ON THE PROJECTIONS OF A CONVEX POLYTOPE

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It is shown that in the class of all centrally symmetric convex bodies in $E^d$ a polytope is uniquely determined, up to a translation, by its brightness (or certain similar functionals) in a suitable, though "arbitrarily small" set of directions.

It is well known that a centrally symmetric convex body (compact, convex set with interior points) in $d$-dimensional Euclidean space $E^d$ is, up to a translation, uniquely determined by its brightness function. To formulate a more general result, let $S^{d-1} = \{x \in E^d : \|x\| = 1\}$ be the unit sphere in $E^d$; for a convex body $K \subset E^d$ and a unit vector $u \in S^{d-1}$ let $K(u)$ be the convex set that arises by orthogonal projection of $K$ onto the $(d-1)$-dimensional linear subspace orthogonal to $u$. For $p \in \{0, 1, \ldots, d-2\}$ let $v_p(K, u)$ denote the $p$-th cross-section measure (Quermassintegral; for a definition see Bonnesen-Fenchel [2, p. 49], or Hadwiger [5, p. 209]) of dimension $d-1$ of the set $K(u)$. Thus, e.g., $v_0(K, u)$ is the brightness of $K$ in the direction $u$, and $v_{d-2}(K, u)$ is, up to a factor depending only on $d$, the mean width of $K(u)$. The following theorem has been proved by A. D. Aleksandrov [1]:

If $K, \bar{K} \subset E^d$ are centrally symmetric convex bodies satisfying $v_p(K, u) = v_p(\bar{K}, u)$ for each $u \in S^{d-1}$ and for some $p \in \{0, 1, \ldots, d-2\}$, then $\bar{K}$ is a translate of $K$.

For another proof and a generalization see Chakerian [3].

One might ask whether in Aleksandrov's theorem it is really necessary to assume the equality $v_p(K, u) = v_p(\bar{K}, u)$ for the set of all directions $u$ or whether some nondense subset thereof might suffice. The latter is, however, not true in general. In fact, given a centrally symmetric convex body $K \subset E^d$ with sufficiently smooth boundary and a symmetric subset $A \subset S^{d-1}$ which is not dense in $S^{d-1}$, there exists a centrally symmetric convex body $\bar{K} \subset E^d$, not a translate of $K$, which satisfies $v_p(K, u) = v_p(\bar{K}, u)$ for each $u \in A$. Examples to this effect have been constructed in [7, §4]. The object of the present note is to exhibit a contrary situation: In case $K$ is a centrally symmetric polytope, there exist sets $A \subset S^{d-1}$ of arbitrarily small (positive) measure such that the assumption

$v_p(K, u) = v_p(\bar{K}, u)$ for each $u \in A$
forces the centrally symmetric convex body $K$ to be a translate of $K$. More precisely, we shall prove the following

**Theorem.** Let $K \subset E^d$ be a centrally symmetric convex polytope. Let $p \in \{0, 1, \ldots, d - 2\}$, and let $A \subset S^{d-1}$ be an open set which contains, corresponding to each $(d - 1 - p)$-dimensional face of $K$, a vector which is parallel to that face. If $\bar{K} \subset E^d$ is a centrally symmetric convex body which satisfies

$$v_p(K, u) = v_p(\bar{K}, u) \text{ for each } u \in A,$$

then $\bar{K}$ is a translate of $K$.

For $p \leq d - 3$ there exist universal sets $A$ with the properties demanded in the theorem. For instance, if $A$ is a neighborhood of an “equator sphere” of $S^{d-1}$, then $A$ contains, corresponding to any $(d - 1 - p)$-face $F$ of any convex polytope, a vector which is parallel to $F$.

