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We present a complete description of the closed sets in the coset ring $\mathscr{R}(G)$ of an abelian topological group G. Using this result we show that every such set in a separable, metrizable, locally compact, abelian group Γ is a strong Ditkin set in the sense of Wik, yielding the converse of a theorem of Rosenthal and thus completing the characterization of the strong Ditkin sets with void interior for certain choices of Γ. These two results were first obtained by J. E. Gilbert. Our development of the former rests on the following theorem, which seems to be of independent interest: If $\varphi: G \to G^*$ is a homomorphism and $A \in \mathscr{R}(G)$, then $\varphi(A) \in \mathscr{R}(G^*)$.

The computations found here are simple, and we hope that our presentation will prove to be more conceptual than those of [2] and [3]. In particular, we show that the characterization of strong Ditkin sets is a direct consequence of earlier results on such sets and the description of closed sets in the coset ring. The results in this paper were obtained independently of the work of Dr. Gilbert.

1. The coset ring.

1.1 The *coset ring* of an abelian group G, denoted by $\mathscr{R}(G)$, is the smallest Boolean algebra of subsets of G containing the cosets of all subgroups of G. It is easy to see (see [8, pp. 81-82]) that every $S \in \mathscr{R}(G)$ has the form

$$S = \bigcup_{j=1}^{N}\left(K_0^j \setminus \bigcup_{i=1}^{n(j)} K_i^j\right) \tag{1}$$

where the K_i^j are (possibly void) cosets of subgroups of G, and the group which is a translate of K_i^j is a subgroup of infinite index of the group which is a translate of K_0^j, $i = 1, \cdots, n(j)$, $j = 1, \cdots, N$.

1.2 We shall need the following restatement of a lemma of Paul Cohen.

LEMMA. ([1, pp. 223-224]) *Let $S \in \mathscr{R}(G)$ and let $\mathscr{A}$ be the smallest Boolean algebra of subsets of G containing S and all of its translates. Then $\mathscr{A}$ contains a finite collection $\mathscr{K}$ of cosets such that the Boolean algebra generated by $\mathscr{K}$ contains S.*

THEOREM 1.3. *Let G and G^* be abelian groups and $\varphi: G \to G^*$ a homomorphism. If $S \in \mathscr{R}(G)$, then $\varphi(S) \in \mathscr{R}(G^*)$.*

Proof. Since φ preserves unions and translations we need only consider, by 1.1, sets of the form $S = G_0 \backslash \bigcup_{i=1}^{n} K_i$, where G_0 is a subgroup of G and the K_i are cosets in G_0 of subgroups of G_0. Write $\varphi | G_0 = \psi \circ \pi$, where $\pi: G_0 \to G_0/(G_0 \cap H)$ is the natural map, $H = \ker \varphi$, and ψ is an isomorphism of $G_0/(G_0 \cap H)$ into G^*. We shall show that $\pi(S) \in \mathscr{R}(G_0/(G_0 \cap H))$; it then follows that $\varphi(S) = \psi(\pi(S)) \in \mathscr{R}(G^*)$. Thus if $K_1, \cdots, K_n$ are cosets in G, H is a subgroup of G, and $\pi: G \to G/H$ is the natural map, then we must show that

$$\pi(\complement K_1 \cap \cdots \cap \complement K_n) \in \mathscr{R}(G/H) ,$$

or equivalently that its complement

$$\{\xi \in G/H: \pi^{-1}(\xi) \subset K_1 \cup \cdots \cup K_n\} \in \mathscr{R}(G/H) .$$

We prove by induction that

$$S = \{x \in G: x + H \subset K_1 \cup \cdots \cup K_n\} \in \mathscr{R}(G) . \tag{2}$$

For such a set S, let K_j be a coset of the group G_j, $j = 1, \cdots, n$. In case $n = 1$, either $S = \emptyset \in \mathscr{R}(G)$ or some coset of H is contained in K_1. In the latter case K_1 is a union of cosets of H, and hence $S = K_1 \in \mathscr{R}(G)$.

