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**SOME REMARKS ON VARIETIES IN POLYDISCS AND
BOUNDED HOLOMORPHIC FUNCTIONS**

EDGAR LEE STOUT

SOME REMARKS ON VARIETIES IN POLYDISCS AND BOUNDED HOLOMORPHIC FUNCTIONS

E. L. STOUT

This note deals with certain questions which arise in connection with the extension problem for bounded holomorphic functions of several complex variables.

An analytic variety V in the unit polydisc

$$U^N = \{(z_1, \dots, z_N) \in \mathbb{C}^N: |z_1|, \dots, |z_N| < 1\}$$

is said to have the H^∞ -extension property if for every $f \in H^\infty(V)$, the space of bounded holomorphic functions on V , there is $F \in H^\infty(U^N)$ such that $F|_V = f$. A related property which V may possess is that of being *defined as a set by bounded functions* in the sense that

$$V = \{z \in U^N: f(z) = 0 \text{ for all } f \in H^\infty(U^N) \text{ which vanish on } V\}.$$

We will begin with an example of a remarkably well behaved variety $V \subset U^N$ which fails to have the H^∞ -extension property, and after this we give an example of a one dimensional disc D embedded as a submanifold of U^N which not only fails to be defined as a set by bounded functions but is, in fact, a *determining set* for $H^\infty(U^N)$ in that if $f \in H^\infty(U^N)$ vanishes on D , then f is the zero function. Positive results obtained include a geometric condition for a U^k embedded in a U^N to be defined as a set by bounded functions and a result to the effect that if a variety V has the H^∞ -extension property and if it satisfies another, possibly redundant, condition, then V is defined as a set by bounded functions.

The general problem of determining which subvarieties of U^N possess the H^∞ -extension property seems to be difficult, but some results in this direction are contained in the papers [1] and [9].

We begin with an example of a disc contained in U^2 which does not have the H^∞ -extension property.

EXAMPLE 1. Denote by \mathbb{C}^* the Riemann sphere and in \mathbb{C}^* let

$$\mathcal{A} = \{\zeta: |1 - \zeta^2|^{-1} < 1\}.$$

The set \mathcal{A} is conformally equivalent to U . If we define $\Phi: \mathbb{C}^* \setminus \{1, -1\} \rightarrow \mathbb{C}^2$ by $\Phi(\zeta) = ((1 - \zeta^2)^{-1}, \varepsilon(1 - \zeta)^{-1})$, then for all choices of $\varepsilon > 0$, the mapping Φ carries $\mathbb{C}^* \setminus \{1, -1\}$ biholomorphically onto a closed, algebraic submanifold M of \mathbb{C}^2 , and if ε is small enough, then $M \cap U^2 = \Phi(\mathcal{A})$ so that $\Phi(\mathcal{A})$ is a disc embedded in U^2 . Assume that $\Phi(\mathcal{A})$ has the

H^∞ -extension property so that if $f \in H^\infty(\mathcal{A})$, there is $F \in H^2(U^2)$ such that $f = F \circ \Phi$. Let $h: \mathcal{A} \rightarrow U$ be a conformal homeomorphism chosen so that $\lim_{y \rightarrow 0^+} h(iy) = 1$ and $\lim_{y \rightarrow 0^-} h(iy) = -1$. Our assumption on $\Phi(\mathcal{A})$ implies that if $\delta > 0$ is small enough, then there exists $F \in H^\infty(U^2)$ such that $\|F\|_{U^2} \leq 1$ and $F(\Phi(\zeta)) = \delta h(\zeta)$ for all $\zeta \in \mathcal{A}$. In particular

$$(1) \quad \lim_{y \rightarrow 0^+} F(\Phi(\pm iy)) = \pm \delta.$$

If z and w are points of U , set

$$[z, w] = \left| \frac{z - w}{1 - \bar{w}_z} \right|.$$

By the invariant form of Schwarz's lemma, we know that if g lies in the unit ball of $H^\infty(U)$, then

$$(2) \quad [g(z), g(w)] \leq [z, w].$$

If we apply (2) to the function h_η given for fixed $\eta \in U$ by $h_\eta(\zeta) = F(\eta, \zeta)$, then for $\eta = (1 - (iy)^2)^{-1}$, $z = \varepsilon/(1 - iy)$, $w = \varepsilon/(1 + iy)$, we are led to

$$\lim_{y \rightarrow 0^+} [F(\Phi(iy)), F(\Phi(-iy))] \leq \lim_{y \rightarrow 0^+} \left[\frac{\varepsilon}{1 - iy}, \frac{\varepsilon}{1 + iy} \right] = 0.$$

However, (1) implies that

$$\lim_{y \rightarrow 0^+} [F(\Phi(iy)), F(\Phi(-iy))] = \frac{2\delta}{1 + \delta^2}.$$

This contradiction shows that the disc $\Phi(\mathcal{A})$ does not have the H^∞ -extension property.