The following remarks are preparatory to the proof of the theorem. For a convex body $K \subset E^d$ let $\mu_p(K, \cdot)$, $p = 1, \ldots, d - 1$, be its $p$-th surface area function; thus $\mu_p$ is a positive Borel measure on $S^{d-1}$ which may be characterized by the fact that

$$V(K, K, \cdots, K, B, \cdots, B) = \frac{1}{d} \int_{S^{d-1}} \bar{h}(v) \mu_p(K, dv)$$

for every convex body $\bar{K} \subset E^d$ (see Fenchel-Jessen [4]); here the left side is a mixed volume, $B$ is the ball bounded by $S^{d-1}$, and $\bar{h}$ is the support function of $\bar{K}$. As a special case of (1) we have the representation

$$v_p(K, u) = \frac{1}{2} \int_{S^{d-1}} |\langle u, v \rangle| \mu_p(K, dv), \quad u \in S^{d-1}.$$ 

For a convex polytope $P \subset E^d$ and $p \in \{1, \ldots, d - 1\}$ let $\sigma_p(P) \subset S^{d-1}$ be the spherical image of the $p$-faces of $P$, thus, by definition, $u \in \sigma_p(P)$ if and only if the supporting hyperplane of $P$ with exterior normal vector $u$ contains a $p$-face of $P$. We assert that the measure $\mu_p(P, \cdot)$ is concentrated on $\sigma_p(P)$. In fact, if $\omega \in S^{d-1}$ is a Borel set having empty intersection with $\sigma_p(P)$, then $\mu_p(P, \omega) = 0$ as may be seen from the last formula of Fenchel-Jessen [4] and an easy estimate of the measure of the "brush set" corresponding to $\omega$.

We shall need two lemmas concerning expressions of the type occurring in (2). Let $\mu$ be a positive Borel measure on $S^{d-1}$ which is
symmetric (i.e., attains the same value at antipodal sets). Then

\( H(x) = \int_{S_{d-1}} |\langle x, v \rangle| \mu(dv) \)

is, for \( x \in E^d \), a (symmetric) convex function. Let \( H'(x; y) \) for \( y \in E^d \setminus \{0\} \) denote the directional derivative (see Bonnesen-Fenchel [2, p. 19]) of \( H \) at \( x \) in the direction \( y \).

**Lemma 1.** If \( H \) is given by (3) with symmetric \( \mu \), then

\[
H'(x; y) = 2 \int_{S_x} \langle y, v \rangle \mu(dv) + \int_{\omega_x} |\langle y, v \rangle| \mu(dv)
\]

where

\[
S_x := \{ v \in S_{d-1} : \langle x, v \rangle > 0 \},
\]

\[
\omega_x := \{ v \in S_{d-1} : \langle x, v \rangle = 0 \}.
\]

For the easy computation, see [6, Lemma 6.1].

**Lemma 2.** If \( \mu \) is a symmetric signed Borel measure on \( S^{d-1} \) which satisfies

\[
\int_{S^{d-1}} |\langle u, v \rangle| \mu(dv) = 0 \quad \text{for each} \quad u \in S^{d-1},
\]

then \( \mu = 0 \).

Essentially, this has been proved by Aleksandrov [1, §8]. In proving his theorem quoted in the introduction, he showed the assertion of Lemma 2 to be true in the case where \( \mu \) is a difference of two \((d - 1 - p)\)-th surface area functions of convex bodies; but this assumption is not needed in the proof. To be sure, this is not a special case, since from the well known existence theorem of Minkowski, Aleksandrov, and Fenchel-Jessen [4, p. 16], it follows that every symmetric Borel measure on \( S^{d-1} \) is the difference of the \((d - 1)\)-st surface area functions of two appropriate centrally symmetric convex bodies; hence Lemma 2 follows also directly from Aleksandrov's theorem cited earlier. For further references and a generalization of Lemma 2, see [6].

We proceed now to the proof of the theorem. It is convenient to write \( d - 1 - p = q \). The assumptions of the theorem together with formula (2) give

\[
\int_{S^{d-1}} |\langle u, v \rangle| \mu_q(K, dv) = \int_{S^{d-1}} |\langle u, v \rangle| \mu_q(\overline{K}, dv)
\]
for each \( u \in A \). Let \( F \) be a \( q \)-dimensional face of the polytope \( K \). We have assumed that the set \( A \) contains a vector \( f \) which is parallel to \( F \). Since \( A \) is an open set it contains a neighborhood of \( f \). If equation (4) holds for a unit vector \( u \), it holds also for every \( \alpha u \), \( \alpha > 0 \); thus there is an open set \( U \) of \( E^d \) containing \( f \) such that (4) holds for each \( u \in U \). Therefore the convex functions which are defined by the left and the right side of (4), respectively, must have equal directional derivatives at \( f \) with respect to every direction \( y \). Then Lemma 1 yields

