

# Pacific Journal of Mathematics

**ON THE CONVERGENCE OF A TRIGONOMETRIC INTEGRAL**

R. MOHANTY AND B. K. RAY

## ON THE CONVERGENCE OF A TRIGONOMETRIC INTEGRAL

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**In the present paper, we shall first establish a theorem concerning the convergence of a trigonometric integral. Then in the final section, we shall evaluate some known definite integrals with the help of our theorem.**

1. DEFINITION. We say that the integral  $\int_0^\infty a(u)du$  is summable  $(C, 1)$  to sum  $S$ , if

$$\lim_{\lambda \rightarrow \infty} \int_0^\lambda \left(1 - \frac{u}{\lambda}\right) a(u) du = S.$$

In [1], a result regarding the  $(C, 1)$  summability of a trigonometric integral was proved which is equivalent to

**THEOREM A.** *Let  $f(t)$  be  $L$  over  $(0, \infty)$ . Then, for  $0 < \alpha < 1$ , the integral*

$$\int_0^\infty u^\alpha du \int_0^\infty f(t) \sin ut dt$$

*is summable  $(C, 1)$  to*

$$\Gamma(\alpha + 1) \text{Cos} \frac{1}{2} \alpha \pi \int_{-0}^\infty \frac{f(t)}{t^{1+\alpha}} dt$$

*whenever this integral exists and whenever*

$$f(t) = O(t^\alpha) \text{ as } t \rightarrow 0.$$

In § 2 of the present paper we establish the following theorem.

**THEOREM.** *Let  $t^{-\alpha} f(t)$  ( $0 < \alpha < 1$ ) be of bounded variation over  $(0, \infty)$  and tend to zero both as  $t \rightarrow 0$  and  $t \rightarrow \infty$ . If the integral  $\int_0^\infty f(t) \sin ut dt$  is uniformly convergent with respect to  $u$  over  $0 < \mu \leq u \leq \lambda < \infty$ , for every  $\mu$  and  $\lambda$ , then*

$$(1.1) \quad \int_{-0}^\infty u^\alpha du \int_0^\infty f(t) \sin ut dt = \Gamma(\alpha + 1) \cos \frac{1}{2} \alpha \pi \int_{-0}^\infty \frac{f(t)}{t^{1+\alpha}} dt$$

*whenever the last integral exists.*

In the present problem  $f(t)$  is not necessarily  $L$  over  $(0, \infty)$ . In

§ 3 we shall evaluate some known definite integrals with the help of the above theorem.

2. For the proof of the theorem, we use the following simple lemma.

LEMMA. *If the function  $g(t)$  is positive and nonincreasing over the interval  $(a, \infty)$ , then*

$$\left| \int_a^{\infty} g(t) \cos t dt \right| \leq Ag(a) .^1$$

*Proof of the theorem.* We write  $h(t) = t^{-\alpha}f(t)$ . For any  $\varepsilon > 0$ , there is a  $\delta$  such that  $|h(t)| < \varepsilon$  for all  $t < \delta$  and for all  $t > 1/\delta$ . For the sake of simplicity, we shall drop the sign  $\rightarrow$  at infinity in the proof. We have

$$(2.1) \quad \int_{\mu}^{\lambda} u^{\alpha} du \int_0^{\infty} f(t) \sin ut dt = \int_0^{\infty} f(t) dt \int_{\mu}^{\lambda} u^{\alpha} \sin ut du$$

since the inversion of order of integration is justified by the uniform convergence of the inner integral of the left side of (2.1). Using integration by parts, we get

$$\int_{\mu}^{\lambda} u^{\alpha} \sin ut du = \frac{\mu^{\alpha}}{t} \cos \mu t - \frac{\lambda^{\alpha}}{t} \cos \lambda t - \frac{\alpha}{t^{\alpha+1}} \int_{\mu t}^{\lambda t} \frac{\cos u}{u^{1-\alpha}} du ,$$

and then the equation (2.1) becomes

$$\begin{aligned} & \int_{\mu}^{\lambda} u^{\alpha} du \int_0^{\infty} f(t) \sin ut dt \\ &= \mu^{\alpha} \int_0^{\infty} \frac{h(t)}{t^{1-\alpha}} \cos \mu t dt - \lambda^{\alpha} \int_0^{\infty} \frac{h(t)}{t^{1-\alpha}} \cos \lambda t dt + \alpha \int_0^{\infty} \frac{h(t)}{t} dt \int_{\mu t}^{\lambda t} \frac{\cos u}{u^{1-\alpha}} du \\ &= I - J + K . \end{aligned}$$

