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Let a linear homeomorphism T from a Banach space X onto the conjugate space Y^* of a Banach space Y be called a conjugate space representation of X. If $T: X \to Y^*$ and $U: X \to Z^*$ are two conjugate space representations of X, say that T and U are essentially different if there is no linear homeomorphism P from Y onto Z satisfying $P^* = T \circ U^{-1}$. It is proven here that if a nonreflexive Banach space has one conjugate space representation, it has uncountably many essentially different conjugate space representations. A Banach space X with norm p will be denoted by (X, p) when it is important to emphasize the norm. The dual of p is the norm p^* defined on the conjugate space $(X, p)^*$ of (X, p) by

$$p^*(f) = \sup\{|f(x)| : x \in X \text{ and } p(x) = 1\} .$$

It is proven here that if $T: (X, p) \to (Y, r)^*$ and $U: (X, p) \to (Z, s)^*$ are two essentially different conjugate space representations of (X, p), then there exists a norm q on X equivalent to p such that $q \circ T^{-1} = r_1^*$ for some norm r_1 on Y equivalent to r, but such that $q \circ U^{-1} \neq s_1^*$ for any norm s_1 on Z equivalent to s.

Williams has shown [7, Th. 1, p. 163] that a Banach space (X, p) is reflexive if and only if every norm q on X^* equivalent to p^* is the dual of some norm on X equivalent to p. We show here that if (X, p) is a nonreflexive Banach space, then there exists a norm q on X^* equivalent to p^* such that q is not the dual of any norm on X equivalent to p, but such that the Banach space (X^*, q) is isometrically isomorphic to a conjugate Banach space. By contrast, Klee [3, Th. 4, p. 21] has exhibited a Banach space (X, p) and a norm q on X^* equivalent to p^* such that (X^*, q) is not isometrically isomorphic to a conjugate Banach space.

We shall use the following notation. If A and B are sets, $A \backslash B$ denotes the set of elements in A but not in B. If x is an element in a linear space, $[x]$ denotes the linear span of x. If A and B are linear subspaces of a linear space X, and if $A \cap B = \{0\}$, then $A \oplus B$ denotes the linear direct sum of A and B. If A is a subset of a normed linear space (X, p), $A^\perp$ donotes the annihilator of A in X^*. If A is a subset of the conjugate space X^* of a normed linear space (X, p), $A_\perp$ denotes the set of elements in X annihilated by A. If (X, p) is a normed linear space, J_X denotes the canonical map from X into X^{**} defined by

$$(J_X x)f = f(x) \text{ for all } x \in X \text{ and } f \in X^* .$$

LEMMA. *If $T: X \rightarrow Y^*$ and $U: X \rightarrow Z^*$ are two conjugate space representations of a Banach space X, then T and U are essentially different if and only if*

$$T^*[J_Y Y] \neq U^* [J_Z Z] .$$

Proof. (i) Suppose T and U are not essentially different. Then there exists a linear homeomorphism P from Y onto Z satisfying $P^* = T \circ U^{-1}$. It is straightforward to verify that $(P^{**}(J_Y y))g = (J_Z(Py))g$ for all $y \in Y$ and $g \in Z^*$; that is, $P^{**} \circ J_Y = J_Z \circ P$. Therefore $T^*[J_Y Y] = (P^* \circ U)^*[J_Y Y] = (U^* \circ P^{**} \circ J_Y)[Y] = U^*[J_Z Z]$.

(ii) Suppose $T^*[J_Y Y] = U^*[J_Z Z]$. Let $P = J_Z^{-1} \circ U^{*-1} \circ T^* \circ J_Y = J_Z^{-1} \circ (T \circ U^{-1})^* \circ J_Y$. Then P is a linear homeomorphism from Y onto Z. It can be verified directly that $(P^* g)y = ((T \circ U^{-1})g)y$ for all $g \in Z^*$ and $y \in Y$. Therefore $P^* = T \circ U^{-1}$.