The present example is not the first known instance of this phenomenon; another, more involved example was given in [9, Example II. 7]. It can be shown, at least for certain choices of the Blaschke product involved, that example has the additional property of being a determining set for $H^\infty(U^2)$. The variety $\Phi(\mathcal{A})$ of the present example is not nearly so pathological, for it is the intersection of U^2 with the zero set of a certain polynomial in two variables. Alexander [1] has also given an example of a variety in U^2 which lacks the H^∞ -extension property. His example is the intersection of U^2 with a certain algebraic curve, but it is not irreducible. In [8] Rudin has also given an example.

In connection with Example 1, it is interesting to consider the composition $\Phi \circ \psi$ where ψ is a conformal homeomorphism from the unit disc to \mathcal{A} such that $\psi(0) = \infty$. If we let $\varphi_1(\zeta) = (1 - \zeta^2)^{-1}$, then $\varphi_1 \circ \psi$ is a two-to-one map from the disc onto itself, and it can be

verified without difficulty that $\varphi_1 \circ \psi(z) = \alpha z^2$ for some α of modulus one. Also, if $\varphi_2(\zeta) = \varepsilon(1 - \zeta)^{-1}$, then $\varphi_2 \circ \psi$ is one-to-one from the disc into itself, and it is not hard to see that if z_0 and z_1 are the two points in the unit circle carried onto 0 by ψ , then $\varphi_2 \circ \psi$ continues across the two arcs of the unit circle determined by z_0 and z_1 . These two points are certain algebraic singularities of the function $\varphi_2 \circ \psi$. These remarks should be compared with the extension theorems for bounded holomorphic functions proved in [9] and [10]; they show that those extension theorems are essentially the best of their kind.

EXAMPLE 2. In this example we will construct in U^N , $N \geq 2$, a disc which is a determining set for $H^\infty(U^N)$.

Let $\Omega = U \setminus [0, 1)$, and let $h: U \rightarrow \Omega$ be a conformal homeomorphism which takes 1 to 1, i to 0 and which has the property that $\text{Im } h(\zeta) \downarrow 0$ as $\zeta \rightarrow e^{i\theta}$ if $\theta \in (0, \pi/2)$. The function h admits a unique extension to a continuous function from \bar{U} to \bar{U} .

Let $r_k = 1 - k^{-1}$, and let $s_k > 0$ be very small so that $\{r_k + is_k\}$ does not satisfy the Blaschke condition, i.e., this sequence is not the zero set of a function bounded and holomorphic in U . If $\{s_k\}$ is chosen properly and if $\alpha_k = h^{-1}(r_k + is_k)$, the sequence $\{\alpha_k\}$ will satisfy the Blaschke condition. Let B be the Blaschke product with $\{\alpha_k\}$ as its zero set, and define Φ by $\Phi(\zeta) = (h(\zeta), B(\zeta))$. The sequence $\{\alpha_k\}$ converges to the point 1, so it follows that at every point of ∂U , either $|B|$ or $|h|$ assumes continuously the value 1. Since h' is zero-free and h is one-to-one, it follows that $\Delta = \Phi(U)$ is an analytic submanifold of U^2 .

We will prove that if $F \in H^\infty(U^2)$ and $F \circ \Phi = 0$, then F is the zero function, i.e., that Δ is a determining set for $H^\infty(U^2)$. If $F \in H^\infty(U^2)$ vanishes on $\Phi(U)$, then $F(r_k + is_k, 0) = 0$ for all k , so since $\{r_k + is_k\}$ does not satisfy the Blaschke condition, F must vanish identically on the disc $D = \{(z, 0): |z| < 1\}$. If F does not vanish identically, there is a factorization $F(z, w) = w^p G(z, w)$ where p is a positive integer and G a bounded holomorphic function which does not vanish identically on D . As F vanishes on $\Phi(U)$, G must also. This implies, as we have just seen, that $G(r_k + is_k, 0) = 0$ whence G vanishes on the disc D , contrary to hypothesis.