Now

$$\begin{aligned} |I| &= \mu^{\alpha} \left| \int_0^{\varepsilon/\mu} + \int_{\varepsilon/\mu}^{\infty} \right| \\ &\leq A \mu^{\alpha} \int_0^{\varepsilon/\mu} \frac{dt}{t^{1-\alpha}} + \left| \int_{\varepsilon}^{\infty} \frac{h(v/\mu)}{v^{1-\alpha}} \cos v dv \right| \\ &\leq A \varepsilon^{\alpha} + o(1) \quad \text{as } \mu \rightarrow 0 , \end{aligned}$$

by applying the lemma for the last integral, after writing  $h(t)$  as the difference of two functions which tend to zero monotonically. Similarly

<sup>1</sup> Throughout the present paper we write  $A$  for an arbitrary constant which is not necessarily the same at each occurrence.

$$\begin{aligned}
 |J| &= \lambda^\alpha \left| \int_0^{1/\lambda} + \int_{1/\lambda}^{1/\varepsilon\lambda} + \int_{1/\varepsilon\lambda}^\infty \right| \\
 &\leq o(1)\lambda^\alpha \int_0^{1/\lambda} \frac{dt}{t^{1-\alpha}} + A \int_1^{1/\varepsilon} |h(v/\lambda)| dv + \left| \int_{1/\varepsilon}^\infty \frac{h(v/\lambda)}{v^{1-\alpha}} \cos v dv \right| \\
 &\leq A\varepsilon^{1-\alpha} + o(1) \quad \text{as } \lambda \rightarrow \infty.
 \end{aligned}$$

Thus it is sufficient to prove that

$$(2.2) \quad \limsup_{\lambda \rightarrow \infty, \mu \rightarrow 0} \left| k - \alpha \int_{1/\lambda}^\infty \frac{h(t)}{t} dt \int_0^\infty \frac{\cos u}{u^{1-\alpha}} du \right| \leq A\varepsilon^\alpha,$$

since

$$\int_0^\infty u^{\alpha-1} \cos u du = \Gamma(\alpha) \cos \frac{1}{2} \alpha \pi.$$

The term inside the absolute value sign of (2.2) is

$$\begin{aligned}
 k - \alpha \int_{1/\lambda}^\infty \frac{h(t)}{t} dt \int_0^\infty \frac{\cos u}{u^{1-\alpha}} du &= \alpha \int_0^{1/\lambda} \frac{h(t)}{t} dt \int_{\lambda t}^{\lambda t} \frac{\cos u}{u^{1-\alpha}} du \\
 &\quad - \alpha \int_{1/\lambda}^\infty \frac{h(t)}{t} dt \int_0^{\mu t} \frac{\cos u}{u^{1-\alpha}} du \\
 &\quad - \alpha \int_{1/\lambda}^\infty \frac{h(t)}{t} dt \int_{\lambda t}^\infty \frac{\cos u}{u^{1-\alpha}} du \\
 &= \alpha(L - M - N).
 \end{aligned}$$

Now,

$$\begin{aligned}
 |L| &\leq \int_0^{1/\lambda} \frac{|h(t)|}{t^1} dt \int_0^{\lambda t} \frac{du}{u^{1-\alpha}} \\
 &\leq A\lambda^\alpha \int_0^{1/\lambda} \frac{|h(t)|}{t^{1-\alpha}} dt = o(1) \quad \text{as } \lambda \rightarrow \infty.
 \end{aligned}$$

By the formula

$$\int_0^t \frac{\cos \mu v}{v^{1-\alpha}} dv = \frac{\sin \mu t}{\mu t^{1-\alpha}} + \frac{1-\alpha}{\mu} \int_0^t \frac{\sin \mu v}{v^{2-\alpha}} dv,$$

we get

$$\begin{aligned}
 M &= \mu^\alpha \int_{1/\lambda}^\infty \frac{h(t)}{t} dt \int_0^t \frac{\cos \mu v}{v^{1-\alpha}} dv \\
 &= \frac{1}{\mu^{1-\alpha}} \int_{1/\lambda}^\infty \frac{h(t) \sin \mu t}{t^{2-\alpha}} dt + \frac{1-\alpha}{\mu^{1-\alpha}} \int_{1/\lambda}^\infty \frac{h(t)}{t} dt \int_0^t \frac{\sin \mu v}{v^{2-\alpha}} dv \\
 &= M_1 + M_2
 \end{aligned}$$

