ON A CLASS OF NODAL ALGEBRAS

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In this paper it is shown that there do not exist nodal algebras $A$ satisfying the conditions:

(I) $x(xy) + (yx)x = 2(xy)x$

(II) $(xy)x - x(yx)$ is in $N$, the set of nilpotent elements of $A$, over any field $F$ of characteristic zero. Also several results regarding algebras satisfying (I) alone are established.

A finite dimensional power-associative algebra $A$ with identity 1 over a field $F$ is called a nodal algebra [7] if every element $x$ of $A$ can be represented in the form $x = a1 + n$ where $a$ is in $F$ and $n$ is nilpotent and if the set $N$ of nilpotent elements of $A$ is not a subalgebra of $A$. It is known [5] that there are no nodal flexible algebras over any field $F$ of characteristic zero. (An algebra is said to be flexible if the identity $(xy)x = x(yx)$ is satisfied). There do exist, however, nodal algebras over fields $F$ of characteristic zero in which $(xy)x - x(yx)$ is in $N$ for all elements $x, y$ of the algebra [3]. Algebras satisfying (I) were first studied by Kosier [6]. The concern, however, was for algebras of degree $> 1$.

Throughout, we shall be using the result of Albert [2, p. 526] who proved that there are no commutative nodal algebras over any field $F$ of characteristic zero by showing that $N$ forms a subalgebra. In the noncommutative case we let $A^+$ be the same vector space as $A$ with multiplication in $A^+$ given by $x \cdot y = 1/2(xy + yx)$, $xy$ the multiplication in $A$. Then $N$ is a subalgebra of $A^+$. In particular, $N$ is a vector space. We use the standard notation, $[x, y]$ for the commutator $xy - yx$ and $(x, y, z)$ for the associator $(xy)z - x(yz)$.

2. It is a well known fact that if an algebra $A$ is power-associative then $A^+$ is power-associative. For algebras satisfying (I) the converse is also true.

**THEOREM 1.** If $A$ is an algebra satisfying (I) over a field $F$ of characteristic $\neq 2$ and if $A^+$ is power-associative then $A$ is power-associative.

**Proof.** The following lemma is due to Witthoft [8].

**LEMMA 1.1.** $xx^n = x^n x$ for all $x$ in $A$ and for all $n$.

The proof is by induction on $n$. Trivially the lemma holds if
\( n = 1 \). Assume it holds for \( n = k - 1 \). Then \( xx^{k-1} = x^{k-1}x = x^k \). By (I), however, \( x(xx^{k-1}) + (x^{k-1}x)x = 2(xx^{k-1})x \) which reduces to \( xx^k = x^k x \) and the lemma holds by mathematical induction.

Now linearize (I) to get:

\[
(1) \quad x(xy) + z(xy) + (yx)z + (yz)x = 2(xy)z + 2(zy)x.
\]

Assume inductively that \( x^ax^b = x^{a+b} \) for all positive integers \( a, b \) such that \( a + b < n \). This is certainly true if \( n = 3 \). The induction hypothesis leads to the following.

**Lemma 1.2.** \( x^{n-k}x^k = x^k x^{n-k} \) for all \( k < n \).

**Proof of Lemma 1.2.** In (1) let \( x = x^{n-k}, y = x^{k-1}, \) and \( z = x \). We get:

\[
\begin{align*}
&x^{n-k}(xx^{k-1}) + x(x^{n-k}x^{k-1}) + (x^{k-1}x^{n-k})x + (x^{k-1}x)x^{n-k} \\
&= 2(x^{n-k}x^{k-1})x + 2(xx^{k-1})x^{n-k}.
\end{align*}
\]

However, by hypothesis \( x^{n-k}x^{k-1} = x^{k-1}x^{n-k} = x^{n-1} \) since the degree of each of these terms is \( n - 1 < n \). Also, by Lemma 1.1 \( xx^{k-1} = x^{n-1}x = x^k \) and \( xx^{n-1} = x^{n-1}x = x^k \). Therefore, the identity is reduced to \( x^{n-k}x^k + x^n + x^nx^{n-k} = 2x^n + 2x^kx^{n-k} \) or \( x^{n-k}x^k = x^kx^{n-k} \) as desired.

Now since \( A^+ \) is power-associative we have \( x^a = x^{n-k}x^k \) for any \( k < n \). Since \( x^{n-k}x^k = x^kx^{n-k} \) we get \( x^n = 2x^{n-k}x^k = x^{n-k}x^k \). Suppose now that \( a + b = n \). Then \( a = n - k, b = k \) for some \( k \leq n \). Then \( x^{a+b} = x^n = x^{n-k}x^k = x^kx^b \) and the result holds for \( a + b = n \). It follows by mathematical induction that \( x^ax^b = x^{a+b} \) for all positive integers \( a, b \) and \( A \) is power-associative.

