Pacific Journal of Mathematics

IN THIS ISSUE—

Vol. 33, No. 1 **March, 1970**

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON SOME EXTREMAL SIMPLEXES

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Let A be a fixed point in n-dimensional Euclidean space. Let $B_1, B_2, \cdots, B_{n+1}$ be the vertices of a simplex S_n of n-dimensions, that is, the $n+1$ vertices do not lie on a $(n-1)$ dimensional subspace. Let d_i, assumed to be positive, be the distance of B_i from A, and let l_{ij} be the cosine of the angle between the straight lines AB_i and AB_j for $i, j = 1, 2, \cdots, n+1$. Let π_i denote the $(n-1)$-dimensional hyperplane passing through all the vertices of S_n except B_i, let p_i, assumed positive, be the perpendicular distance of π_i from A, and let m_{ij} denote the cosine of the angle between the normals from A to π_i and π_j for $i, j = 1, 2, \cdots, n+1$. The present paper deals with the following problems.

(a) An expression for the content of S_n, $C(S_n)$ say, in terms of d_i and l_{ij} for $i, j = 1, 2, \cdots, n+1$ is first obtained. Then leaving $d_1, d_2, \cdots, d_{n+1}$ fixed, values of l_{ij}, say l_{ij}^*, are determined in such a manner that $C(S_n)$ is a maximum, and the maximum value of $C(S_n)$ is obtained for the two cases that arise: (i) when A is inside S_n, (ii) when A is outside S_n. The latter case does not arise when $d_1 = d_2 = \cdots = d_{n+1}$.

(b) An expression for $C(S_n)$ is obtained in terms of p_i and m_{ij}, $i, j = 1, 2, \cdots, n+1$. Then leaving $p_1, p_2, \cdots, p_{n+1}$ fixed, values for m_{ij}, say m_{ij}^*, are determined in such a manner that $C(S_n)$ is a minimum, and such $C(S_n)$ is computed for the two cases that arise depending on (i) whether A is inside S_n or (ii) A is outside S_n. The latter case does not arise when

$$p_1 = p_2 = \cdots = p_{n+1} .$$

The results are stated below.

(a) The content of S_n, max $C(S_n)$ and l_{ij}^* are given by

$$n!C(S_n) = |(l_{ij}d_id_j + 1)|^{1/2} \tag{1.1}$$

$$\max (n!C(S_n))^2 = -u^{-1}\prod_{i=1}^{n+1}(d_i^2 - u) \tag{1.2}$$

$$l_{ij}^* = u/(d_id_j) \text{ for } i, j = 1, 2, \cdots, n+1\,;\quad i \neq j\,, \tag{1.3}$$

where u satisfies the equation

$$1 + u\sum_{i=1}^{n+2}(d_i^2 - u)^{-1} = 0\,. \tag{1.4}$$

The unique negative root for u in (1.4) corresponds to the case when A is inside S_n. When the relation

$$d_1 = d_2 = \cdots = d_{n+1}$$

is not satisfied, the smallest positive root for u in (1.4) corresponds to the case when A is outside S_n. Other roots for u in (1.4), if any, are inadmissible.

(b) The content $C(S_n)$, min $(C(S_n))$ and m_{ij}^* are given by

$$(n!C(S_n))^2 = |(p_i p_j + m_{ij})|^n / \prod_{i=1}^{n+1} |M_{ii}| \tag{1.5}$$

where $|M_{ii}|$ is the cofactor of m_{ii} in $|(m_{ij})|$ and

$$\min (n!C(S_n))^2 = -v^{-1} n^{2n} \prod_{i=1}^{n+1} (p_i^2 - v) \tag{1.6}$$

and

$$m_{ij}^* = v/(p_i p_j) \text{ for } i \neq j \;;\; i, j = 1, 2, \cdots, n+1 \;; \tag{1.7}$$

where v satisfies the equation

$$1 + v \sum_{i=1}^{n+1} (p_i^2 - v)^{-1} = 0 \,. \tag{1.8}$$

The unique negative root for v in (1.8) corresponds to the case when A is inside S_n. When the relation

$$p_1 = p_2 = \cdots = p_{n+1}$$

is not satisfied, the smallest positive root for v in (1.8) corresponds to the case when A is outside S_n. All other roots, if any, are inadmissible.

When $d_1 = d_2 = \cdots = d_{n+1}$, we obtain the special result that the largest simplex inscribed in a sphere of n-dimensions is a regular one, while when $p_1 = p_2 = \cdots = p_{n+1}$ the smallest simplex circumscribing a sphere is a regular one.

The coordinates of B_i referred to a n-dimensional Cartesian coordinate system with origin at A will be denoted by $(x_{i,1}, x_{i,2}, \cdots, x_{i,n})$. $(x_1, x_2, \cdots, x_n)$ will denote a general point in the n-space.

2. Extremal simplex determined by the distance of vertices. The content of S_n is given by (Sommerville, p. 124) $n!C(S_n) = |V|$ where

$$V = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} & 1 \\ x_{2,1} & \cdots & x_{2,n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} & 1 \end{bmatrix} \tag{2.1}$$

so that $(n!C(S_n))^2 = |VV'| = |(w_{ij})|$ say, where

$$w_{ij} = 1 + s_{ij} \text{ for } i, j = 1, 2, \cdots, n+1; \text{ and} \tag{2.2}$$

$$(s_{ij}) = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} \end{bmatrix} \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & & \vdots \\ x_{n+1,1} & \cdots & v_{n+1,n} \end{bmatrix}' \tag{2.3}$$

$$= (l_{ij} d_i d_j) \,. \tag{2.4}$$

Hence we have proved (1.1).

We note that $s_{ii} = d_i^2$, for $i = 1, 2, \cdots, n+1$. From (2.3) we also note that the rank of (s_{ij}) is less than $n+1$ so that $|(s_{ij})| = 0$ and (s_{ij}) is semi-positive definite. Further we note that both (s_{ij}) and (w_{ij}) are symmetric matrices and since $B_1, \cdots, B_{n+1}$ do not lie on a $(n-1)$-dimensional subspace, we must have $|(w_{ij})| \neq 0$, in fact, $|(w_{ij})| > 0$ since (w_{ij}) is positive definite. Our problem of maximizing $C(S_n)$ with respect to the l_{ij}, $i \neq j$, for given values of d_i, $d_i > 0$, may be re-stated as follows.