\[
2 \int_{\omega_f} \langle y, v \rangle \mu_q(K, dv) + \int_{\omega_f} |\langle y, v \rangle| \mu_q(K, dv) \\
= 2 \int_{\omega_f} \langle y, v \rangle \mu_q(\bar{K}, dv) + \int_{\omega_f} |\langle y, v \rangle| \mu_q(\bar{K}, dv)
\]

for each \( y \in E^d \). If we replace \( y \) by \(-y\) and add the resulting equation to the former one we see that

\[
\int_{\omega_f} |\langle y, v \rangle| \mu_q(K, dv) = \int_{\omega_f} |\langle y, v \rangle| \mu_q(\bar{K}, dv) .
\]

Since \( K \) and \( \bar{K} \) are centrally symmetric, the measures \( \mu_q(K, \cdot) \) and \( \mu_q(\bar{K}, \cdot) \) are symmetric. We can now apply Lemma 2 with the dimension \( d \) replaced by \( d - 1 \), with \( S^{d-1} \) replaced by \( \omega_f \), and with \( \mu \) replaced by the restriction of \( \mu_q(K, \cdot) - \mu_q(\bar{K}, \cdot) \) to \( \omega_f \). We deduce that

\[
\mu_q(K, \omega \cap \omega_f) = \mu_q(\bar{K}, \omega \cap \omega_f)
\]

for every Borel set \( \omega \) of \( S^{d-1} \). Now observe that the vector \( f \) has been chosen parallel to the \( q \)-face \( F \). Thus every unit vector which is orthogonal to \( F \) is contained in \( \omega_f \), hence \( \omega_f \) contains the spherical image of the face \( F \). Therefore equation (6) is especially true if \( \omega_f \) is replaced by the spherical image of \( F \). Now \( F \) is an arbitrary \( q \)-face of \( K \), hence the the additivity of the measures allows us to further replace the spherical image of \( F \) by the union of the spherical images of the \( q \)-faces of \( K \):

\[
\mu_q(K, \omega \cap \sigma_q(K)) = \mu_q(\bar{K}, \omega \cap \sigma_q(K)) .
\]

It has already been noticed that the measure \( \mu_q(K, \cdot) \) is concentrated on \( \sigma_q(K) \), therefore to intersect \( \omega \) with \( \sigma_q(K) \) on the left side of (7) is indeed superfluous; we have

\[
\mu_q(K, \omega) = \mu_q(\bar{K}, \omega \cap \sigma_q(K))
\]

for every Borel set \( \omega \) on \( S^{d-1} \). Write
\[ \nu(\omega) := \mu_q(\overline{K}, \omega) - \mu_q(K, \omega), \]

then (8) gives
\[ \nu(\omega) = \mu_q(\overline{K}, \omega \cap [S^{d-1}\setminus \sigma_q(K)]) \]

so that \( \nu \) is still a positive measure. Hence the function
\[ H(x) := \int_{S^{d-1}} \langle x, v \rangle \nu(dv), \]

defined for \( x \in E^d \), is the support function of a compact convex set \( C \).

By (4) we have \( H(u) = 0 \) for each \( u \in A \), where \( A \) is an open set on \( S^{d-1} \), and since \( H \) is even, we have also \( H(u) = 0 \) for each \( u \) in the set antipodal to \( A \). Thus \( C \) cannot contain a point different from 0.

This gives \( H(x) = 0 \) for each \( x \in E^d \), and another application of Lemma 2 shows that \( \nu \), being symmetric, must vanish identically. We have proved that the convex bodies \( K \) and \( \overline{K} \) have the same \( q \)-th surface area function, hence they differ at most by a translation (Aleksandrov [1], Fenchel-Jessen [4]).

**REFERENCES**


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