Clearly, Theorem 1 would also hold for a ring \( A \) in which the equation \( 2x = a \) is solvable for all \( a \) in \( A \). It should be noted that (I) alone is not sufficient to guarantee power-associativity of \( A \) since Albert [1, p. 25] has shown that commutativity does not guarantee power-associativity.

3. In this section we shall be considering finite dimensional, power-associative algebras with 1 every element of which is of the form \( \alpha 1 + n \) with \( n \) nilpotent. We call a nilpotent element \( w \) of such an algebra a commutator nilpotent if there are elements \( u, v \) in the algebra such that \( [u, v] = \alpha 1 + w \) for some \( \alpha \) in the base field. We write \( \text{tr.}(T) \) for the trace of an operator \( T \).

**Theorem 2.** Let \( A \) be a finite dimensional algebra satisfying (I) over a field \( F \) of characteristic zero in which every element \( z \) is
of the form \( z = \alpha 1 + n \) where \( \alpha \) is in \( F \) and \( n \) is nilpotent. Then a necessary and sufficient condition for the set \( N \) of nilpotent elements to form an ideal of \( A \) is that \( \text{tr.} (R(w)) = 0 \) for every commutator nilpotent \( w \). \( (R(w)) \) is the operator which takes any \( x \) into \( xw \).

**Proof.** Gerstenhaber [4, p. 29] has shown that in a commutative power-associative algebra over a field of characteristic zero, the assumption that an element \( n \) is nilpotent implies that \( R(n) \) is nilpotent. We apply this result to the algebra \( A^+ \) so that if \( a \) is a nilpotent element of \( A \) then \( R(a) = 1/2(R(a) + L(a)) \) is nilpotent and thus \( \text{tr.} [R(a)] + \text{tr.} [L(a)] = 0 \). Writing (1) in terms of operators we get:

\[
(2) \quad R(y)L(x) + R(xy) + L(y)R(x) = 2L(xy) + 2R(y)R(x)
\]

If we interchange \( x \) and \( y \) in (2) and subtract the result from (2) we get:

\[
[L(y), R(x)] + [R(y), L(x)] + R([x, y]) + L([y, x]) = 2L([x, y]) + 2[R(y), R(x)]
\]

which gives rise to:

\[
(3) \quad \text{tr.} R([x, y]) + \text{tr.} L([y, x]) = 2\text{tr.} L([x, y]) .
\]

Assume that \( \text{tr.} R(w) = 0 \) for all commutator nilpotents \( w \) of \( A \). Then \( \text{tr.} L(w) = \text{tr.} R(w) = 0 \) also. Let \( x \) and \( y \) be arbitrary elements of \( N \). Then \( [x, y] = \alpha 1 + n \) for some \( \alpha \) in \( F \) and \( n \) in \( N \) and \( n \) is a commutator nilpotent. Therefore (3) reduces to \( \text{tr.} [R(\alpha 1)] - \text{tr.} [L(\alpha 1)] = 2\text{tr.} [L(\alpha 1)] \) or \( \text{tr.} [R(\alpha 1)] = 3\text{tr.} [L(\alpha 1)] \) a contradiction unless \( \alpha = 0 \). Therefore, \( [x, y] \) is in \( N \) and by [2], \( xy \) and \( yx \) are in \( N \). Thus \( N \) is an ideal of \( A \).

Conversely, let \( N \) be an ideal of \( A \). Therefore \( [x, y] \) is in \( N \) for all \( x, y \) in \( N \) and consequently for all \( x, y \) in \( A \). Thus if \( w \) is a commutator nilpotent of \( A \) there is an \( x, y \) such that \( w = [x, y] \). From (3) we have that \( \text{tr.} R(w) - \text{tr.} L(w) = 2\text{tr.} L(w) \). But \( \text{tr.} R(w) + \text{tr.} L(w) = 0 \). Therefore \( \text{tr.} R(w) = 0 \) and the result holds.

**Theorem 3.** There are no nodal Lie-admissible algebras satisfying (I) over any field \( F \) of characteristic zero.

**Proof.** For if \( A \) is such a Lie-admissible algebra then for all \( u, v \) in \( N \) and \( w \) in \( A \) we have \( [[u, v], w] + [[v, w], u] + [[w, u], v] = 0 \). In operator form this becomes:

\[
L([[u, v], w]) - R([[u, v], w]) + [L(v), R(u)] + [R(u), R(v)]
+ [L(u), L(v)] + [R(v), L(u)] = 0 .
\]

Therefore, \( \text{tr.} L([[u, v], w]) = \text{tr.} R([[u, v], w]) \).