We must maximize $|(w_{ij})|$ over the class of symmetric matrices (s_{ij}) or (w_{ij}) with respect to s_{ij}, $i, j = 1, \cdots, n+1$, subject to the conditions: $|(s_{ij})| = 0$ and $s_{ii} = d_i^2$ for $i = 1, \cdots, n+1$. Further (s_{ij}) should be semipositive definite and $|w_{ij}| \neq 0$.

Let θ and $\mu_1, \cdots, \mu_{n+1}$ be Lagrange multipliers. We seek the extreme values of the function L with respect to s_{ij}, $i, j = 1, \cdots, n+1$, where

$$L = |w_{ij}| - \theta |s_{ij}| + \sum_{i=1}^{n+1} \mu_i (s_{ii} - d_i^2) .$$

Hence s_{ij} must satisfy

$$\frac{1}{2}\frac{\partial L}{\partial s_{ij}} = |W_{ij}| - \theta |S_{ij}| = 0 \text{ for } i \neq j,\ i, j, = 1, \cdots, n+1$$

$$\text{and } \frac{\partial L}{\partial s_{ii}} = |W_{ii}| - \theta |S_{ii}| + \mu_i = 0 \text{ for } i = 1, \cdots, n+1 ;$$

where $|W_{kl}|$ and $|S_{kl}|$ denote co-factors of w_{kl} and s_{kl} in $|(w_{ij})|$ and $|(s_{ij})|$ respectively.

This implies that

$$\sum_{j=1}^{n+1} w_{kj} \cdot \frac{1}{2}\frac{\partial L}{\partial s_{ij}} + w_{ki}\frac{\partial L}{\partial s_{ii}} = 0$$

so that

$$\sum_{j=1}^{n+1} w_{kj} |W_{ij}| - \theta \sum_{j=1}^{n+1} w_{kj} |S_{ij}| + \mu_i w_{ki} = 0 .$$

Let $k \neq i$; then using (2.2), $w_{kj} = 1 + s_{kj}$ and by the well-known property that expansions in terms of alien co-factors vanish identically (Aitken, p. 51) we finally obtain

$$-\theta \sum_{j=1}^{n+1} |S_{ij}| + \mu_i w_{ki} = 0$$

so that $s_{ki} = w_{ki} - 1 = \theta/\mu_i \sum_{j=1}^{n+1} |S_{ij}| - 1$, for all $k \neq i$. Since the above expression for s_{ki} is constant for values of $k = 1, \cdots, n+1$, $k \neq i$, we conclude that the elements of the ith column of (s_{ij}), except

$s_{ii} = d_i^2$, must be equal. Since s_{ij} is a symmetric matrix, the above property extends to the rows of (s_{ij}) and it is easily seen that the extreme values of L correspond to values s_{ij}^* of s_{ij} where

$$s_{ij}^* = u \text{ for } i \neq j,\ i, j = 1, \cdots, n+1 \tag{2.5}$$

while

$$s_{ii}^* = d_i^2,\ i = 1, \cdots, n+1 .$$

Now u can be determined from the relation $|s_{ij}| = 0$ so that we must have

$$\begin{vmatrix} d_1^2 & u & \cdot & \cdot & u \\ u & d_2^2 & \cdot & \cdot & u \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u & u & \cdot & \cdot & d_{n+1}^2 \end{vmatrix} = 0 . \tag{2.6}$$

Let us define the determinant

$$D_k(x; a_1, \cdots, a_k) = \begin{vmatrix} a_1 & x & \cdot & \cdot & x \\ x & a_2 & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & a_k \end{vmatrix} . \tag{2.7}$$

From the relation due to Grabeiri (1874) (see Muir, vol. 3, 4, p. 110), or by subtracting the first row of the above determinant from the remaining rows and by the use of Cauchy expansion in terms of the first row and first column, we have

$$D_k(x; a_1, \cdots, a_k) = \left(1 + x \sum_{i=1}^{k} (a_i - x)^{-1}\right) \prod_{i=1}^{k} (a_i - x) . \tag{2.8}$$

Hence from (2.6) u must satisfy the equation

$$\left(1 + u \sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right) \prod_{i=1}^{n+1} (d_i^2 - u) = 0 . \tag{2.9}$$

From (2.2) and (2.5) the extreme value of $(n!C(S_n))^2$ for any u satisfying (2.9) is equal to

$$\begin{aligned} & D_{n+1}(1 + u; 1 + d_1^2, \cdots, 1 + d_{n+1}^2) \\ & = \left(1 + (1 + u) \sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right)\left(\prod_{i=1}^{n+1} (d_i^2 - u)\right) \\ & = \left(\sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right)\left(\prod_{i=1}^{n+1} d_i^2 - u)\right) \end{aligned} \tag{2.10}$$

by the use of (2.9).

Since $u = 0$ does not satisfy (2.6), we immediately obtain from

(2.9) that the expression (2.10) is equal to

$$-u^{-1}\prod_{i=1}^{n+1}(d_i^2-u) \tag{2.11}$$

which is the extreme value of $(n!C(s_n))^2$ in terms of u. In order that the content is nonzero we must have $u \neq d_i^2$ for $i = 1, \cdots, n+1$. This statement along with (2.9) implies that u must satisfy the equation

$$1 + u\sum_{i=1}^{n+1}(d_i^2-u)^{-1} = 0 \ . \tag{2.12}$$

The roots for u, temporarily assuming that $d_1, \cdots, d_{n+1}$ are distinct, can be located by Decartes rule of signs by checking the signs of the left-handside of (2.12) for values of u, equal to $-\infty$, 0, $+\infty$ and in the neighborhood of d_i^2, $i = 1, \cdots, n+1$. Relabelling d_i such that $d_1 < d_2 < \cdots < d_{n+1}$, it is easily verified that all the roots for u are real, say $u_1, \cdots, u_{n+1}$ and may be labelled in such a manner that

$$u_1 < 0 < d_1^2 < u_2 < d_2^2 < \cdots < u_{n+1} < d_{n+1}^2 \ . \tag{2.13}$$

Consider the characteristic roots of (s_{ij}^*) given by $|s_{ij}^* - \lambda I| = 0$. By (2.5) and (2.7) λ must satisfy $D_{n+1}(u; d_1^2 - \lambda, \cdots, d_{n+1}^2 - \lambda) = 0$. Hence from (2.9)

$$\left(1 + u\sum_{i=1}^{n+1}(d_i^2-\lambda-u)^{-1}\right)\prod_{i=1}^{n+1}(d_i^2-\lambda-u) = 0 \ .$$