Assume the induction has been carried out to some n and

$$S = \{x \in G: x + H \subset K_1 \cup \cdots \cup K_{n+1}\} .$$

If $S \neq \emptyset$ we may translate it and thus assume that $H \subset K_1 \cup \cdots \cup K_{n+1}$. Let $H_j = H \cap K_j$, $j = 1, \cdots, n + 1$. Since $x + H = \bigcup_{j=1}^{n+1} (x + H_j)$, $x \in G$, we see that

$$S = \bigcap_{j=1}^{n+1} \{x: x + H_j \subset K_1 \cup \cdots \cup K_{n+1}\} .$$

And

$$\begin{aligned} \{x: x + H_j \subset K_1 \cup \cdots \cup K_{n+1}\} &= \{x: x + H_j \subset K_j\} \cup \{x: x + H_j \subset \bigcup_{i \neq j} K_i\} \\ &= G_j \cup \{x: x + H_j \subset \bigcup_{i \neq j} K_i\} , \end{aligned}$$

which is in $\mathscr{R}(G)$ by the induction hypothesis. Hence $S \in \mathscr{R}(G)$, and the induction is complete.

Now for S a nonvoid set of the form (2), S is a union of cosets of H, and so is every member of the Boolean algebra $\mathscr{A}$ generated by S and all of its translates. By 1.2 $\mathscr{A}$ contains a finite collection $\mathscr{K}$ of cosets such that the Boolean algebra $\mathscr{B}$ generated by $\mathscr{K}$ contains S. π clearly induces a Boolean algebra homomorphism on $\mathscr{A}$, hence on $\mathscr{B}$, so $\pi(S) \in \mathscr{R}(G/H)$. Since

$$\pi(S) = \{\xi \in G/H: \pi^{-1}(\xi) \subset K_1 \cup \cdots \cup K_n\} ,$$

the theorem is proved.

LEMMA 1.4. *Let G be an abelian topological group, G_0 a dense subgroup of G, and $K_1, \cdots, K_n$ cosets in G_0. Let $S = G_0\setminus\bigcup_{i=1}^n K_i$. Then there is an open subgroup H of G such that $\bar{S}$ is a union of cosets of H.*

Proof. Let Σ denote the smallest (and necessarily finite) collection of subgroups of G satisfying $G_0 \in \Sigma$, $G_i \in \Sigma$ where K_i is a coset of G_i, $i = 1, \cdots, n$, and Σ is closed under intersections. Choose $K \in \Sigma$ minimal with respect to the property that $\bar{K}$ is open in G. Then there is a (perhaps void) subset F of $\{1, \cdots, n\}$ such that:

(i) $K = G_0 \cap \bigcap_{i \in F} G_i$,

(ii) $i \in F$ and $G_i = G_j$ imply $j \in F$.

Set $H = \bar{K}$.

Let $\tilde{H}$ be any coset of H; we must show that either $\bar{S} \cap \tilde{H} = \varnothing$ or $\tilde{H} \subset \bar{S}$. Thus suppose $y \in \tilde{H} \cap S$. Then $y + K$ is a dense subset of $\tilde{H} = y + H$; and we shall show that

$$(y + K)\setminus\bigcup_{i=1}^n K_i = (y + K)\setminus\bigcup_{i=1}^n L_i$$

is dense in $\tilde{H}$, where $L_i = K_i \cap (y + K)$, $i = 1, \cdots, n$. For $i \in F$, K is a subgroup of G_i; thus $L_i = \varnothing$ since $y + K \not\subset K_i$. If $i \notin F$ then L_i is either void or a coset of $K \cap G_i$. By the choice of K and $F (K \cap G_i)^-$ is not open, so $K \cap G_i$ is nowhere dense in G. Thus $\bigcup_{i \notin F} L_i$ is nowhere dense, whence

$$(y + K)\setminus\bigcup_{i=1}^n K_i = (y + K)\setminus\bigcup_{i \notin F} L_i$$

is dense in $\tilde{H}$. We now have

$$\tilde{H} = \Big[(y + K)\setminus\bigcup_{i=1}^n K_i\Big]^- \subset \Big[G_0\setminus\bigcup_{i=1}^n K_i\Big]^- = \bar{S}\,.$$

COROLLARY 1.5. *Let G be an abelian, connected topological group and $S \in \mathscr{R}(G)$. If S is not dense in G then S is contained in some finite union of cosets of proper closed subgroups of G.*

Proof. Let S be written in the form (1). Either each $\bar{K}_0^j$ is a coset of a proper closed subgroup of G, or else some K_0^k is dense in G. In the latter case we may translate $S_k = K_0^k\setminus\bigcup_{i=1}^{n(k)} K_i^k$ and apply 1.4, concluding that $\bar{S}_k$ is a union of cosets of some open subgroup of G. Since G is connected, we must have $H = G$; so S_k, and hence S, is dense in G.