Thus we have a disc in U^2 which is a determining set for $H^\infty(U^2)$. It is quite simple, using the existence of this disc, to find a disc in U^N , $N \geq 2$, which is a determining set for $H^\infty(U^N)$. We proceed inductively. Suppose that $\Delta \subset U^k$ is a disc which is a determining set for $H^\infty(U^k)$. Then $U^{k+1} = U^k \times U \supset \Delta \times U$. The set $\Delta \times U$ is bi-holomorphically equivalent to U^2 so there is a one dimensional disc Δ' which is a determining set for $\Delta \times U$. Suppose that $F \in H^\infty(U^{k+1})$ vanishes on Δ' . If we take on U^{k+1} the coordinates (\mathfrak{z}, ζ) , $\mathfrak{z} \in U^k$, $\zeta \in U$, then since Δ' is a determining set for $\Delta \times U$, it follows that for each

$\zeta \in U$, $F(\cdot, \zeta)$ vanishes identically. As this holds for every $\zeta \in U$, F must be the zero function.

Our next example is a direct consequence of the construction given in Example 2.

EXAMPLE 3. In U^N , $N \geq 3$, there exist irreducible varieties V which are at positive distance, in the sense of the usual metric on C^N , from the distinguished boundary, T^N , of U^N and yet which are not defined, as sets, by bounded functions. To obtain such an example, let Δ be an irreducible variety, e.g., a disc, which is a determining set for $H^\infty(U^2)$. The set $\Delta \times \{0\} \subset U^2 \times U^{N-2} = U^N$ is an example of a variety of the desired kind.

This example is of interest because it contracts markedly with a theorem of Rudin [7] according to which *if $V \subset U^N$ is a variety of codimension 1 which is at positive distance from T^N , then not only is V defined as a set by a single bounded function, but, in addition, there is an $F \in H^\infty(U^N)$ with the property that every function holomorphic in U^N and vanishing on V admits a factorization $G = FH$, H holomorphic in U^N .*¹

Our next result gives a sufficient condition for a disc or polydisc contained in U^N to be defined, as a set, by bounded holomorphic functions.

THEOREM 4. Let $\Phi: U^k \rightarrow U^N$ be a proper, holomorphic map, $k \leq N$, say $\Phi(z) = (\varphi_1(z), \dots, \varphi_N(z))$. If there is a $\delta > 0$ with the property that for each $z \in U^k$ at least $N - k$ of $|\varphi_1(z)|, \dots, |\varphi_N(z)|$ are no more than $1 - \delta$, then the variety $\Phi(U^k)$ is defined as a set by bounded holomorphic functions.

Let us remark that since Φ is proper, $\Phi(U^k)$ is a variety by [4, Th. V.C. 5].

Proof. Consider first the case that $k = 1$. Let

$$K = \{(z_1, \dots, z_N) \in U^N: |z_1|, \dots, |z_N| \leq 1 - \delta\}.$$

The set $\Phi^{-1}(K)$ is compact, and by the maximum modulus theorem no component of the set $\Sigma = U \setminus \Phi^{-1}(K)$ can be bounded away from ∂U , so Σ is connected. If $\zeta \in \Sigma$, then for some j , $|\varphi_j(\zeta)| > 1 - \delta$. Let

$$\Sigma_j = \{\zeta \in \Sigma: |\varphi_j(\zeta)| > 1 - \delta\}.$$

¹ Added in proof. Y.-T. Siu in his paper Sheaf cohomology with bounds and bounded holomorphic functions, Proc. Amer. Math. Soc. **21** (1969), 226-229, has given a cohomological proof of this and a related result.

The sets Σ_j are all open, they are pairwise disjoint, and their union is the connected set Σ . Thus one of them, say Σ_1 , is the whole of Σ and all the other Σ_j are empty. The map Φ is proper so it follows that $|\varphi_1(\zeta)| \rightarrow 1$ as $|\zeta| \rightarrow 1$. (Although we do not need this fact, it follows that φ_1 is a finite Blaschke product.)