Suppose that \( [u, v] = \alpha 1 + z \) with \( \alpha \) in \( F \) and \( z \) in \( N \). Then \( \text{tr.} L(\alpha 1) + \text{tr.} L(z) = \text{tr.} R(\alpha 1) + \text{tr.} R(z) \). Therefore, \( \text{tr.} R(z) = \text{tr.} L(z) \)
for all commutator nilpotents z. From [4] we conclude that tr. $R(z) = 0$
and by Theorem 2, $N$ is an ideal of $A$. Therefore $A$ is not a nodal algebra.

We say that $N$ has nilindex $p$ if $p$ is the smallest positive integer such that $n^p = 0$ for all $n$ in $N$.

**Lemma 1.** There are no nodal algebras satisfying (I) over a field $F$ of characteristic zero for which the nilindex of $N$ is two.

**Proof.** For if $N$ has nilindex two, then $xy + yx = 0$ for all $x, y$ in $N$. Applying (I) to $x$ and $y$ in $N$ we have $x(xy) - (xy)x = 2(xy)x$ or $x(xy) = 3(xy)x$. If $xy = \alpha 1 + z$ with $\alpha$ in $F$ and $z$ in $N$ the preceding identity becomes $\alpha xx + zz = 3\alpha xx + 3zx$. But $zz = -zx$. Therefore it reduces to $2\alpha xx = 4zx$ and since characteristic $F \neq 2$ to $\alpha xx = 2zx$. Multiplying on the left by $x$ we have $0 = \alpha xx^2 = 2x(xz)$ or $x(xz) = 0$. But $x[x(xy)] = x[x(\alpha 1 + z)] = x[\alpha xx + xz] = \alpha xx^2 + x(xz) = 0$. Therefore we have $yL(x)^2 = 0$ for all $x, y$ in $N$.

Let $\alpha 1 + n$ be a typical element of the algebra $A$. Then $(\alpha 1 + n)L(x)^3 = \alpha xx^3 + nL(x)^3$ and $nL(x)^3 = 0$ as above. Therefore $L(x)^3 = 0$, $L(x)$ is a nilpotent operator of $A$ and tr. $L(x) = 0$. As before, this implies that tr. $R(x) = 0$. By Theorem 2, $N$ is an ideal of $A$ and $A$ is not a nodal algebra.

Anderson [3] has shown the existence of simple nodal algebras over a field of characteristic zero for which the associators $(x, y, z)$ are nilpotent for all $x, y,$ and $z$. The following theorem shows that no such algebras exist which satisfy (I).

**Theorem 4.** There are no simple nodal algebras satisfying (I) and (II) over any field $F$ of characteristic zero.

**Proof.** We first prove the following lemmas.

**Lemma 4.1.** If $x$ and $y$ are in $N$ then $xy^2$ and $y^2x$ are also in $N$.

For if we let $xy = \alpha 1 + n$ with $\alpha$ in $F$ and $n$ in $N$, then $yx = 2x \cdot y - \alpha 1 - n$ and $(x, y, x) = 2xx + nx + xn - 2x(y \cdot y)$. But $xn + nx = 2x \cdot n$ is in $N$, $2ax$ is in $N$, and by hypothesis $(x, y, x)$ is in $N$. Therefore, $2x(x \cdot y)$ and consequently $x(x \cdot y)$ is in $N$. Linearizing this we have:

\[ (4) \quad x(x \cdot y) + z(x \cdot y) \text{ is in } N \text{ if } x, y, z \text{ in } N. \]

Let $z = y$ in (4). Then $xy^2 + y(x \cdot y)$ is in $N$. But $y(y \cdot x)$ is in $N$ from the previous remark and we conclude that $xy^2$ is in $N$. Since $x \cdot y^2$ is in $N$ $y^2x$ is also in $N$. 
It can be further shown by mathematical induction that $x^j y^k$ is in $N$ if $j > 1$ or $k > 1$.

**Lemma 4.2.** For any $x, y$ in $N$ the following elements are in $N$: $(xy)x$, $x(xy)$, $(yx)x$, and $x(yx)$.

For, since $A$ is power-associative we have

$$ (x, x, y) + (y, x, x) + (x, y, x) = 0. $$

But $(x, y, x)$ is in $N$. So we have that $(x, x, y) + (y, x, x)$ is in $N$ for all $x, y$ in $A$. If $x$ and $y$ are in $N$ then by Lemma 4.1, $x^2 y - y x^2$ is in $N$. Thus,

$$ (yx)x - x(xy) $$

is in $N$ for all $x, y$ in $N$.