EXAMPLE 1.6. ([6, pp. 22–24 and Appendix AO]) Let S be a closed nonvoid set in $\mathscr{R}(G)$, where $G = R$, the additive group of real numbers, or $G = T$, the circle group. Then S has one of the following forms.

(i) S is finite

(ii) $S = G$

(iii) $G = R$ and there exist a finite number $Z_1, \cdots, Z_n$ of arithmetic progressions (i.e., $Z_i = \{nx_i + y_i : n \in Z\}$ for some $0 \leqq y_i < x_i$) such that $S \triangle (Z_1 \cup \cdots \cup Z_n)$ is finite.

Proof. Since every closed, proper subgroup of R [resp., T] is cyclic [resp., finite], we need only apply 1.5 and an easy description of $\mathscr{R}(Z)$ (cf. [8, 3.1.6, p. 61]).

THEOREM 1.7. *Let G be an abelian topological group. If $S \in \mathscr{R}(G)$ then $\bar{S} \in \mathscr{R}(G)$. If $S \in \mathscr{R}(G)$ is closed, then S has the form* (1) *where the K_i^j are closed (possibly void) cosets in G such that for each $j = 1, \cdots, N$ K_i^j is relatively open in K_0^j, $i = 1, \cdots, n(j)$.*

Proof. Let $S \in \mathscr{R}(G)$ of the form $S = G_0 \backslash \bigcup_{i=1}^{n} K_i$, where G_0 is a subgroup of G and the K_i are cosets contained in G_0. By 1.4 there is a relatively open subgroup H of $\bar{G}_0$ such that $\bar{S}$ is a union of cosets of H. If $\pi : \bar{G}_0 \rightarrow \bar{G}_0/H$ is the natural homomorphism, then by Theorem 1.3 $\pi(S) \in \mathscr{R}(\bar{G}_0/H)$, say

$$\pi(S) = \bigcup_{j=1}^{M} \left(L_0^j \backslash \bigcup_{i=1}^{m(j)} L_i^j \right),$$

the L_i^j being cosets in $\bar{G}_0/H$. And $\pi(S) = \pi(\bar{S})$ since π is continuous and $\bar{G}_0/H$ is discrete. Thus

$$\begin{aligned} \bar{S} &= \pi^{-1}(\pi(\bar{S})) = \pi^{-1}(\pi(S)) \\ &= \bigcup_{j=1}^{M} \left[\pi^{-1}(L_0^j) \backslash \bigcup_{i=1}^{m(j)} \pi^{-1}(L_i^j) \right], \end{aligned}$$

where each $\pi^{-1}(L_i^j)$ is open in $\bar{G}_0$.

If S is an arbitrary set in $\mathscr{R}(G)$, then $S = S_1 \cup \cdots \cup S_N$, where each S_k is a translate of a set of the type just described. Thus $\bar{S} = \bar{S}_1 \cup \cdots \cup \bar{S}_N \in \mathscr{R}(G)$ and has the desired form.

COROLLARY 1.8. *Let G and S be as in* 1.7, *and suppose S is compact. Then S is a finite union of compact cosets.*

Proof. S has the form (1) as in 1.7; fix j and denote $n(j)$ by n. Let K_i^j be a nonvoid coset of the group G_i, $i = 1, \cdots, n$, and let $H =$

$G_1 \cap \cdots \cap G_n$. Then $K_0^j \setminus \bigcup_{i=1}^n K_i^j$ is a compact set which is a union of relatively open cosets of H; this union must therefore be a finite one, and the corollary follows.

EXAMPLE 1.9. Let $G = R^p \times T^q$, p and q nonnegative integers. The structure of all closed subgroups of G is well known (e.g., see [4, Th. 9.11, pp. 92–94]), and this structure along with Theorem 1.7 allows one to give a complete description of the closed sets in $\mathscr{R}(G)$ in the manner of 1.6.