Now consider the case of general k . By [8, Th. II. 3], we may reindex the functions $\varphi_1, \dots, \varphi_N$, so that if $z = (z_1, \dots, z_N)$, then for $1 \leq j \leq k$, $\varphi_j(z)$ depends only on z_j and so that if

$$\varphi_j^*(e^{i\theta}) = \lim_{r \rightarrow 1^-} \varphi_j(re^{i\theta}),$$

then $|\varphi_j^*(e^{i\theta})| = 1$ on a set of θ 's of positive measure. It follows that we can choose z_2^0, \dots, z_k^0 , of modulus less than one so that for some $\eta, 1 > \eta > |\varphi_j(z_j^0)| > 1 - \delta$. Define $\psi: U \rightarrow U^k$ by

$$\psi(\zeta) = (\zeta, z_2^0, \dots, z_k^0).$$

The map ψ is proper, so $\Phi \circ \psi$ is a proper map from U into U^N . We have that of the N coordinates of $\Phi(\psi(\zeta))$, $k - 1$, viz., $\varphi_2(z_2^0), \dots, \varphi_k(z_k^0)$, exceed $1 - \delta$, so by our hypothesis on Φ , at least $N - k$ of $|\varphi_1(\zeta)|, |\varphi_{k+1}(\psi(\zeta))|, \dots, |\varphi_N(\psi(\zeta))|$ are less than $1 - \delta$. Thus the map $\Phi \circ \psi$ has the property that if $\Phi \circ \psi(\zeta) = (w_1, \dots, w_N)$, then at least $N - 1$ of $|w_1|, \dots, |w_N|$ are less than η . By our consideration of the case $k = 1$, it follows that one of $|\varphi_1|, |\varphi_{k+1} \circ \psi|, \dots, |\varphi_N \circ \psi|$, tends to one at the boundary of the unit disc while the others remain bounded away from one. As $|\varphi_1^*| = 1$ on a set of positive measure, we may conclude that $|\varphi_1(z_1)| \rightarrow 1$ as $|z_1| \rightarrow 1$. In the same way we can show that $|\varphi_j(z_j)| \rightarrow 1$ as $|z_j| \rightarrow 1, 2 \leq j \leq k$.

Let $\pi: U^N \rightarrow U^k$ be the natural projection onto the first k coordinates, and set $V = \Phi(U^k)$. We know that V is a variety, and what we have done implies that π carries V properly onto U^k . Thus the triple $(V, \pi|_V, U^k)$ is an analytic cover so our result is a consequence of the following general fact.

LEMMA 5. *Let $\Omega \subset \mathbb{C}^m$ and $\Omega' \subset \mathbb{C}^n$ be bounded domains, let $V \subset \Omega \times \Omega'$ be a purely m dimensional variety, and let $\pi: \Omega \times \Omega' \rightarrow \Omega$ be the natural projection. If $(V, \pi|_V, \Omega')$ is an analytic cover, then V is defined as a set by bounded holomorphic functions on $\Omega \times \Omega'$.*

This lemma is contained in the proof of [4, III. B. 19].

We finish with a result which partially-only partially-answers an obvious question: If the variety $V \subset U^N$ has the H^∞ -extension property, does it necessarily follow that V is defined as a set by bounded holomorphic functions? It seems probable that this question has an

affirmative answer without qualification on the variety V , but we are able to prove a result in this direction only by making an additional assumption.

THEOREM 6. *If $V \subset U^N$ is a variety with the H^∞ -extension property and if V is open in the spectrum of $H^\infty(V)$, then V is defined as a set by bounded holomorphic functions.*

We understand by the *spectrum* of a commutative Banach algebra A the space consisting of the nonzero complex homomorphisms of A taken with the weak* topology. We denote the spectrum of A by $\Sigma(A)$.

Proof. We define an ideal $I^\infty(V)$ and a variety \tilde{V} by

$$I^\infty(V) = \{f \in H^\infty(U^N): f \text{ vanishes on } V\}$$

and

$$\tilde{V} = \{\zeta \in U^N: f(\zeta) = 0 \text{ for all } f \in I^\infty(V)\}.$$

The variety \tilde{V} evidently contains V and we will prove, under the hypotheses of the theorem, that $\tilde{V} = V$. The restriction map ρ from $H^\infty(U^N)$ to $H^\infty(V)$ is onto and consequently $\Sigma(H^\infty(V))$ can be identified with the set

$$\{\varphi \in \Sigma(H^\infty(U^N)): \varphi f = 0 \text{ if } f \in I^\infty(V)\}.$$

This set contains \tilde{V} in a natural way and as V is assumed to be open in $\Sigma(H^\infty(V))$, it follows that V is an open subset of \tilde{V} . Plainly, V is closed in \tilde{V} .

