

Pacific Journal of Mathematics

CONJUGATE SPACE REPRESENTATIONS OF BANACH SPACES

EMILE B. ROTH

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Let a linear homeomorphism T from a Banach space X onto the conjugate space Y^* of a Banach space Y be called a conjugate space representation of X . If $T: X \rightarrow Y^*$ and $U: X \rightarrow Z^*$ are two conjugate space representations of X , say that T and U are essentially different if there is no linear homeomorphism P from Y onto Z satisfying $P^* = T \circ U^{-1}$. It is proven here that if a nonreflexive Banach space has one conjugate space representation, it has uncountably many essentially different conjugate space representations. A Banach space X with norm p will be denoted by (X, p) when it is important to emphasize the norm. The dual of p is the norm p^* defined on the conjugate space $(X, p)^*$ of (X, p) by

$$p^*(f) = \sup \{ |f(x)| : x \in X \text{ and } p(x) = 1 \}.$$

It is proven here that if $T: (X, p) \rightarrow (Y, r)^*$ and $U: (X, p) \rightarrow (Z, s)^*$ are two essentially different conjugate space representations of (X, p) , then there exists a norm q on X equivalent to p such that $q \circ T^{-1} = r_1^*$ for some norm r_1 on Y equivalent to r , but such that $q \circ U^{-1} \neq s_1^*$ for any norm s_1 on Z equivalent to s .

Williams has shown [7, Th. 1, p. 163] that a Banach space (X, p) is reflexive if and only if every norm q on X^* equivalent to p^* is the dual of some norm on X equivalent to p . We show here that if (X, p) is a nonreflexive Banach space, then there exists a norm q on X^* equivalent to p^* such that q is not the dual of any norm on X equivalent to p , but such that the Banach space (X^*, q) is isometrically isomorphic to a conjugate Banach space. By contrast, Klee [3, Th. 4, p. 21] has exhibited a Banach space (X, p) and a norm q on X^* equivalent to p^* such that (X^*, q) is not isometrically isomorphic to a conjugate Banach space.

We shall use the following notation. If A and B are sets, $A \setminus B$ denotes the set of elements in A but not in B . If x is an element in a linear space, $[x]$ denotes the linear span of x . If A and B are linear subspaces of a linear space X , and if $A \cap B = \{0\}$, then $A \oplus B$ denotes the linear direct sum of A and B . If A is a subset of a normed linear space (X, p) , A^\perp denotes the annihilator of A in X^* . If A is a subset of the conjugate space X^* of a normed linear space (X, p) , A_\perp denotes the set of elements in X annihilated by A . If (X, p) is a normed linear space, J_X denotes the canonical map from X into X^{**} defined by

$$(J_x x)f = f(x) \text{ for all } x \in X \text{ and } f \in X^* .$$

LEMMA. *If $T: X \rightarrow Y^*$ and $U: X \rightarrow Z^*$ are two conjugate space representations of a Banach space X , then T and U are essentially different if and only if*

$$T^*[J_Y Y] \neq U^*[J_Z Z] .$$

Proof. (i) Suppose T and U are not essentially different. Then there exists a linear homeomorphism P from Y onto Z satisfying $P^* = T \circ U^{-1}$. It is straightforward to verify that $(P^{**}(J_Y y))g = (J_Z(Py))g$ for all $y \in Y$ and $g \in Z^*$; that is, $P^{**} \circ J_Y = J_Z \circ P$. Therefore $T^*[J_Y Y] = (P^* \circ U)^*[J_Y Y] = (U^* \circ P^{**} \circ J_Y)[Y] = U^*[J_Z Z]$.

(ii) Suppose $T^*[J_Y Y] = U^*[J_Z Z]$. Let $P = J_Z^{-1} \circ U^{*-1} \circ T^* \circ J_Y = J_Z^{-1} \circ (T \circ U^{-1})^* \circ J_Y$. Then P is a linear homeomorphism from Y onto Z . It can be verified directly that $(P^*g)y = ((T \circ U^{-1})g)y$ for all $g \in Z^*$ and $y \in Y$. Therefore $P^* = T \circ U^{-1}$.

