POSITIVE HOLOMORPHIC DIFFERENTIALS ON KLEIN SURFACES

Newcomb Greenleaf and Walter Read
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NEWCOMB GREENLEAF AND WALTER READ

Let $\mathcal{X}$ be a compact Klein surface with boundary $\partial X$, and let $\mathcal{O}$ be an orientation of $\partial X$. We conjecture that there is a holomorphic differential which is positive on $\mathcal{O}$ if and only if $\mathcal{O}$ is not induced by an orientation of $X$, and we prove this when $\mathcal{X}$ is elliptic or hyperelliptic.

Let $\mathcal{X}$ be a Klein surface, with underlying topological space $X$, and let $\eta$ be a meromorphic differential on $X$ (for basic definitions and results see [1], [2]). If $g \in E(\mathcal{X})$ is a nonconstant meromorphic function, then there is a unique $f \in E(\mathcal{X})$ such that $\eta = f \cdot dg$.

Let $B$ be an oriented component of $\partial X$, and let $-B$ be the same component with the opposite orientation. For $x \in B$ choose a local parameter $g \in E(\mathcal{X})$ such that $g$ is increasing on $B$ near $x$. We say that $\eta$ is positive on $B$ at $x$ if $\eta = f \cdot dg$ with $0 < f(x) < \infty$, and that $\eta$ is positive on $B$ if it is positive at all $x \in B$. It is easily checked that this definition does not depend on the choice of local parameters. Further $\eta$ is positive on $B$ or $-B$ if and only if it has no zeros or poles on $B$, and if $\eta$ is positive on $B$, then $-\eta$ is positive on $-B$.

By an orientation $\mathcal{O}$ of $\partial X$ we mean an orientation of each component of $\partial X$. If $\partial X$ has $r$ components, then it has $2^r$ orientations. If $X$ is orientable, then two of these are induced by the two possible orientations of $X$. If $\eta$ is positive on each component of $\mathcal{O}$, we will say that it is positive on $\mathcal{O}$, and that $\mathcal{O}$ has a positive differential.

In this note we investigate the following question: if $\mathcal{X}$ is a compact Klein surface and $\mathcal{O}$ is an orientation of $\partial X$, does $\mathcal{O}$ have a positive holomorphic differential. Our first result is in the negative direction.

**Theorem 1.** Let $\mathcal{X}$ be a compact orientable Klein surface, and let $\mathcal{O}$ be an orientation of $\partial X$ induced by an orientation of $X$. Then $\mathcal{O}$ has no positive holomorphic differentials.

**Proof.** Let $\mathcal{X}_i$ be the analytic structure which is contained in the dianalytic structure $\mathcal{X}$ and which corresponds to the orientation of $\mathcal{X}$ which induces $\mathcal{O}$. If $\eta$ is a holomorphic differential on $X$, we can as well regard it as a differential on $\mathcal{X}_i$, and we can then apply the Cauchy integral theorem to obtain $\int_{\mathcal{O}} \eta = 0$. If $\eta$ were positive on $\mathcal{O}$, this integral would be strictly positive. Note that this proof
extends to meromorphic differentials of the second kind which have no poles on $\partial X$.

We conjecture that if an orientation $\mathcal{O}$ is not induced by an orientation of $X$, then it has a positive holomorphic differential, but we can so far prove this only in the cases $X$ elliptic or $X$ hyperelliptic (i.e., when $X$ can be represented as a double cover of the compactified upper half plane $\mathbb{D}$).

**Theorem 2.** Let $X$ be an elliptic or hyperelliptic Klein surface and let $\mathcal{O}$ be an orientation of $\partial X$ not induced by any orientation of $X$. Then $\mathcal{O}$ has a positive holomorphic differential.

**Proof.** Let $X$ be an elliptic or hyperelliptic with $r \geq 1$ boundary components. We can find meromorphic functions $f, g$ which generate $E(\mathfrak{x})$ over the reals, with $f^2 = H(g)$, where $H$ is a real polynomial of degree $n$ without multiple factors. Then the mapping associated with $g$ represents $X$ as a double cover of $\mathbb{D}$, which is ramified at the zeros of $H$, and also at $\infty$ if $n$ is odd. If $H$ has no real zeros, then $r = 1$ or $r = 2$, depending on whether $n/2$, the number of ramified points in the interior of $\mathfrak{x}$, is odd or even. If $H$ has $m \geq 1$ real zeros, then $r = [(m + 1)/2]$.

The genus of $X$ is $\gamma = [(n - 1)/2]$, and the differentials $\{dg/f, g \cdot dg/f, \ldots, g^{-1} \cdot dg/f\}$ form a basis over $R$ for the space of holomorphic differentials on $X$ (see [3], p. 293). $X$ may have two real points, one real point, or one complex point at infinity. The differential $dg/f$ has all of its zeros at infinity. In the first case it has zeros of order $\gamma - 1$ at each such point, in the second a zero of order $2\gamma - 2$, and in the third a zero of order $\gamma - 1$.

Assume now that $H$ has no real zeros. Then $X$ is orientable. If $r = 1$, then every orientation of $\partial X$ comes from an orientation of $X$, so there is nothing to prove. If $r = 2$, then $\gamma - 1 = n/2 - 2$ is even. The differential $(g^{-1} + 1) \cdot dg/f$ has no zeros on $\partial X$ and hence is positive with respect to some orientation $\mathcal{O}$, and its negative is positive on $-\mathcal{O}$.

Now assume that $H$ has $m \geq 1$ real zeros. By choosing, if necessary, a new generator for $R(g)$, we may assume that $X$ has a single complex point at infinity. Then $H$ has $2r$ real zeros, and $n = 2(r + s)$, where $s$ is the number of irreducible quadratic factors of $H$. Let the real zeros of $H$, in increasing order, be $a_1, b_1, \ldots, a_r, b_r$, and pick $c_j$ between $b_j$ and $a_{j+1}$, $j = 1, \ldots, r - 1$. Then the components of $\partial X$ lie over the intervals $[a_j, b_j], j = 1, \ldots, r$. Let $J \subset \{1, \ldots, r - 1\}$ be any set of cardinality at most $\gamma - 1$, and set
\[ \eta_J = \prod_{j \in J} (g - c_j) \cdot d\eta^j. \]

Each of the differentials \( \pm \eta_J \) is positive with respect to a different orientation of \( \partial X \). Hence for \( \gamma \geq r \) we obtain positive differentials for all \( 2^r \) possible orientations of \( \partial X \), and the theorem is proved. So assume that \( \gamma < r \). Since \( r + 1 = n/2 = r + s \), we must have \( s = 0 \) and \( \gamma = r - 1 \). Because \( s = 0 \), \( X \) is orientable, and because \( \gamma = r - 1 \) we can use all subsets \( J \) except \( J = \{1, \ldots, r - 1\} \). We have thus obtained positive differentials for \( 2^r - 2 \) different orientations of \( \partial X \), and have completed the proof of the theorem.

REFERENCES


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