THEOREM 1. *Suppose that (X, p) is a nonreflexive Banach space which is linearly homeomorphic to the conjugate (Y^*, r^*) of a Banach space (Y, r). Then there exists an uncountable collection of essentially different conjugate space representations $U_\alpha: (X, p) \rightarrow (Z_\alpha^*, s_\alpha^*)$ such that each space (Z_α, s_α) is linearly homeomorphic to (Y, r).*

Proof. By hypothesis there exists a linear homeomorphism T from (X, p) onto (Y^*, r^*). Let $M = T^*[J_Y Y]$. Then [4, p. 577] M is a minimal total norm-closed subspace of (X^*, p^*). That is, M is total and norm-closed, and no proper subspace of M is both total and norm-closed. If L is any norm-closed subspace of X^*, let Q_L denote the canonical map from X into L^* defined by

$$(Q_L x)f = f(x) \text{ for all } x \in X \text{ and } f \in L .$$

In particular, Q_{X^*} is the canonical map J_X from X into X^{**}. By [4, p. 577], the map Q_L is a linear homeomorphism from (X, p) onto $(L^*, (p^* | L)^*)$ if and only if L is a minimal total norm-closed subspace of (X^*, p^*). Since (X, p) is not reflexive, Q_{X^*} is not a linear homeomorphism from (X, p) onto (X^{**}, p^{**}), and X^* is not a minimal total norm-closed subspace of X^*.

Let $f \in X^*$. Let us show that there is a minimal total norm-closed subspace B of X^* such that $f \in B$ and such that B is linearly homeomorphic to (Y, r). If $f \in M$, we may take $B = M$. Now suppose $f \notin M$. By a theorem of Dixmier [1, Th. 11, p. 1065] a norm-closed total subspace V of the conjugate E^* of a Banach space E is a minimal total norm-closed subspace of E^* if and only if $E^{**} = J_E E \oplus V^\perp$. Thus we have $X^{**} = J_X X \oplus M^\perp$. Let $H = [f]^\perp$. By the Hahn-Banach Theorem [6, Corollary 2, p. 67], $H \not\supseteq M^\perp$ so $J_X X \oplus (H \cap M^\perp)$ is a maximal subspace of X^{**}. Now $H \not\supseteq J_X X$ since $f \neq 0$, so $J_X X \oplus (H \cap M^\perp) \neq H$. Since $J_X X \oplus (H \cap M^\perp)$ and H are distinct maximal subspaces of X^{**}, there exists $G \in H$ such that $X^{**} = J_X X \oplus (H \cap M^\perp) \oplus [G]$. Let $D = (H \cap M^\perp) \oplus [G]$, and let $B = D_\perp$. Then [6, (x), p. 238] B is a norm-closed subspace of (X^*, p^*). The subspaces H and $M^\perp$ are $w(X^{**}, X^*)$-closed [6, (x), p. 238], so $H \cap M^\perp$ is $w(X^{**}, X^*)$-closed. Therefore [6, Corollary 5, p. 192] D is $w(X^{**}, X^*)$closed, and [6, Th. 1, p. 238] $B^\perp = (D_\perp)^\perp = D$. Now B is a total subspace of X^* since $B^\perp \cap J_X X = \{0\}$. By the theorem of Dixmier mentioned above, B is a minimal total norm-closed subspace of X^*. By the Hahn-Banach Theorem, we have $f \in B$ since $B^\perp \subseteqq H = [f]^\perp$. Now observe that both B and M are maximal subspaces of $(H \cap M^\perp)_\perp$, and consequently each of them is linearly homeomorphic to the topological direct sum of $B \cap M$ with a one-dimensional space. Therefore B is linearly homeomorphic to M which is in turn linearly homeomorphic to Y.