As $\tilde{V} \supset V$, the hypotheses of the theorem imply that the restriction map ρ' from $H^\infty(\tilde{V})$ to $H^\infty(V)$ is onto so we can identify $\Sigma(H^\infty(V))$ with

$$\{\varphi \in \Sigma(H^\infty(\tilde{V})): \varphi f = 0 \text{ if } f \in \ker \rho'\}.$$

The characteristic function χ of $\tilde{V} \setminus V$ lies in $H^\infty(\tilde{V})$ since V is open and closed in \tilde{V} . Since $\chi \in \ker \rho'$, it follows that $\tilde{V} \setminus V$ cannot meet $\Sigma(H^\infty(V))$. We know that $\tilde{V} \subset \Sigma(H^\infty(V))$ so we conclude that $\tilde{V} = V$ as was to be proved.

Our formulation of Theorem 6 suggests another question: *If V is an analytic variety, is it open in $\Sigma(H^\infty(V))$?* This question does not seem to have an obvious answer even for subvarieties of a polydisc though it does seem likely that generally V is open in $\Sigma(H^\infty(V))$. The following remarks are relevant.

REMARKS 7. (a) It is well known that the unit disc is open in $\Sigma(H^\infty(U))$. (See [5].) Similarly, U^N is open in $\Sigma(H^\infty(U^N))$.

(b) It is not hard to see that for many familiar open sets S in C^N , e.g., balls, special analytic polyhedra [4], S is open in $\Sigma(H^\infty(S))$.

(c) For general one dimensional varieties V , we do not know that V is open in $\Sigma(H^\infty(V))$, but the following rather ad hoc argument settles the question for certain Riemann surfaces. Let R be an open connected Riemann surface of finite genus so that $R = R_1 \setminus E$, R_1 a compact Riemann surface and E a closed subset thereof. If $H^\infty(R)$ contains a nonconstant function, then R is open in $\Sigma(H^\infty(R))$. Since R is contained in a compact surface and $H^\infty(R)$ contains a nonconstant function, it follows easily from the Riemann-Roch theorem that $H^\infty(R)$ separates points on R . (In the case that R_1 is of genus zero, this sort of result is in papers of Rudin [6] and Wermer [11]; the case of general, finite, genus follows in an analogous way.)

Let $\zeta_0 \in R$. By the Riemann-Roch theorem there exists a function h meromorphic on the ambient surface R_1 which has only one pole, that at ζ_0 and of assigned order p if p is large enough. Thus, for a suitable function $h_1 \in H^\infty(R)$, the function $H = hh_1$ will have at ζ_0 a simple pole, it will be holomorphic on $R \setminus \{\zeta_0\}$, and it will be bounded off a neighborhood of ζ_0 . Define an operator $T: H^\infty(R) \rightarrow H^\infty(R)$ by

$$T(f) = (f - f(\zeta_0))H.$$

The properties of the function H show that T is a bounded linear operator on $H^\infty(R)$ and that

$$T(fg) = gT(f) + f(\zeta_0)T(g).$$

Thus in the terminology of Banaschewski [3], T is a *bounded derivation of type* (I, ζ_0) . By Proposition 1 of [3], a result previously obtained by Bishop [2], there is a homeomorphism Φ from the open unit disc U onto an open set in $\Sigma(H^\infty(R))$ such that $\Phi(0) = \zeta_0$ and such that if $f \in H^\infty(R)$, then $\hat{f} \circ \Phi$ is holomorphic on U . Since there is a disc in R through ζ_0 , it follows from the openness of $\Phi(U)$ in $\Sigma(H^\infty(R))$ that some neighborhood of ζ_0 in R is at the same time a neighborhood of ζ_0 in $\Sigma(H^\infty(R))$. It follows that R is open in $\Sigma(H^\infty(R))$ as was to be proved.

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