THEOREM 1. *Suppose that (X, p) is a nonreflexive Banach space which is linearly homeomorphic to the conjugate (Y^*, r^*) of a Banach space (Y, r) . Then there exists an uncountable collection of essentially different conjugate space representations $U_\alpha: (X, p) \rightarrow (Z_\alpha^*, s_\alpha^*)$ such that each space (Z_α, s_α) is linearly homeomorphic to (Y, r) .*

Proof. By hypothesis there exists a linear homeomorphism T from (X, p) onto (Y^*, r^*) . Let $M = T^*[J_Y Y]$. Then [4, p. 577] M is a minimal total norm-closed subspace of (X^*, p^*) . That is, M is total and norm-closed, and no proper subspace of M is both total and norm-closed. If L is any norm-closed subspace of X^* , let Q_L denote the canonical map from X into L^* defined by

$$(Q_L x)f = f(x) \text{ for all } x \in X \text{ and } f \in L .$$

In particular, Q_{X^*} is the canonical map J_X from X into X^{**} . By [4, p. 577], the map Q_L is a linear homeomorphism from (X, p) onto $(L^*, (p^*|L)^*)$ if and only if L is a minimal total norm-closed subspace of (X^*, p^*) . Since (X, p) is not reflexive, Q_{X^*} is not a linear homeomorphism from (X, p) onto (X^{**}, p^{**}) , and X^* is not a minimal total norm-closed subspace of X^* .

Let $f \in X^*$. Let us show that there is a minimal total norm-closed subspace B of X^* such that $f \in B$ and such that B is linearly homeomorphic to (Y, r) . If $f \in M$, we may take $B = M$. Now suppose $f \notin M$. By a theorem of Dixmier [1, Th. 11, p. 1065] a norm-closed total subspace V of the conjugate E^* of a Banach space E is a minimal total norm-closed subspace of E^* if and only if $E^{**} = J_E E \oplus V^\perp$. Thus

we have $X^{**} = J_X X \oplus M^\perp$. Let $H = [f]^\perp$. By the Hahn-Banach Theorem [6, Corollary 2, p. 67], $H \not\cong M^\perp$ so $J_X X \oplus (H \cap M^\perp)$ is a maximal subspace of X^{**} . Now $H \not\cong J_X X$ since $f \neq 0$, so $J_X X \oplus (H \cap M^\perp) \neq H$. Since $J_X X \oplus (H \cap M^\perp)$ and H are distinct maximal subspaces of X^{**} , there exists $G \in H$ such that $X^{**} = J_X X \oplus (H \cap M^\perp) \oplus [G]$. Let $D = (H \cap M^\perp) \oplus [G]$, and let $B = D_\perp$. Then [6, (x), p. 238] B is a norm-closed subspace of (X^*, p^*) . The subspaces H and M^\perp are $w(X^{**}, X^*)$ -closed [6, (x), p. 238], so $H \cap M^\perp$ is $w(X^{**}, X^*)$ -closed. Therefore [6, Corollary 5, p. 192] D is $w(X^{**}, X^*)$ -closed, and [6, Th. 1, p. 238] $B^\perp = (D_\perp)^\perp = D$. Now B is a total subspace of X^* since $B^\perp \cap J_X X = \{0\}$. By the theorem of Dixmier mentioned above, B is a minimal total norm-closed subspace of X^* . By the Hahn-Banach Theorem, we have $f \in B$ since $B^\perp \subseteq H = [f]^\perp$. Now observe that both B and M are maximal subspaces of $(H \cap M^\perp)_\perp$, and consequently each of them is linearly homeomorphic to the topological direct sum of $B \cap M$ with a one-dimensional space. Therefore B is linearly homeomorphic to M which is in turn linearly homeomorphic to Y .