where

$$\begin{aligned}
 |M_1| &= \left| \int_{\mu/\lambda}^{\infty} \frac{h(v/\mu) \sin v}{v^{2-\alpha}} dv \right| \\
 &\leq \left| \int_{\mu/\lambda}^{\epsilon} \right| + \left| \int_{\epsilon}^{\infty} \right| \leq A\epsilon^\alpha + o(1) \quad \text{as } \mu \rightarrow 0
 \end{aligned}$$

and

$$\begin{aligned}
 M_2 &= \frac{A}{\mu^{1-\alpha}} \int_{1/\lambda}^{\infty} \frac{h(t)}{t} dt \int_0^{1/\lambda} \frac{\sin \mu v}{v^{2-\alpha}} dv \\
 &\quad + \frac{A}{\mu^{1-\alpha}} \int_{1/\lambda}^{\infty} \frac{\sin \mu v}{v^{2-\alpha}} dv \int_v^{\infty} \frac{h(t)}{t} dt \\
 &= o(1) + A \int_{\mu/\lambda}^{\infty} \frac{\sin \omega}{\omega^{2-\alpha}} d\omega \int_{\omega/\mu}^{\infty} \frac{h(t)}{t} dt
 \end{aligned}$$

where the change of order of integration is easily proved, and then

$$|M_2| \leq A\epsilon^\alpha + o(1).$$

Finally

$$\begin{aligned}
 |N| &\leq \frac{A}{\lambda^{1-\alpha}} \int_{1/\lambda}^{\delta} \frac{|h(t)|}{t^{2-\alpha}} dt + \int_{\delta}^{1/\delta} \frac{|h(t)|}{t} dt \int_{\lambda t}^{\infty} \frac{\cos v}{v^{1-\alpha}} dv \\
 &\quad + \frac{A}{\lambda^{1-\alpha}} \int_{1/\delta}^{\infty} \frac{|h(t)|}{t^{2-\alpha}} dt \\
 &\leq A\epsilon + o(1) \quad \text{as } \lambda \rightarrow \infty.
 \end{aligned}$$

Thus we get the required inequality (2.2) and the theorem is completely proved.

3. Evaluation of integrals. Let us consider the function

$$f(t) = t/(1 + t^2) \quad (0, \infty).$$

Then the integral of the left side of (1.1) for the present function reduces to

$$\begin{aligned}
 &\int_0^{\infty} u^\alpha du \int_0^{\infty} \frac{t}{1 + t^2} \sin ut dt \\
 &= \frac{\pi}{2} \int_0^{\infty} u^\alpha e^{-u} du = \frac{\pi}{2} \Gamma(\alpha + 1).
 \end{aligned}$$

Obviously, the function satisfies all the conditions of the theorem, so we have

$$\frac{\pi}{2} \Gamma(\alpha + 1) = \Gamma(\alpha + 1) \cos \frac{1}{2} \alpha \pi \int_0^{\infty} \frac{t^{-\alpha}}{1 + t^2} dt$$

i.e.,

$$(3.1) \quad \int_0^\infty \frac{t^{-\alpha}}{1+t^2} dt = \frac{\pi}{2 \cos \frac{1}{2} \alpha \pi} \quad \text{for } 0 < \alpha < 1.$$

Next we consider the function

$$f(t) = t^3/(1+t^4) \quad (0, \infty).$$

Obviously, this function satisfies the hypotheses of the theorem of the present paper. The integral on the left side of (1.1) for the present function reduces to

$$\begin{aligned} & \int_0^\infty u^\alpha du \int_0^\infty \frac{t^3}{1+t^4} \sin ut dt \\ &= \frac{\pi}{2} \int_0^\infty u^\alpha e^{-u/\sqrt{2}} \cos \frac{u}{\sqrt{2}} du = \frac{\pi}{2} \Gamma(\alpha + 1) \cos(\alpha + 1) \frac{\pi}{4}. \end{aligned}$$

Now by the theorem of the present note, we have

$$\frac{\pi}{2} \Gamma(\alpha + 1) \cos(\alpha + 1) \frac{\pi}{4} = \Gamma(\alpha + 1) \cos \frac{1}{2} \alpha \pi \int_0^\infty \frac{t^{2-\alpha}}{1+t^4} dt.$$

Therefore

$$\int_0^\infty \frac{t^{2-\alpha}}{1+t^4} dt = \frac{\pi \cos(\alpha + 1)\pi/4}{\cos \frac{1}{2} \alpha \pi} \quad \text{for } 0 < \alpha < 1.$$

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### REFERENCE

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