Let $\{Z_\alpha : \alpha \in \Phi\}$ be the collection of all minimal total norm-closed subspaces of X^* which are linearly homeomorphic to Y. For each $\alpha \in \Phi$ let $s_\alpha = p^* \mid Z_\alpha$. We have established that every element $f \in X^*$ is contained in some Z_α. Each Z_α is nowhere dense in X^* since each Z_α is a proper norm-closed subspace of X^*. Since X^* is a complete linear metric space, the Baire Category Theorem guarantees that X^* is not a countable union of nowhere dense sets. Therefore $\{Z_\alpha : \alpha \in \Phi\}$ is an uncountable collection. For every $\alpha \in \Phi$, let $U_\alpha = Q_{Z_\alpha}$. It is straightforward to verify that $U_\alpha^* \circ J_{Z_\alpha}$ is the identity map on Z_α. Thus $U_\alpha^*[J_{Z_\alpha} Z_\alpha] = Z_\alpha$. Therefore by the lemma U_β and U_α are essentially different whenever $\beta, \alpha \in \Phi$ and $\beta \neq \alpha$.

THEOREM 2. *Suppose that* $T: (X, p) \to (Y^*, r^*)$ *and* $U: (X, p) \to (Z^*, s^*)$ *are two essentially different conjugate space representations. Then there exists a norm* q *on* X *equivalent to* p *such that* $q \circ T^{-1}$ *is the dual of some norm* r_1 *on* Y *equivalent to* r, *but such that* $q \circ U^{-1}$ *is not the dual of any norm* s_1 *on* Z *equivalent to* s.

REMARK. An interesting example may be obtained by letting X, Y and Z be the sequence spaces l, c, and c_0, respectively.

Proof of Theorem 2. Let $A = T^*[J_Y Y]$ and let $B = U^*[J_Z Z]$. Then $A \neq B$ by the lemma. By [4, p. 577], A and B are minimal total norm-closed subspaces of X^*. The map T is a vector space isomorphism from X onto Y^* and $T^*[J_Y Y] = A$; it follows that T is a $w(X, A) - w(Y^*, J_Y Y)$-homeomorphism. Let $S = T^{-1}[\{g \in Y^* : r^*(g) \leqq 1\}]$. Then S is $w(X, A)$-compact, because $\{g \in Y^* : r^*(g) \leqq 1\}$ is $w(Y^*, J_Y Y)$-

compact by the Banach-Alaoglu Theorem [6, Th. 1, p. 239]. Since $A \neq B$ and since A and B are both minimal with respect to certain properties, we must have $A \not\subseteq B$. Thus there exists $f \in A \backslash B$. Let $L = f^{-1}(0)$ and let $V = L \cap S$. The subspace L is $w(X, A)$-closed [6, Th. 3, p. 186] since $f \in A$. Thus V is $w(X, A)$-compact.

Now f is not $w(X, B)$-continuous [2, Th. 9, p. 421] since $f \notin B$, so [6, Th. 3, p. 186] L is not $w(X, B)$-closed. However, L is norm-closed [6, Th. 3, p. 186] since $f \in X^*$. Thus $U[L]$ is a norm-closed subspace of (Z^*, s^*), but $U[L]$ is not $w(Z^*, J_Z Z)$-closed. Let K be the $w(X, B)$-closure of V. Then [2, Lemma 4, p. 415] K is convex since V is convex. Now $U[K]$ is a convex $w(Z^*, J_Z Z)$-closed subset of Z^*. By a corollary of the Krein-Šmulian Theorem [2, Corollary 9, p. 429], the linear span of a convex, weak* closed set is weak* closed if and only if it is norm-closed. Therefore $U[L] \neq \operatorname{span}(U[K])$. Consequently, $L \neq \operatorname{span}(K)$. However, $\operatorname{span}(K) \supseteqq \operatorname{span}(V) = L$, so there exists an element $x_0 \in K \backslash L$. Let W be the convex balanced hull of $V \cup \{(1/2)x_0\}$. Then if co denotes convex hull and bal denotes balanced hull, we have

$$W = \operatorname{co}\left(\operatorname{bal}\left(V \cup \left\{\frac{1}{2}x_0\right\}\right)\right) = \operatorname{co}\left(\operatorname{bal}(V) \cup \operatorname{bal}\left\{\frac{1}{2}x_0\right\}\right)$$
$$= \operatorname{co}\left(V \cup \operatorname{bal}\left\{\frac{1}{2}x_0\right\}\right)$$