Let $\{Z_\alpha: \alpha \in \Phi\}$ be the collection of all minimal total norm-closed subspaces of X^* which are linearly homeomorphic to Y . For each $\alpha \in \Phi$ let $s_\alpha = p^* | Z_\alpha$. We have established that every element $f \in X^*$ is contained in some Z_α . Each Z_α is nowhere dense in X^* since each Z_α is a proper norm-closed subspace of X^* . Since X^* is a complete linear metric space, the Baire Category Theorem guarantees that X^* is not a countable union of nowhere dense sets. Therefore $\{Z_\alpha: \alpha \in \Phi\}$ is an uncountable collection. For every $\alpha \in \Phi$, let $U_\alpha = Q_{Z_\alpha}$. It is straightforward to verify that $U_\alpha^* \circ J_{Z_\alpha}$ is the identity map on Z_α . Thus $U_\alpha^*[J_{Z_\alpha} Z_\alpha] = Z_\alpha$. Therefore by the lemma U_β and U_α are essentially different whenever $\beta, \alpha \in \Phi$ and $\beta \neq \alpha$.

THEOREM 2. *Suppose that $T: (X, p) \rightarrow (Y^*, r^*)$ and $U: (X, p) \rightarrow (Z^*, s^*)$ are two essentially different conjugate space representations. Then there exists a norm q on X equivalent to p such that $q \circ T^{-1}$ is the dual of some norm r_1 on Y equivalent to r , but such that $q \circ U^{-1}$ is not the dual of any norm s_1 on Z equivalent to s .*

REMARK. An interesting example may be obtained by letting X, Y and Z be the sequence spaces l, c , and c_0 , respectively.

Proof of Theorem 2. Let $A = T^*[J_Y Y]$ and let $B = U^*[J_Z Z]$. Then $A \neq B$ by the lemma. By [4, p. 577], A and B are minimal total norm-closed subspaces of X^* . The map T is a vector space isomorphism from X onto Y^* and $T^*[J_Y Y] = A$; it follows that T is a $w(X, A) - w(Y^*, J_Y Y)$ -homeomorphism. Let $S = T^{-1}[\{g \in Y^*: r^*(g) \leq 1\}]$. Then S is $w(X, A)$ -compact, because $\{g \in Y^*: r^*(g) \leq 1\}$ is $w(Y^*, J_Y Y)$ -

compact by the Banach-Alaoglu Theorem [6, Th. 1, p. 239]. Since $A \neq B$ and since A and B are both minimal with respect to certain properties, we must have $A \not\subseteq B$. Thus there exists $f \in A \setminus B$. Let $L = f^{-1}(0)$ and let $V = L \cap S$. The subspace L is $w(X, A)$ -closed [6, Th. 3, p. 186] since $f \in A$. Thus V is $w(X, A)$ -compact.

Now f is not $w(X, B)$ -continuous [2, Th. 9, p. 421] since $f \notin B$, so [6, Th. 3, p. 186] L is not $w(X, B)$ -closed. However, L is norm-closed [6, Th. 3, p. 186] since $f \in X^*$. Thus $U[L]$ is a norm-closed subspace of (Z^*, s^*) , but $U[L]$ is not $w(Z^*, J_Z Z)$ -closed. Let K be the $w(X, B)$ -closure of V . Then [2, Lemma 4, p. 415] K is convex since V is convex. Now $U[K]$ is a convex $w(Z^*, J_Z Z)$ -closed subset of Z^* . By a corollary of the Krein-Šmulian Theorem [2, Corollary 9, p. 429], the linear span of a convex, weak* closed set is weak* closed if and only if it is norm-closed. Therefore $U[L] \neq \text{span}(U[K])$. Consequently, $L \neq \text{span}(K)$. However, $\text{span}(K) \supseteq \text{span}(V) = L$, so there exists an element $x_0 \in K \setminus L$. Let W be the convex balanced hull of $V \cup \{(1/2)x_0\}$. Then if co denotes convex hull and bal denotes balanced hull, we have

$$\begin{aligned} W &= \text{co} \left(\text{bal} \left(V \cup \left\{ \frac{1}{2}x_0 \right\} \right) \right) = \text{co} \left(\text{bal}(V) \cup \text{bal} \left\{ \frac{1}{2}x_0 \right\} \right) \\ &= \text{co} \left(V \cup \text{bal} \left\{ \frac{1}{2}x_0 \right\} \right) \end{aligned}$$