By similar method as used to obtain (2.13) we find that the roots for λ may be so labelled that $\lambda_1 = 0$ and

$$d_i^2 < \lambda_{i+1} + u < d_{i+1}^2 \qquad i = 1, \cdots, n \ .$$

In order that all the roots for λ are nonnegative it is easily seen that the relation

$$d_2^2 - u > \lambda_2 \geqq 0 \tag{2.14}$$

must be satisfied so that we must have $u < d_2^2$. From (2.13) we find that the only admissible roots for u are u_1 and u_2.

To establish (1.4) it only remains to show that u_1 corresponds to the case when A is inside the extremal simplex whereas u_2 corresponds to the case when A is outside the extremal simplex.

Consider the equation of π_i, passing through all the vertices of S_n except B_i having the coordinates $(x_{i,1}, \cdots, x_{i,n})$, given by

$$L_i(x_1, \cdots, x_n) = 0 \ ,$$

where

$$L_i(x_1, \cdots, x_n) = \begin{vmatrix} x_{1,1} & \cdots & x_{1,n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{i-1,1} & \cdots & x_{i-1,n} & 1 \\ x_1 & \cdots & x_n & 1 \\ x_{i+1,1} & \cdots & x_{1+1,n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} & 1 \end{vmatrix} .$$

Now A and B_i lie on the same side of π_i if and only if $L_i(x_{i,1}, \cdots, x_{i,n})$. $L_i(0, \cdots, 0) > 0$ while A and B_i lie on opposite sides of π_i if and only if $L_i(x_{i,1}, \cdots, x_{i,n})$. $L_i(0, \cdots, 0) < 0$.

Now by direct multiplication of the determinant $L_i(x_{i,1}, \cdots, x_{i,n})$ with the transpose of the determinant $L_i(0, 0, \cdots, 0)$ we obtain

$$L_i(x_{i,1}, \cdots, x_{i,n}) \cdot L_i(0, 0, \cdots, 0)$$
$$= \begin{vmatrix} 1 + s_{11} & 1 + s_{12} & \cdots 1 \cdots & 1 + s_{1\ n+1} \\ 1 + s_{21} & 1 + s_{22} & \cdots 1 \cdots & 1 + s_{2\ n+1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 + s_{n+11} & 1 + s_{n+12} & \cdots 1 & 1 + s_{n+1\ n+1} \end{vmatrix} .$$

We now assume that S_n is an extremal simplex so that from (2.5) $s_{\nu\nu} = d_\nu^2$, $\nu = 1, \cdots, n+1$ and $s_{\nu k} = u$, $\nu \neq k$, $\nu, k = 1, \cdots, n+1$. Then in the last determinant each entry in the i-th column is 1, the jth diagonal entry is $d_j^2 + 1$ for $j \neq i, j = 1, \cdots, n+1$ while the remaining entries are $1 + u$. Subtracting $(1 + u)$ times the i-th column from the remaining columns we immediately obtain

$$L_i(x_{i,1}, \cdots, x_{i,n}) \cdot L_i(0, \cdots, 0) = (d_i^2 - u)^{-1} \prod_{j=1}^{n+1} (d_j^2 - u)$$
$$= \frac{-u^{-1} \prod_{1}^{n+1} (d_j^2 - u)}{(-u^{-1}(d_i^2 - u))} .$$

Since from (2.11) the numerator of the last expression is positive, we find that A and B_i lie on the same side of π_i if and only if

$$-u^{-1}(d_i^2 - u) > 0 ,$$

while they lie on opposite sides of π_i if and only if $-u^{-1}(d_i^2 - u) < 0$.

Since $-u_2^{-1}(d_2^2 - u_2) < 0$ and $-u_1^{-1}(d_1^2 - u_1) > 0$, it is readily checked that we have proved (1.2), (1.3) and (1.4) in the case when $d_1, \cdots, d_{n+1}$ are distinct.

Necessary modifications are easily made when some or all of the d_i are not distinct.

Finally we remark that the simplex corresponding to u_1 has larger content than that for u_2. This is because

$$d_i^2 - u_1 > d_i^2 - u_2 > 0 \text{ for } i = 2, \cdots, n-1$$

and

$$-u_1^{-1}(d_1^2 - u_1) = 1 - \frac{d_1^2}{u_1} > 1 - d_1^2/u_2 = -u_2^{-1}(d_2^2 - u_2) ,$$

so that

$$-u_1^{-1} \prod_1^{n+1} (d_j^2 - u_1) > -u_2^{-1} \prod_1^{n+1} (d_j^2 - u_2) . \tag{2.15}$$

We also note that when $d_1 = d_2 = \cdots = d_{n+1}$ (1.4) has a unique negative root for u and the point A corresponding to this value of u must lie inside the extremal simplex.

3. Simplex determined by distances of faces. We recall that the $(n-1)$-dimensional hyperplane π_i passes through all the vertices of S_n except B_i. The distance of π_i from A is p_i. The point B_i does not lie on π_i but does lie on all the remaining n hyperplanes

$$\pi_j, j \neq i, j = 1, \cdots, n+1 .$$

Let π_i be given by (in normal form)

$$\pi_i\colon\ e_{i,1}x_1 + e_{i,2}x_2 + \cdots + e_{i,n}x_n = e_{i,n+1} \tag{3.1}$$

where for notational convenience we have written

$$p_i = e_{i,n+1} , \tag{3.2}$$

and $e_{i,1}, \cdots, e_{i,}n$ are the direction cosines of the normal to π_i, so that we have

$$\sum_{j=1}^{k} e_{i,j}e_{k,j} = m_{ik};\ i, k = 1, 2, \cdots, n+1;\ m_{ii} = 1 . \tag{3.3}$$

The notations used in this section will be listed first and some relations needed later will be established in order to avoid future digression.