which is $w(X, A)$-closed [2, Lemma 5, p. 415] since the sets V and $\operatorname{bal}\{(1/2)x_0\}$ are convex and $w(X, A)$-compact. The set W is norm-closed since the norm topology is stronger than the $w(X, A)$ topology. Also W is norm-bounded since V and $\operatorname{bal}\{(1/2)x_0\}$ are norm-bounded. Now $\operatorname{span}(W) = X$ since $\operatorname{span}(W)$ properly contains the maximal subspace L. Thus for any $x \in X$ there exist elements $w_1, \cdots w_N \in W$ and nonzero numbers $t_1, \cdots, t_N$ such that $x = t_1 w_1 + \cdots + t_N w_N$. Let $t = \sum_{i=1}^N |t_i|$. Then $x/t \in W$ since W is convex and balanced. Thus W is absorbing. We have shown that W is a convex, balanced, absorbing, norm-closed norm-bounded subset of the Banach space (X, p). Therefore W is a norm-neighborhood of zero since Banach spaces are barrelled. By [6, p. 58] the gauge q of W is a norm on X equivalent to p, and $W = \{x \in X: q(x) \leqq 1\}$.

Let $q_1 = q \circ T^{-1}$. Then q_1 is a norm on Y^* equivalent to r^* since T is a linear $p - r^*$ homeomorphism. Also $\{g \in Y^*: q_1(g) \leqq 1\} = T[W]$, and $\{g \in Y^*: q_1(g) \leqq 1\}$ is $w(Y^*, J_Y Y)$-closed since W is $w(X, A)$-closed. Singer has shown [5, Lemma 2, p. 450] that if (E, h) is a Banach space, and if h_1 is a norm on E^* equivalent to h^*, then h_1 is the dual of some norm on E equivalent to h if and only if the set $\{g \in E^*: h_1(g) \leqq 1\}$ is $w(E^*, J_E E)$-closed. (In one direction, of course, this is the well-known Banach-Alaoglu Theorem.) Therefore there exists a norm r_1 on Y equivalent to r such that $r_1^* = q_1 = q \circ T^{-1}$.

Let $q_2 = q \circ U^{-1}$. Then q_2 is a norm on Z^* equivalent to s^* since U is a linear $p - s^*$-homeomorphism. Also $\{g \in Z^*: q_2(g) \leqq 1\} = U[W]$. Now $x_0 \notin W$, for if $x_0 = cv + (1 - c)(d)((1/2)x_0)$ with $v \in V$, $0 \leqq c \leqq 1$, and $|d| \leqq 1$, then $(1 - 1/2(1 - c)d)x_0 = cv \in L$, so that $x_0 \in L$, contrary to the definition of x_0. However, x_0 belongs to the $w(X, B)$-closure of W since the $w(X, B)$-closure of W contains the $w(X, B)$-closure of V, namely K. Therefore W is not $w(X, B)$-closed. Thus $U[W] = \{g \in Z^*: q_2(g) \leqq 1\}$ is not $w(Z, J_Z Z)$-closed. By the Banach-Alaoglu Theorem, there is no norm s_1 on Z equivalent to s such that $s_1^* = q_2 = q \circ U^{-1}$.

COROLLARY. *If (X, p) is a nonreflexive Banach space, there is a norm q on X^* equivalent to p^* such that q is not the dual of any norm on X equivalent to p, but such that the Banach space (X^*, q) is isometrically isomorphic to a conjugate Banach space.*

Proof. Suppose that (X, p) is a nonreflexive Banach space. By Theorem 1 there exists a conjugate space representation $T: (X^*, p^*) \to (Y^*, r^*)$ such that T is essentially different from the identity map I on X^*. By Theorem 2 there exists a norm q on X^* equivalent to p^* such that $q \circ T^{-1} = r_1^*$ for some norm r_1 on Y equivalent to r, but such that $q \circ I^{-1}$ is not the dual of any norm on X equivalent to p. Now T is an isometric isomorphism from (X^*, q) onto the conjugate Banach space (Y^*, r_1^*).

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