} \]

Let

(4.5) \[ K_{ab}(s, t) = \begin{cases} K(s, t), & a \leq t \leq s \leq b \\ 0, & a \leq s \leq t \leq b \end{cases} \]

where $[a, b]$ is any closed sub-interval of $(\omega_- , \omega_+)$. Then from the above remarks it follows that

(4.6) \[ G_{ab}(s, t) = K_{ab}(s, t) + \sum_{k=1}^{n} A_k(t) \psi_k(s) \]

where $A_k(t)$, $k = 1, 2, \cdots, n$, is chosen in such a way that $G_{ab}(s, t)$, as a function of $s$, satisfies (1.10). Compare (4.6) with [4, Th. 8, p. 1319]. In particular, one obtains by Cramer’s rule that

\[ A_k(t) = \frac{\Omega_k^a(t)}{\Omega(a, b)} \]

where $\Omega_k^a(t)$ denotes the determinant of the matrix obtained from $A(a, b)$ by replacing the $k$-th column by the column whose $r$-th component $v_r$ is given by

\[ v_r = \begin{cases} 0, & r = 1, 2, \cdots, m \\ -\sum_{\alpha, \beta=1}^{n} \gamma_{\alpha\beta} \psi_\alpha(t) U_{\beta-r, \gamma}^m \psi_\gamma, & r = m + 1, \cdots, n . \end{cases} \]
Since $\psi_k \in H$, $k = 1, 2, \ldots, n$ it follows immediately from (2.4) and (2.5) that there exists a constant $C$ such that

$$\| A_k(t) \|^2 \leq C, \quad k = 1, 2, \ldots, n$$

whenever $a \leq a_0, b_0 \leq b$.

It follows from (4.1) that for $a \leq s \leq b$

$$g_{ab}(s) \leq \left\{ \int_a^s | G_{ab}(s, t) |^2 k(t) dt \right\}^{1/2} + \left\{ \int_s^b | G_{ab}(s, t) |^2 k(t) dt \right\}^{1/2}.$$

By (4.4), (4.6) and the triangle inequality we obtain that

$$\left\{ \int_a^s | G_{ab}(s, t) |^2 k(t) dt \right\}^{1/2} \leq \sum_{j=1}^n \left| \gamma_{ij} \psi_j(s) \right| \left\| \psi_j(t) \right\|^2_a + \sum_{j=1}^n \left| \psi_j(s) \right| \left\| A_j(t) \right\|^2_a.$$

But $\psi_j$ is bounded on $(\omega_-, \omega_+)$ and $\psi_j \in H$, $j = 1, 2, \ldots, n$; hence by (4.7) the first quantity on the right in (4.8) is uniformly bounded on $a \leq s \leq b$ provided $a \leq a_0$, $b_0 \leq b$. A similar proof shows that the second integral on the right in (4.8) is also uniformly bounded on $a \leq s \leq b$ provided $a \leq a_0$, $b_0 \leq b$. This gives the desired result. The next result gives uniform estimates for the eigenfunctions of Theorem 3.

**Theorem 4.** If in addition to the hypotheses of Theorem 3, $\psi_j$ is bounded on $(\omega_-, \omega_+)$, $j = 1, 2, \ldots, n$, then the eigenfunctions $x^j$ corresponding to $\lambda$ and $y^i_{ab}$ corresponding to $\mu^i_{ab}$ of Theorem 3 are such that

$$y^i_{ab}(s) = x^j(s) - f^i(s) + O[\Gamma_a(x)] + O[\Gamma_b(x)], \quad j = 1, 2, \ldots, k,$$

where $f^i(s)$ is the unique solution of the boundary problem

$$Lf = l_0 f, \quad U_i^j f = U^i_{ab} x^j, \quad i = 1, 2, \ldots, m,$$

$$U_i^j f = U^i_{ab} x^j, \quad i = 1, 2, \ldots, n - m.$$

**Proof.** The Schwarz inequality for $H[a, b]$ yields

$$| y^i_{ab}(s) - (\lambda - l_0)G_{ab} x^j(s) | = | G_{ab}[(\mu^i_{ab} - l_0)y^i_{ab}(s) - (\lambda - l_0)x^j(s)] | \leq g_{ab}(s) \left( | \mu^i_{ab} - l_0 | \left\| y^i_{ab} - x^j \right\|^2_a + | \mu^i_{ab} - \lambda | \left\| x^j \right\|^2_a \right).$$

Hence Lemma 2 and Theorems 2 and 3 show that there exists a constant $C$ such that

$$| y^i_{ab}(s) - (\lambda - l_0)G_{ab} x^j(s) | \leq C[\Gamma_a(x) + \Gamma_b(x)].$$
on \( a \leq s \leq b \), whenever \( a \leq a_0, b_0 \leq b \).