We define the $(n+1) \times (n+1)$ matrix E in double suffix notation as

$$E = (e_{i,j}) \tag{3.4}$$

and $E_{i,j}$ will denote the co-factor of $e_{i,j}$ in E. We also define the $(n+1) \times (n+1)$ matrix M as

$$M = (m_{ij}) \tag{3.5}$$

and M_{ij} as co-factor of m_{ij} in M.
Let σ_i denote the signature of $|E_{i,n+1}|$ so that

$$\sigma_i = \begin{cases} 1 \text{ if } |E_{i,n+1}| > 0 \\ -1 \text{ if } |E_{i,n+1}| < 0 \end{cases} \quad \text{for } i = 1, \cdots, n+1 . \tag{3.6}$$

We remark here that $E_{i,n+1}$ is nonsingular. This is because

$$\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_{n+1}$$

have one and only one point in common, namely $(x_{i,1}, \cdots, x_{i,n})$. Since π_i does not pass through the above common point, it is easily seen that the matrix E is also nonsingular, so that

$$|E| \neq 0 \text{ and } |E_{i,n+1}| \neq 0,\ i = 1, \cdots, n+1 . \tag{3.7}$$

Furthermore it is easily seen that

$$|E_{i,n+1}| = \sigma_i |E_{i,n+1}E'_{i,n+1}|^{1/2} = \sigma_i |M_{ii}|^{1/2} \quad \text{for } i = 1, \cdots, n+1 \tag{3.8}$$

where the radical above as well as all radicals appearing in this paper will be always taken as positive. Hence from (3.2) and (3.4) we have

$$|E| = \sum_{i=1}^{n+1} p_i |E_{i,n+1}| = \sum_{i=1}^{n+1} \sigma_i p_i |M_{ii}|^{1/2} = \rho \text{ (say)} . \tag{3.9}$$

D will denote the diagonal matrix

$$D = \text{Diag. } (p_1, \cdots, p_{n+1}) \tag{3.10}$$

and let

$$R = (r_{ij}) = D^{-1}MD^{-1} \tag{3.11}$$

so that $r_{ii} = p_i^{-2}$ for $i = 1, \cdots, n+1$. Since

$$M = \begin{bmatrix} e_{1,1} & \cdots & e_{1,n} \\ \vdots & & \vdots \\ e_{n+1,1} & \cdots & e_{n+1,n} \end{bmatrix} \begin{bmatrix} e_{1,1} & \cdots & e_{1,n} \\ \vdots & & \vdots \\ e_{n+1,1} & \cdots & e_{n+1,n} \end{bmatrix}'$$

we also remark that M and consequently R are symmetric positive semi-definite matrices, so that $|M| = 0$ and $|R| = 0$.

Finally, it follows that

$$|M_{ii}| = |R_{ii}| \left(\prod_{j=1}^{n+1} p_j^2 \right) \Big/ p_i . \tag{3.12}$$

To obtain the content $C(S_n)$, we will use the formula (2.1). Since $(x_{i,1}, \cdots x_{i,n})$ lies on π_j; $j \neq i, j = 1, \cdots, n+1$, we may directly solve for $x_{i,j}$ from the following n linear equations:

$$\begin{bmatrix} e_{1,1}, & \cdots, & e_{1,n} \\ \vdots & & \vdots \\ e_{i-1,1}, & \cdots, & e_{i-1,n} \\ e_{i+1,1}, & \cdots, & e_{i+1,n} \\ \vdots & & \vdots \\ e_{n+1,1}, & \cdots, & e_{n+1,n} \end{bmatrix} \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \cdot \\ \cdot \\ \cdot \\ x_{i,n} \end{bmatrix} = \begin{bmatrix} e_{1,n+1} \\ \vdots \\ e_{i-1,n+1} \\ e_{i+1,n+1} \\ \vdots \\ e_{n+1,n+1} \end{bmatrix} .$$

A simple calculation shows that (see (3.4))

$$x_{i,j} = (-1)^{n-j}(-1)^{i+j} \mid E_{i,j} \mid / ((-1)^{n+1+i} \mid E_{i,n+1} \mid) .$$

Hence we obtain

$$x_{i,j} = -\mid E_{i,j} \mid / \mid E_{i,n+1} \mid ; \; i, j = 1, \cdots, n+1 .$$

Substituting these values in $| V |$ of (2.1) and factoring out -1 from each of the first n columns of V and also factoring out $| E_{i,n+1} |^{-1}$ from the ith row of V for $i = 1, \cdots, n+1$, we readily obtain

$$\begin{aligned} n! C(S_n) &= (-1)^n \mid \operatorname{Adj} E \mid \Big/ \prod_{i=1}^{n+1} \mid E_{i,n+1} \mid \\ &= (-1)^n \mid E \mid^n \Big/ \prod_{i=1}^{n+1} \mid E_{i,n+1} \mid \end{aligned} \tag{3.13}$$

where $| \operatorname{Adj} E |$ is the adjoint determinant of $| E |$. In order to avoid the ambiguity of sign in $C(S_n)$ we consider $(n! C(S_n))^2$ instead and from (3.9) and (3.12) we obtain

$$\begin{aligned} (n! C(S_n))^2 &= \mid E \mid^{2n} \Big/ \prod_{i=1}^{n+1} \mid E_{i,n+1} \mid^2 \\ &= \left(\sum_{i=1}^{n+1} \sigma_i p_i \mid M_{ii} \mid^{1/2} \right)^{2n} \Big/ \prod_{i=1}^{n+1} \mid M_{ii} \mid \\ &= \left(\sum_{i=1}^{n+1} \sigma_i \mid R_{ii} \mid^{1/2} \right)^{2n} \Big/ \prod_{i=1}^{n+1} \mid R_{ii} \mid . \end{aligned}$$

Our problem of minimization is equivalent to minimizing

$$ln \left[\left(\sum_{i=1}^{n+1} \sigma_i \mid R_{ii} \mid^{1/2} \right)^2 \Big/ \prod_{i=1}^{n+1} \mid R_{ii} \mid^{1/n} \right]$$

with respect to r_{ij}, $i, j = 1, \cdots, n+1$, subject to the restriction that $r_{ii} = p_i^{-2}$, $i = 1, \cdots, n+1$ and $| R | = 0$ over the class of symmetric matrices R.