1.10. Another example of the structure developed in 1.7 is afforded by the additive groups of Δ_p, the p-adic integers, and Ω_p, the p-adic number field, p a prime. (see [4, §10]). Recall that every closed proper subgroup of Ω_p is one of the open, compact, canonical subgroups Λ_n, $n \in Z$, which are topologically isomorphic with Δ_p. Notice the similarity of the following description and the case $G = Z(p^\infty)$, where the fact that every proper subgroup is finite implies that every set in $\mathscr{R}(G)$ is finite or the complement of a finite set.

EXAMPLE. If S is a closed, nonvoid set in $\mathscr{R}(\Omega_p)$, then S has one of the following forms:

(i) S is finite.

(ii) $S = \Omega_p$.

(iii) S is the union of a finite set and a finite collection of cosets of some one Λ_n.

(iv) S is the union of a finite set and the complement of a finite union of cosets of some one Δ_n.

In particular, every closed $S \in \mathscr{R}(\Omega_p)$ is the union of a finite set and an open and closed set.

Proof. Let $\Omega_p \neq S \in \mathscr{R}(\Omega_p)$, and let S be infinite and written in the form (1) as in 1.7. We may assume that each of the K_i^j is infinite. Let $H = \bigcap_{j=1}^N \bigcap_{i=0}^{n(j)} G_i^j$, each K_i^j being a coset of the open subgroup G_i^j of Ω_p. Then $H = \Lambda_n$ for some $n \in Z$, and $[G_i^j : H] < \infty$ unless $i = 0$ and $G_0^j = \Omega_p$. For any fixed j, if $G_0^j \neq \Omega_p$, then $K_0^j \setminus \bigcup_{i=1}^{n(j)} K_i^j$ has the form (iii). And if $G_0^j = K_0^j = \Omega_p$ we have $K_0^j \setminus \bigcup_{i=1}^{n(j)} K_i^j$ satisfying (iv). The result follows.

REMARK 1.11. In [5] we have shown that the examples given in this section are the only locally compact ones whose coset rings have the respectively indicated properties. More precisely:

THEOREM. ([5]) *Let G be a locally compact abelian group. Every nontrivial, proper closed subgroup of G is*

(i) *finite*
(ii) *of finite index*
(iii) *compact*
(iv) *open*
(v) *discrete*

if and only if G *is, respectively,*

(i) T, $Z(p^\infty)$ (*for some prime* p), *or finite.*
(ii) Z, Δ_p (*some* p), *or finite.*
(iii) Ω_p, $Z(p^\infty)$, *or compact.*
(iv) Δ_p, Ω_p, *or discrete.*
(v) T *or* R.

2. Strong Ditkin sets.

2.1. Let G be a locally compact, metrizable, separable, abelian group (so that its character group Γ has these properties also), and let E be a closed subset of Γ. Let

$$I(E) = \{f \in L^1(G) : \hat{f} \equiv 0 \quad \text{on} \quad E\}$$

($\hat{f}$ denotes the Fourier transform of f) and

$$I_0(E) = \{f \in L^1(G) : \hat{f} \equiv 0 \quad \text{on a neighborhood of} \quad E\}\ ,$$

and recall that E is said to be of *spectral synthesis* if $[I_0(E)]^- = I(E)$. E is called a *Ditkin set* (C-set in [8]) if for every $f \in I(E)$ we can find a sequence $(u_n)_{n=1}^\infty$ in $I_0(E)$ such that $\| u_n * f - f \|_1 \to 0$ as $n \to \infty$. Following [9], if the sequence $(u_n)_{n=1}^\infty$ may be chosen independently of f, then E is called a *strong Ditkin set.* We follow notation in [8] throughout this section.

LEMMA 2.2. ([7, Lemma 2.2 (b)]) *Let* E *be a closed subset of* Γ. E *is a strong Ditkin set if and only if there is a sequence* $(\mu_n)_{n=1}^\infty$ *in* $M(G)$ *such that* $\hat{\mu}_n \equiv 1$ *on a neighborhood of* E, $n = 1, 2, \cdots$, *and* $\| \mu_n * f \|_1 \to 0$ *for all* $f \in I(E)$.

