

Pacific Journal of Mathematics

**POSITIVE HOLOMORPHIC DIFFERENTIALS ON KLEIN
SURFACES**

NEWCOMB GREENLEAF AND WALTER READ

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Let \mathfrak{X} be a compact Klein surface with boundary ∂X , and let \mathcal{O} be an orientation of ∂X . We conjecture that there is a holomorphic differential which is positive on \mathcal{O} if and only if \mathcal{O} is not induced by an orientation of X , and we prove this when \mathfrak{X} is elliptic or hyperelliptic.

Let \mathfrak{X} be a Klein surface, with underlying topological space X , and let η be a meromorphic differential on \mathfrak{X} (for basic definitions and results see [1], [2]). If $g \in E(\mathfrak{X})$ is a nonconstant meromorphic function, then there is a unique $f \in E(\mathfrak{X})$ such that $\eta = f \cdot dg$.

Let B be an oriented component of ∂X , and let $-B$ be the same component with the opposite orientation. For $x \in B$ choose a local parameter $g \in E(\mathfrak{X})$ such that g is increasing on B near x . We say that η is positive on B at x if $\eta = f \cdot dg$ with $0 < f(x) < \infty$, and that η is positive on B if it is positive at all $x \in B$. It is easily checked that this definition does not depend on the choice of local parameters. Further η is positive on B or $-B$ if and only if it has no zeros or poles on B , and if η is positive on B , then $-\eta$ is positive on $-B$.

By an orientation \mathcal{O} of ∂X we mean an orientation of each component of ∂X . If ∂X has r components, then it has 2^r orientations. If X is orientable, then two of these are induced by the two possible orientations of X . If η is positive on each component of \mathcal{O} , we will say that it is positive on \mathcal{O} , and that \mathcal{O} has a positive differential.

In this note we investigate the following question: if \mathfrak{X} is a compact Klein surface and \mathcal{O} is an orientation of ∂X , does \mathcal{O} have a positive holomorphic differential. Our first result is in the negative direction.

THEOREM 1. *Let \mathfrak{X} be a compact orientable Klein surface, and let \mathcal{O} be an orientation of ∂X induced by an orientation of X . Then \mathcal{O} has no positive holomorphic differentials.*

Proof. Let \mathfrak{X}_1 be the analytic structure which is contained in the dianalytic structure \mathfrak{X} and which corresponds to the orientation of \mathfrak{X} which induces \mathcal{O} . If η is a holomorphic differential on X , we can as well regard it as a differential on \mathfrak{X}_1 , and we can then apply the Cauchy integral theorem to obtain $\int_{\mathcal{O}} \eta = 0$. If η were positive on \mathcal{O} , this integral would be strictly positive. Note that this proof

extends to meromorphic differentials of the second kind which have no poles on ∂X .

We conjecture that if an orientation \mathcal{O} is not induced by an orientation of X , then it has a positive holomorphic differential, but we can so far prove this only in the cases \mathfrak{X} elliptic or \mathfrak{X} hyperelliptic (i.e., when \mathfrak{X} can be represented as a double cover of the compactified upper half plane \mathfrak{D}).

THEOREM 2. *Let \mathfrak{X} be an elliptic or hyperelliptic Klein surface and let \mathcal{O} be an orientation of ∂X not induced by any orientation of X . Then \mathcal{O} has a positive holomorphic differential.*

Proof. Let \mathfrak{X} be an elliptic or hyperelliptic with $r \geq 1$ boundary components. We can find meromorphic functions f, g which generate $E(\mathfrak{X})$ over the reals, with $f^2 = H(g)$, where H is a real polynomial of degree n without multiple factors. Then the mapping associated with g represents \mathfrak{X} as a double cover of \mathfrak{D} , which is ramified at the zeros of H , and also at ∞ if n is odd. If H has no real zeros, then $r = 1$ or $r = 2$, depending on whether $n/2$, the number of ramified points in the interior of \mathfrak{D} , is odd or even. If H has $m \geq 1$ real zeros, then $r = [(m + 1)/2]$.

The genus of \mathfrak{X} is $\gamma = [(n - 1)/1]$, and the differentials $\{dg/f, g \cdot dg/f, \dots, g^{r-1} \cdot dg/f\}$ form a basis over R for the space of holomorphic differentials on \mathfrak{X} (see [3], p. 293). \mathfrak{X} may have two real points, one real point, or one complex point at infinity. The differential dg/f has all of its zeros at infinity. In the first case it has zeros of order $\gamma - 1$ at each such point, in the second a zero of order $2\gamma - 2$, and in the third a zero of order $\gamma - 1$.

Assume now that H has no real zeros. Then X is orientable. If $r = 1$, then every orientation of ∂X comes from an orientation of X , so there is nothing to prove. If $r = 2$, then $\gamma - 1 = n/2 - 2$ is even. The differential $(g^{r-1} + 1) \cdot dg/f$ has no zeros on ∂X and hence is positive with respect to some orientation \mathcal{O} , and its negative is positive on $-\mathcal{O}$.

Now assume that H has $m \geq 1$ real zeros. By choosing, if necessary, a new generator for $R(g)$, we may assume that \mathfrak{X} has a single complex point at infinity. Then H has $2r$ real zeros, and $n = 2(r + s)$, where s is the number of irreducible quadratic factors of H . Let the real zeros of H , in increasing order, be $a_1, b_1, \dots, a_r, b_r$, and pick c_j between b_j and a_{j+1} , $j = 1, \dots, r - 1$. Then the components of ∂X lie over the intervals $[a_j, b_j]$, $j = 1, \dots, r$. Let $J \subset \{1, \dots, r - 1\}$ be any set of cardinality at most $\gamma - 1$, and set

$$\eta_J = \prod_{j \in J} (g - c_j) \cdot dg / f .$$

Each of the differentials $\pm \eta_J$ is positive with respect to a different orientation of ∂X . Hence for $\gamma \geq r$ we obtain positive differentials for all 2^r possible orientations of ∂X , and the theorem is proved. So assume that $\gamma < r$. Since $r + 1 = n/2 = r + s$, we must have $s = 0$ and $\gamma = r - 1$. Because $s = 0$, X is orientable, and because $\gamma = r - 1$ we can use all subsets J except $J = \{1, \dots, r - 1\}$. We have thus obtained positive differentials for $2^r - 2$ different orientations of ∂X , and have completed the proof of the theorem.

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