which is $w(X, A)$ -closed [2, Lemma 5, p. 415] since the sets V and $\text{bal} \{(1/2)x_0\}$ are convex and $w(X, A)$ -compact. The set W is norm-closed since the norm topology is stronger than the $w(X, A)$ topology. Also W is norm-bounded since V and $\text{bal} \{(1/2)x_0\}$ are norm-bounded. Now $\text{span}(W) = X$ since $\text{span}(W)$ properly contains the maximal subspace L . Thus for any $x \in X$ there exist elements $w_1, \dots, w_N \in W$ and nonzero numbers t_1, \dots, t_N such that $x = t_1 w_1 + \dots + t_N w_N$. Let $t = \sum_{i=1}^N |t_i|$. Then $x/t \in W$ since W is convex and balanced. Thus W is absorbing. We have shown that W is a convex, balanced, absorbing, norm-closed norm-bounded subset of the Banach space (X, p) . Therefore W is a norm-neighborhood of zero since Banach spaces are barreled. By [6, p. 58] the gauge q of W is a norm on X equivalent to p , and $W = \{x \in X: q(x) \leq 1\}$.

Let $q_1 = q \circ T^{-1}$. Then q_1 is a norm on Y^* equivalent to r^* since T is a linear $p - r^*$ homeomorphism. Also $\{g \in Y^*: q_1(g) \leq 1\} = T[W]$, and $\{g \in Y^*: q_1(g) \leq 1\}$ is $w(Y^*, J_Y Y)$ -closed since W is $w(X, A)$ -closed. Singer has shown [5, Lemma 2, p. 450] that if (E, h) is a Banach space, and if h_1 is a norm on E^* equivalent to h^* , then h_1 is the dual of some norm on E equivalent to h if and only if the set $\{g \in E^*: h_1(g) \leq 1\}$ is $w(E^*, J_E E)$ -closed. (In one direction, of course, this is the well-known Banach-Alaoglu Theorem.) Therefore there exists a norm r_1 on Y equivalent to r such that $r_1^* = q_1 = q \circ T^{-1}$.

Let $q_2 = q \circ U^{-1}$. Then q_2 is a norm on Z^* equivalent to s^* since U is a linear $p - s^*$ -homeomorphism. Also $\{g \in Z^*: q_2(g) \leq 1\} = U[W]$. Now $x_0 \notin W$, for if $x_0 = cv + (1 - c)(d)((1/2)x_0)$ with $v \in V, 0 \leq c \leq 1$, and $|d| \leq 1$, then $(1 - 1/2(1 - c)d)x_0 = cv \in L$, so that $x_0 \in L$, contrary to the definition of x_0 . However, x_0 belongs to the $w(X, B)$ -closure of W since the $w(X, B)$ -closure of W contains the $w(X, B)$ -closure of V , namely K . Therefore W is not $w(X, B)$ -closed. Thus $U[W] = \{g \in Z^*: q_2(g) \leq 1\}$ is not $w(Z, J_Z Z)$ -closed. By the Banach-Alaoglu Theorem, there is no norm s_1 on Z equivalent to s such that $s_1^* = q_2 = q \circ U^{-1}$.

COROLLARY. *If (X, p) is a nonreflexive Banach space, there is a norm q on X^* equivalent to p^* such that q is not the dual of any norm on X equivalent to p , but such that the Banach space (X^*, q) is isometrically isomorphic to a conjugate Banach space.*

Proof. Suppose that (X, p) is a nonreflexive Banach space. By Theorem 1 there exists a conjugate space representation $T: (X^*, p^*) \rightarrow (Y^*, r^*)$ such that T is essentially different from the identity map I on X^* . By Theorem 2 there exists a norm q on X^* equivalent to p^* such that $q \circ T^{-1} = r_1^*$ for some norm r_1 on Y equivalent to r , but such that $q \circ I^{-1}$ is not the dual of any norm on X equivalent to p . Now T is an isometric isomorphism from (X^*, q) onto the conjugate Banach space (Y^*, r_1^*) .

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