Let $\lambda, \mu_1, \cdots, \mu_{n+1}$ be Lagrange multipliers and we seek the extreme value of

$$L = ln \left(\sum_{i=1}^{n+1} \sigma_i \mid R_{ii} \mid^{1/2} \right)^2 - \frac{1}{n} \sum_{i=1}^{n+1} ln \mid R_{ii} \mid - \lambda \mid R \mid + \sum_{i=1}^{n+1} \mu_i (r_{ii} - p_i^{-2}) .$$

r_{ij} must satisfy:

$$\frac{\partial L}{\partial r_{ij}} = \rho^{-1} \sum_{\nu=1}^{n+1} \frac{\partial |R_{\nu\nu}|}{\partial r_{ij}} \frac{\sigma_\nu}{|R_{\nu\nu}|^{1/2}} - \frac{1}{n} \sum_{\nu=1}^{n+1} \frac{1}{R_{\nu\nu}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ij}} - \lambda \frac{\partial |R|}{\partial r_{ij}} = 0 , \quad i \neq j,\ i, j = 1, \cdots, n+1$$

and

$$\frac{\partial L}{\partial r_{ii}} = \rho^{-1} \sum_{\nu=1}^{n+1} \frac{\sigma_\nu}{|R_{\nu\nu}|^{1/2}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ii}} - \frac{1}{n} \sum_{\nu=1}^{n+1} \frac{1}{R_{\nu\nu}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ii}} - \lambda \frac{\partial |R|}{\partial r_{ii}} + \mu_i = 0$$

where ρ is as defined in (3.9).

These equations reduce to

$$\frac{1}{2} \frac{\partial L}{\partial r_{ij}} = \sum_{\substack{\nu=1 \\ \nu \neq i,j}}^{n+1} (\rho^{-1}\sigma_\nu |R_{\nu\nu}|^{-1/2} - n^{-1} |R_{\nu\nu}|^{-1}) |R_{\nu\nu|ij}| - \lambda |R_{ij}| = 0 \quad \text{for } i \neq j;\ i, j = 1, \cdots, n+1$$

and

$$\frac{\partial L}{\partial r_{ii}} = \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n+1} (\rho^{-1}\sigma_\nu |R_{\nu\nu}|^{-1/2} - n^{-1} |R_{\nu\nu}|^{-1}) |R_{\nu\nu|ii}| - \lambda |R_{ii}| + \mu = 0$$

where $|R_{\nu\nu|ij}|$ is the co-factor of r_{ij} in $|R_{\nu\nu}|$.

Hence the minimizing values of r_{ij}, r_{ij}^*, say, must satisfy the equations in r_{ij}:

$$r_{ii} = p_i^{-2}$$

and

$$\sum_{\substack{j=1 \\ j \neq i}}^{n+1} r_{ij} \frac{1}{2} \frac{\partial L}{\partial r_{ij}} + r_{ii} \frac{\partial L}{\partial r_{ii}} = 0 \tag{3.14}$$

and

$$\sum_{\substack{j=1 \\ j \neq i}}^{n+1} r_{kj} \frac{1}{2} \frac{\partial L}{\partial r_{ij}} + r_{kj} \frac{\partial L}{\partial r_{ii}} = 0 . \tag{3.15}$$

After obvious simplification (3.14) yields

$$\sum_{\substack{\nu=1 \\ \nu \neq i}}^{n+1} (\rho^{-1}\sigma_\nu |R_{\nu\nu}|^{-1/2} - n^{-1} |R_{\nu\nu}|^{-1}) |R_{\nu\nu}| + \mu_i p_i^{-2} = 0 ,$$

or

$$\mu_i = p_i^2 \rho^{-1} \sigma_i R_{ii} . \tag{3.16}$$

From (3.15) we obtain for $k \neq i$,

$$(3.17)\quad \sum_{j=1}^{n+1}\sum_{\substack{\nu=1\\ \nu\neq i,j}}^{n+1}(\sigma_\nu|R_{\nu\nu}|^{-1/2}\rho^{-1}-n^{-1}|R_{\nu\nu}|^{-1})r_{kj}|R_{\nu\nu|ij}|+\mu_i r_{ki}=0\,.$$

After some calculations we obtain

$$(3.18)\quad r_{ki}=\mu_i^{-1}(\sigma_k|R_{kk}|^{-1/2}\rho^{-1}-n^{-1}|R_{kk}|^{-1})|R_{ik}|\,.$$

It is easily seen from (3.11) that $|R_{ik}|=p_ip_k|M_{ik}|$ and

$$M_{ik}=|E_{i,n+1}||E_{k,n+1}|$$

and hence from (3.8),

$$|R_{ik}|=\sigma_{ik}|R_{ii}|^{1/2}|R_{kk}|^{1/2}$$

so that substituting for μ_i from (3.16) in (3.18) we obtain

$$(3.19)\quad p_i^2 r_{ki}=1-n^{-1}\rho\sigma_k|R_{kk}|^{-1/2}\,.$$

In obtaining (3.18) from (3.17), we illustrate the case for $i=1$, $n+1=4$ and $k=2$, for the expression, for example:

$$\begin{aligned}&\sum_{j=1}^{4}\sum_{\substack{\nu=1\\ \nu\neq 1,j}}^{4}\sigma_\nu|R_{\nu\nu}|^{-1/2}r_{kj}|R_{\nu\nu|ij}|\\ &\quad= r_{21}(\sigma_2|R_{22|11}||R_{22}|^{-1/2}+\sigma_3|R_{33|11}||R_{33}|^{-1/2}+\sigma_4|R_{44|11}||R_{44}|^{-1/2})\\ &\qquad+r_{22}(\sigma_3|R_{33|12}||R_{33}|^{-1/2}+\sigma_4|R_{44|12}||R_{44}|^{-1/2})\\ &\qquad+r_{23}(\sigma_2|R_{22|13}||R_{22}|^{-1/2}+\sigma_4|R_{44|13}||R_{44}|^{-1/2})\\ &\qquad+r_{24}(\sigma_2|R_{22|14}||R_{22}|^{-1/2}+\sigma_3|R_{33|14}||R_{33}|^{-1/2})\\ &\quad=\sigma_2|R_{21}||R_{22}|^{-1/2}\,.\end{aligned}$$

The last expression is obtained from the coefficients of $|R_{22}|^{-1/2}$; the coefficients of $|R_{33}|^{-1/2}$ or $|R_{11}|^{-1/2}$ are easily seen to vanish identically, since they represent expansion by alien co-factors.