The solution \( f^j(s) \) of the boundary problem (4.10) is given by (3.5) or (3.7) with \( x \) replaced by \( x^j \). The function \( F^j \) defined by

\[
F^j(s) = (\lambda - l_0)G_{ab}x^j(s) - x^j(s) + f^j(s)
\]
satisfies

\[
LF^j = l_0F^j, \quad U_i^jF^j = 0, \quad i = 1, 2, \ldots, m,
\]

\[
U_i^jF^j = 0, \quad i = 1, 2, \ldots, n - m
\]

and hence \( F^j \) is the zero solution on \( a \leq s \leq b \) for \( j = 1, 2, \ldots, k \). This with (4.11) immediately gives the uniform estimates (4.9).

5. Asymptotic variational formulae for eigenvalues. The purpose here is to derive formulae for the change \( \mu_{ab}^j - \lambda \) of eigenvalues under the perturbation \( D_0 \to D[a, b] \), valid for \( a, b \) in neighbourhoods of \( \omega_- \), \( \omega_+ \) respectively. Let \( x^j, y^j \) denote the normalized eigenfunctions associated with \( \lambda \) and \( \mu^j = \mu_{ab}^j \) as described in Theorem 3 and let \( f^j \) be the unique solution of (4.10). One obtains the following theorem:

**Theorem 5.** Under the assumptions of Theorem 4 the following asymptotic variational formulae for the eigenvalues \( \lambda, \mu_{ab}^j \) are valid:

\[
\lambda - \mu_{ab}^j = [f^jx^j](b) - [f^jx^j](a)
\]

\[
+ (l_0 - \lambda)(f^j, f^j) + [\Gamma_\alpha(x) + \Gamma_\beta(x)](f^j, 1)O(1)
\]

as \( a, b \to \omega_-, \omega_+ \).

**Proof.** Let \( Uy = 0 \) denote the self-adjoint set of boundary conditions given by (1.10). Then by [3, Chapter 11] there exist boundary forms \( U_o, U_o^+ \) of rank \( n \) such that

\[
[uv](b) - [uv](a) = Uu \cdot U_o^+v + U_o^+u \cdot Uv
\]

for any pair \( u, v \in C_{a-1}^n[a, b] \), where \( \cdot \) represents the scalar product.

Now \( Uy^j = 0 \) by (1.10) and (1.11) and \( Ux^j = Uf^j \) by (4.10); hence (dropping the superscripts \( j \))

\[
[xy](b) - [xy](a) = Ux \cdot U_o^+y
\]

\[
= [fy](b) - [fy](a) .
\]

Then, application of Green's formula (1.8) to the differential equations \( Lx = \lambda x, \quad Lf = l_0f \) and \( Ly = \mu y \) on \([a, b]\), leads to

\[
(\lambda - \mu)(x, y)_a^b = (l_0 - \mu)(f, y)_a^b ;
\]

\[
[f^jx](b) - [f^jx](a) = (l_0 - \lambda)(f, x)_a^b .
\]
Hence one obtains as a consequence of Theorems 1, 2 and 3 that \( \mu = \lambda + o(1) \) and

\[
|x, y|^k - |x, x|^k \leq ||x||^k ||y - x||^k = o(1)
\]
as \( a, b \to \omega_-, \omega_+ \). Hence

\[
|\chi, y|^k = 1 + o(1), \quad a, b \to \omega_-, \omega_+
\]
and (5.2) yield

\[
(5.4) \quad \lambda - \mu = (\ell_0 - \lambda)(f, y)^k(1 + o(1)) .
\]

We now appeal to the uniform estimate (4.9) to obtain

\[
(5.5) \quad (f, y)^k = (f, x)^k - (f, f)^k + [\Gamma_\alpha(x) + \Gamma_\beta(x)](f, 1)^kO(1) .
\]

Then applying (5.3) and (5.5) to (5.4) the result (5.1) follows easily.

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