LEMMA 2.3. ([9, Th. 3]) *Finite unions of strong Ditkin sets are strong Ditkin sets.*

LEMMA 2.4. ([7, Th. 2.3]) *Every closed coset in* Γ *is a strong Ditkin set.*

LEMMA 2.5. *Let* Λ *be a closed subgroup of* Γ *and* Δ *a relatively open subgroup of* Λ. *There exists* $\mu \in M(G)$ *such that* $\hat{\mu} \equiv 0$ *on* Δ *and* $\hat{\mu} \equiv 1$ *on a neighborhood of* $\Lambda \backslash \Delta$.

Proof. Let K denote the annihilator of $\varDelta$ in G, and $\pi: \varGamma \to \varGamma/\varDelta$ the natural homomorphism. Since $\varLambda/\varDelta$ is discrete in $\varGamma/\varDelta$, we may choose an open set U in $\varGamma/\varDelta$ such that $\bar{U} \cap (\varLambda/\varDelta) = \{0\}$ and $f \in L^1(K)$ such that $\hat{f}(0) = 1$ and $\hat{f} \equiv 0$ off U. Setting $\mu = \delta_0 - f dm_K \in M(G)$ (m_K being the chosen Haar measure on K), we have $\hat{\mu}(\gamma) = 1 - \hat{f}(\pi(\gamma)) = 0, \gamma \in \varDelta$, and $\hat{\mu}(\gamma) = 1$ if $\gamma \in \complement\pi^{-1}(\bar{U})$, a neighborhood of $\varLambda \backslash \varDelta$.

THEOREM 2.6. *Every closed set in* $\mathscr{R}(\varGamma)$ *is a strong Ditkin set.*

Proof. First note that 2.5 holds, by translation, if $\varLambda$ and $\varDelta$ are cosets in $\varGamma$. If $\varLambda$ is a closed coset in $\varGamma$ and $\varDelta_1, \cdots, \varDelta_m$ are relatively open sub-cosets in $\varLambda$, let μ_i be the measure constructed in 2.5 for $\varDelta_i$ and $\varLambda$, $i = 1, \cdots, m$. Let $S = \varLambda \backslash \bigcup_{i=1}^{m} \varDelta_i$, and set $\mu = \mu_1 * \cdots * \mu_m$. Then $\hat{\mu} \equiv 0$ on each $\varDelta_i$ and $\hat{\mu} \equiv 1$ on a neighborhood of S. By 2.2 and 2.4 there exists a sequence $(\nu_n)_{n=1}^{\infty}$ in $M(G)$ such that $\hat{\nu}_n \equiv 1$ on a neighborhood of $\varLambda$ and $\|\nu_n * f\|_1 \to 0$ if $f \in I(\varLambda)$. Let $\sigma_n = \nu_n * \mu$, $n = 1, 2, \cdots$. Then $\hat{\sigma}_n \equiv 1$ on a neighborhood of S, $n = 1, 2, \cdots$, and if $f \in I(S)$, then $\mu * f \in I(\varLambda)$, so $\|\sigma_n * f\|_1 = \|\nu_n * \mu * f\|_1 \to 0$. Thus S is a strong Ditkin set, and the theorem follows from 1.7 and 2.3.

2.7. If we combine [7, Th. 1.3] and 2.6 we obtain the following theorem, which is [7, Th. 2.5] in case $G = \varGamma = R$.

THEOREM. *Let* $\varGamma$ *be* R^n, T^n, *or any compact metrizable group such that the union of all of its finite subgroups is dense. Let* E *be a closed, nowhere-dense subset of* $\varGamma$. *Then* E *is a strong Ditkin set if and only if* $E \in \mathscr{R}(\varGamma)$.

REMARK 2.8. As noted in [7], the notion of strong Ditkin sets may be extended to arbitrary locally compact abelian groups by replacing the sequence in 2.1 by a net $(u_\delta)_{\delta \in D}$ which is uniformly bounded in convolution operator norm on $I(E)$, and the analogous results in [7] remain valid. If we make the stronger assumption that $\sup_{\delta \in D} \|u_\delta\|_1 < \infty$ we also obtain the analog of [7, Th. 1.3] for every G and Γ, without any restriction on the set E. Since 2.6 also holds under this stronger definition, we can obtain a theorem like 2.7 for arbitrary G and $\varGamma$: $I(E)$ has a norm-bounded approximate identity if and only if $E \in \mathscr{R}(\varGamma)$.

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