In the summation appearing in (3.17) only the term with $\nu=k$ survives;

$$\sum_{\substack{j=1\\ j\neq k}}^{n+1}r_{kj}|R_{kk|ij}|$$

is the expansion of the determinant obtained by replacing the elements of the i-th row of $|R|$ by those of the k-th row of $|R|$ with the k-th row and k-th column deleted. Transferring the elements r_{ki} appearing in the i-th row to the k-th row, there results the minor of r_{ki} in $|R|$. Hence multiplying by $(-1)^{i-k}$ and $(-1)^{i+k}$ we obtain $|R_{ki}|$. It is thus seen that

$$\sum_{\substack{j=1\\ j\neq k}}^{n+1}r_{kj}|R_{kk|ij}|=|R_{ki}|=|R_{ik}|\,.$$

From (3.19) it is easily checked that we have

$$p_i^2p_k^2r_{ik} = p_j^2p_k^2r_{jk} , \tag{3.20}$$

for all $i, j = 1, \cdots, n+1$, with $i \neq k, j \neq k$.

Since the matrix

$$(p_i^2r_{ij}p_j^2) = D^2RD^2 = D^2D^{-1}MD^{-1}D^2 = DMD = (p_im_{ij}p_j)$$

is symmetric, and (3.20) implies that nondiagonal elements of each row or column of this matrix are equal we conclude, (in a manner analogous to (2.5)) that $r_{ii}^* = p_i^{-2}$, $i = 1, \cdots, n+1$ and

$$p_i^2r_{ij}^*p_j^2 = p_ip_jm_{ij}^* = v ,$$

say, for $i \neq j$; $i, j = 1, \cdots, n+1$ so that

$$\begin{cases} m_{ii}^* = 1 & \text{for } i = 1, \cdots, n+1 \\ m_{ij}^* = \dfrac{v}{p_ip_j} & \text{for } i \neq j;\ i, j = 1, \cdots, n+1. \end{cases} \tag{3.21}$$

We obtain values of v by equating $|r_{ij}^*| = 0$ or equivalently by setting $|DMD| = |(p_ip_jm_{ij}^*)| = 0$, where $p_ip_jm_{ij}^* = v$, $i \neq j$ and $p_i^2m_{ii}^* = p_i^2$, and it is seen from (2.7) that v must satisfy

$$D_{n+1}(v; p_1^2, \cdots, p_{n+1}^2) = 0 .$$

and hence

$$\left(1 + v\sum_{i=1}^{n+1}(p_i^2 - v)^{-1}\right)\prod_{i=1}^{n+1}(p_i^2 - v) = 0 . \tag{3.22}$$

We also note from (3.13), (3.8), (3.9) and (3.12) that

$$\begin{aligned}(n!C(S_n))^2 &= \rho^{2n}\cdot\prod_{i=1}^{n+1}(|R_{ii}|^{-1}\cdot p_i^2) \\ &= p_1^2|R_{11}|^{-1}\cdot\prod_{i=2}^{n+1}(\rho|R_{ii}|^{-1/2})^2 .\end{aligned} \tag{3.23}$$

But from (3.19) we have

$$\rho\sigma_k|R_{kk}|^{-1/2} = n(1 - p_i^2r_{ki}^*)$$

so that $\rho\sigma_k|R_{ii}|^{-1/2} = n(p_i^2 - v)/p_i$, from (3.21). Also from (3.21), since $r_{ij}^* = v/(p_i^2p_j^2)$ and $r_{ii}^* = p_i^{-2}$ it is easily seen that

$$\begin{aligned}|R_{11}|\prod_{i=2}^{n+1}p_i^2 &= D_n(v; p_2^2, \cdots, p_{n+1}^2) \\ &= \left(1 + v\sum_{i=2}^{n+1}(p_i^2 - v)^{-1}\right)\prod_{i=2}^{n+1}(p_i^2 - v) \\ &= (p_1^2 - v)^{-1}\Big(-v(p_1^2 - v)^{-1} +\end{aligned}$$

$$+ 1 + v\sum_{i=1}^{n+1}(p_1^2 - v)^{-1}\Big)\prod_{i=1}^{n+1}(p_i^2 - v)$$

$$= -v(p_1^2 - v)^{-2}\prod_{i=1}^{n+1}(p_i^2 - v) \quad \text{from} \quad (3.22)\,.$$

Substituting in (3.23) we readily find that

$$(n!C(S_n))^2 = v^{-1}n^{2n}\prod_{i=1}^{n+1}(p_i^2 - v)\,. \tag{3.24}$$

Thus (1.6) is proved.

In order that S_n is nondegenerate $v \neq p_i^2$, $i = 1, \cdots, n+1$. Hence from (3.22) v must satisfy

$$1 + v\sum_{i=1}^{n+1}(p_i^2 - v)^{-1} = 0\,. \tag{3.25}$$

Thus we have exactly the same equation as (2.12) with d_i replaced by p_i and u replaced by v. By exactly the same argument that follows (2.12) we conclude that, when $p_1, \cdots, p_{n+1}$ are distinct, if the roots of (3.25) are so labelled that the unique negative root of (3.25) is v_1 and the smallest positive root for v is v_2 and if the p_i are labelled so that p_1 is the smallest and p_2 the second smallest p_i, $i = 1, \cdots, n+1$, we have the two eligible roots of (3.26) as v_1 and v_2 satisfying

$$v_1 < 0 < p_1^2 < v_2 < p_2^2\,. \tag{3.26}$$

It remains to prove that v_1 corresponds to the case when A is inside S_n while v_2 corresponds to the case when A is outside S_n.