We write $x(xy) = (yx)x + n$ for some $n$ in $N$. Adding (I) to this we get that $2x(xy) = 2(xy)x + n$. But characteristic $F \neq 2$. Therefore, $x(xy) - (xy)x$ is in $N$. But $x \cdot (xy)$ is in $N$. Thus, $x(xy)$ and $(xy)x$ are in $N$ if $x$ and $y$ are in $N$. Applying (I) again $(yx)x = 2(xy)x - x(xy)$.

By the previous remark the right side is in $N$. We conclude, therefore, that $(yx)x$ and hence $x(yx)$ is in $N$ completing the proof of the lemma.

Since $x(xy)$ is in $N$, it follows that:

$$ x(zy) + z(xy) $$

is in $N$ if $x, y, z$ are in $N$.

Also $(yx)x$ in $N$ implies that:

$$ (yx)z + (yz)x $$

is in $N$ if $x, y, z$ are in $N$.

Now, let $y$ be an element of $N$. Then $y^i$ is in $N$. We shall analyze the ideal $I$ generated by the element $y^i$. $I$ is the set of all sums of terms, each term being a product of elements of $A$ at least one element of which is the element $y^i$. Consider the number of multiplications on $y^i$ in a typical summand. If we multiply $y^i$ by a single element in $N$, say $z$, we have either $y^iz$ or $zy^i$ which are in $N$ by Lemma 4.1.

We prove by mathematical induction that any number of multiplications on $y^i$ by elements of $N$ maintains nilpotency. The result has been shown for one multiplication. Assume that $n$ multiplications on $y^i$ maintains nilpotency and consider $n + 1$ multiplications by elements $q_1, q_2, \cdots, q_n, q_{n+1}$ of $N$. There are only four cases to consider:

1. $[[[\cdots \cdots (y^3) \cdots \cdots )]]q_n]q_{n+1}$
2. $q_{n+1}[[[\cdots \cdots (y^3) \cdots \cdots )]]q_n$
3. $q_{n+1}[q_n[[[\cdots \cdots (y^3) \cdots \cdots )]]]
4. $[q_n[[[\cdots \cdots (y^3) \cdots \cdots )]]]q_{n+1}$

for all other arrangements would involve $n$ or less multiplications. Let
\( b = (((\cdots (y^2) \cdots ))) \). By hypothesis \( b \) is in \( N \). We must show then, that

\[
(1) \quad bq_n/q_{n+1} \quad (2) \quad q_{n+1}(bqn) \quad (3) \quad q_{n+1}(qnb) \quad (4) \quad (qnb)q_{n+1}
\]

are all in \( N \).

In (6) let \( x = q_{n+1}, \ z = b, \) and \( y = q_n \). Then we have that \( q_{n+1}(bqn) + b(q_{n+1}q_n) \) is in \( N \). But \( b(q_{n+1}q_n) \) involves only \( n \) multiplications on \( y^2 \). Therefore, by the induction hypothesis it is in \( N \) and we conclude that \( q_{n+1}(bqn) \) and therefore by [2] \( (bqn)q_{n+1} \) are in \( N \). Similarly, in (7) let \( x = b, \ y = q_n, \) and \( z = q_{n+1}. \) Then we have \( (q_nb)q_{n+1} + (q_nq_{n+1})b \) are in \( N \). As before this implies that \( (q_nb)q_{n+1} \) and consequently \( q_{n+1}(q_nb) \) are in \( N \). Therefore \( n + 1 \) multiplications on \( y^2 \) by elements of \( N \) maintains nilpotency and the result holds for any number of multiplications. It follows easily that any number of multiplications on \( y^2 \) by elements of \( A \) preserve nilpotency.

Now every element of \( I \) is a sum of terms of the above type and consequently nilpotent. Thus \( I \subseteq N \). Hence, \( I \) is an ideal of \( A \) which does not encompass all of \( A \) and by the simplicity of \( A, I = 0. \) But \( y^2 \) is in \( I \). Therefore \( y^2 = 0. \) This holds for all \( y \) in \( N \) and so the nilindex of \( N \) is two. By Lemma 1, \( A \) is not nodal.

**Theorem 5.** There are no nodal algebras satisfying (I) and (II) over any field \( F \) of characteristic zero.

**Proof.** For let \( A \) be such an algebra. By Theorem 4, \( A \) is not simple. Let \( B \) be a maximal ideal of \( A. \) Then \( A/B \) is a simple nodal algebra satisfying (I) and (II) contradicting Theorem 4.

**References**


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