We will prove that, for the extremal simplexes obtained above, the vertex B_i and the fixed point A lie on the same side of π_i if

$$p_i^2 - v > 0$$

while A and B_i lie on opposite sides if $p_i^2 - v < 0$.

Let

$$L_i(x_1, \cdots, x_n) = e_{i,1}x_1 + \cdots + e_{i,n}x_n - e_{i,n+1}\,.$$

Then $L_i(0, \cdots, 0) = -e_{i,n+1} = -p_i$, and

$$\begin{aligned} &L_i(x_{i1}, \cdots, x_{in}) \\ &\quad = -\sum_{j=1}^{n+1} e_{i,j}|E_{i,j}|/|E_{i,n+1}| && \text{(by virtue of (3.5))} \\ &\quad = -|E|/|E_{i,n+1}| \\ &\quad = -p_i\rho/\sigma_i|R_{ii}|^{1/2} && \text{(from (3.8) and (3.12))} \\ &\quad = -np_i(1 - p_k^2 r_{ki}^*) && \text{(from (3.19))} \\ &\quad = -np_i(1 - v/p_i^2) && \text{(from (3.21))} \end{aligned}$$

Hence $L_i(0, \cdots, 0) \cdot L_i(x_{i,1}, \cdots, x_{i,n}) = n(p_i^2 - v)$. Now the equation of π_i is $L_i(x_1, \cdots, x_n) = 0$. Hence $p_i^2 - v > 0$ implies that A and B_i lie on the same side of π_i while $p_i^2 - v < 0$ implies that A and B_i lie on opposite sides of π_i. Since $p_i^2 - v_1$ is positive for $i = 1, \cdots, n + 1$ we conclude from (3.26) that corresponding to v_1, A is inside S_n. Also from (3.26) we find $p_1^2 - v_2$ is negative so that corresponding to v_2 the point A lies outside S_n. Hence it is readily checked that we have proved (1.5), (1.6), (1.7) and (1.8).

Finally, using an argument analogous to that used to obtain (2.15) we find that

$$-v_1^{-1} \prod_{i=1}^{n+1} (p_i^1 - v_1) > - v_2^{-1} \prod_{i=1}^{n+1} (p_i^2 - v_2)$$

so that from (3.24) we conclude that the content of S_n corresponding to v_1 is greater than the content of S_n corresponding to v_2.

Obvious modifications in the foregoing proofs are easily made. when some or all the $p_1, \cdots, p_{n+1}$ are equal.

When $p_1 = p_2 = \cdots = p_{n+1}$, (3.25) has a unique negative solution for v and in this case A must lie inside the extremal simplex.

The author expresses his thanks to Professor H. S. M. Coxeter for his valuable association which led to this problem and for his keen interest in this work.

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Received July 23, 1969. This work was done at the University of Toronto, Department of Industrial Engineering, and the author was on leave of absence from the University of Western Ontario, Department of Mathematics.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON NORMED RINGS WITH MONOTONE MULTIPLICATION

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It is shown that if a normed division ring has a norm which is "multiplication monotone" in the sense that $N(x) < N(x')$ and $N(y) < N(y')$ imply $N(xy) \leqq N(x'y')$, and if the norm is "commutative" in the sense that $N(\cdots xy \cdots) = N(\cdots yx \cdots)$ for all x and y, then the topology of that ring is given by an absolute value. A consequence of this result is that if the norm of a connected normed ring with unity is multiplication monotone and commutative then the ring is embeddable in the system of quaternions.

Pontrjagin has shown [7] that the only locally compact connected fields are the field of real numbers and the field of complex numbers. A theorem of A. Ostrowski [6] implies that if the topology of a connected field is given by an absolute value then the field is (isomorphic to) a subfield of the field of complex numbers. Both results are contributions toward the solution of the problem of determining what connected fields exist.

In this note the more restricted question of studying connected normed fields is considered. (It is recalled that a *normed ring* has its topology induced by a *norm* function N; that is, N is a real-valued function defined on the ring such that: (i) $N(0) = 0$ and $N(x) > 0$ for $x \neq 0$, (ii) $N(-x) = N(x)$ for all x, (iii) $N(x + y) \leqq N(x) + N(y)$ for all x and y, (iv) $N(xy) \leqq N(x)N(y)$ for all x and y.) Ostrowski's results may be regarded as the treatment of the special case of this problem in which the norm N satisfies the additional condition

$$N(xy) = N(x)N(y)$$

for all x and y. This extra requirement is replaced here by the weaker condition that N be *multiplication monotone* in the sense that whenever $N(x) < N(x')$ and $N(y) < N(y')$ then $N(xy) \leqq N(x'y')$.

Specifically, it is shown in the corollary of Theorem 3 that if a commutative connected normed ring with unity has a multiplication monotone norm then that ring is (algebraically and topologically isomorphic to) a subring of the field of complex numbers. (The version of this statement which appears below actually includes the noncommutative case as well.) The basic device employed in obtaining this result is Theorem 2, which asserts that if a normed division ring has a multiplication monotone norm N such that

$$N(\cdots xy \cdots) = N(\cdots yx \cdots)$$

for all x and y then there is an absolute value which induces the topology of the ring.

2. Preliminaries. It is recalled that a *norm* for a ring A is a real-valued function N on A such that: (i) $N(0) = 0$ and $N(x) > 0$ for all nonzero x in A, (ii) $N(-x) = N(x)$ for all x in A, (iii) $N(x + y) \leqq N(x) + N(y)$ for all x, y in A, (iv) $N(xy) \leqq N(x)N(y)$ for all x, y in A. If a norm N for a ring A also has the property that $N(xy) = N(x)N(y)$ for all x, y in A then N is called an *absolute value* for A.

By a *normed ring* is meant a ring A, together with a norm N for A. The norm for a normed ring induces a metric, and therefore a topology, in A.

A topological ring is called a *Q-ring* of its set of quasiinvertible elements is open; for a topological ring A with unity to be a Q-ring it is necessary and sufficient that the set of invertible elements be open. In particular, it can be shown that every complete normed ring with unity is a Q-ring.

Further details on these concepts can be found in [1] and [4], where the term *metric ring* is employed for a normed ring.

If a norm N for a ring A has the property that $N(\cdots xy \cdots) = N(\cdots yx \cdots)$ for all x, y in A then N will be called a *commutative* norm. For instance, absolute values are always commutative, and every norm for a commutative ring is also commutative.

In addition to the above notions, we shall also refer to the concepts which figure in [5], and we shall make use of the criteria given by Kaplansky in that paper for a topological division ring to admit an equivalent absolute value.

Two elementary lemmas will help to translate Kaplansky's criteria to the special case of normed division rings. The proofs are routine.

LEMMA 1. *An element x of a normed ring is topologically nilpotent if and only if there exists a natural number n such that $N(x^n) < 1$.*

LEMMA 2. *The set of topologically nilpotent elements of a normed ring is open.*

Kaplansky's criteria can now be rephrased to fit the needs of the present discussion.

THEOREM 1. *Let K be a normed division ring whose norm is commutative. In order for K to admit an equivalent absolute value* *(that is, an absolute value whose induced topology coincides with the topology induced by the norm for K), it is necessary and sufficient that the set of elements which are either topologically nilpotent or neutral be right bounded.*

Proof. The necessity of the conditions is obvious. For the sufficiency of the conditions, we first note that the commutativity of the norm implies that $N(x) = N(1)$ whenever x is an element of the commutator subgroup of the multiplicative group of nonzero elements of K; this commutator subgroup is therefore metrically bounded and is consequently right bounded. Lemma 2 and [5; Th. 2] imply that there is an equivalent absolute value for K.

3. **Rings with multiplication monotone norm.** We shall subject the norm for a normed ring to a monotonicity condition which is of interest because it implies the existence of an absolute value equivalent to the given norm.

DEFINITION. A norm N for a ring A is said to be *multiplication monotone* provided that whenever $N(x) < N(x')$ and $N(y) < N(y')$ then $N(xy) \leqq N(x'y')$.

Clearly every absolute value is multiplication monotone, while the following theorem indicates that under suitable conditions a multiplication monotone norm for a division ring must have an equivalent absolute value.

THEOREM 2. *Let K be a normed division ring whose norm is commutative and multiplication monotone. Then there is an equivalent absolute value for K.*

Proof. The theorem obviously holds for discrete division rings, so we may confine our attention to nondiscrete division rings.

Let x be a fixed element of K such that $0 < N(x) < N(1)^{-1}$. Then if $N(y) > N(x^{-2})$ it follows that $N(y^{-1}) \leqq N(x) < 1$, and y is therefore inversely nilpotent. Thus whenever y is topologically nilpotent or neutral we have $N(y) \leqq N(x^{-2})$, so that the set of elements of K which are topologically nilpotent or neutral is metrically bounded and therefore right bounded. Theorem 1 yields the desired result.

It is possible to relax the requirement that the ring in question be a division ring, provided that the ring is connected. In order to achieve this we introduce the notion of *generalized zero-divisors*.

DEFINITION. An element b of a normed ring A will be called a *generalized left zero-divisor* (*generalized right zero-divisor*) provided that the greatest lower bound of the set $\{N(bx)/N(x) \mid x \neq 0\}$ ($\{N(xb)/N(x) \mid x \neq 0\}$) is zero.

These are essentially the definitions which were employed in [1], but we may also note that b is a generalized left zero-divisor (generalized right zero-divisor) if and only if there exists a sequence $\{x_n\}$ of nonzero elements of A such that

$$\lim N(bx_n)/N(x_n) = 0 \ (\lim N(x_n b)/N(x_n) = 0) .$$

Although normed rings usually have many generalized zero-divisors it can be shown that a connected normed ring whose norm is multiplication monotone has no generalized zero-divisors other than zero.

LEMMA 3. *Let A be a connected normed ring with unity such that the norm for A is multiplication monotone. Then A has no generalized left zero-divisors or generalized right zero-divisors other than zero.*

Proof. Suppose b is a generalized left zero-divisor in A. Let $\{x_n\}$ be a sequence of nonzero elements of A such that

$$\lim N(bx_n)/N(x_n) = 0 .$$

Choose a sequence $\{y_n\}$ in A such that $(1/2)N(x_n) < N(y_n) < N(x_n)$ for every natural number n.

If I is the set of all elements c of A such that

$$\lim N(cy_n)/N(y_n) = 0$$

then I is clearly a left ideal in A. Also, whenever c is an element of A such that $N(c) < N(b)$ then $N(cy_n)/N(y_n) \leqq N(bx_n)/((1/2)N(x_n))$ for all n, so that c is an element of I. Thus, if b were not zero then an entire neighborhood of zero would be contained in the left ideal I, and I would therefore be open and closed in the connected ring A; consequently I would coincide with A, in contradiction to the fact that I can not contain the unity of A. We conclude that b is zero.

Similarly, every generalized right zero-divisor is zero.

In order to obtain the desired results concerning connected normed rings we first dispose of a special case in the following lemma.

LEMMA 4. *Let A be a connected ring with unity such that the set $A^{\#}$ of nonzero elements of A is disconnected. Then A is a division ring.*

Proof. If c is a nonzero element of A then the mapping $x \to cx$ is clearly a continuous endomorphism of the additive group of A, so that its image H is a connected nonzero subgroup of the additive group of A. But it can be shown that the additive group of A is continuously isomorphic to the additive group of real numbers (for instance, a proof is outlined in [3; Chap. 5, p. 28, Exercise 4]), and H must therefore coincide with the additive group of A. Thus, 1 is in H, so that $1 = cd$ for some d in A, and c has a right inverse in A.

Since every nonzero element of A has a right inverse in A we conclude that A is a division ring.

It is now possible to pass to the general case.

THEOREM 3. *Let K be a connected normed Q-ring with unity such that the norm for K is commutative and multiplication monotone. Then A is algebraically and topologically isomorphic to the field $\mathfrak{R}$ of real numbers, a dense connected subfield of the field $\mathfrak{C}$ of complex numbers, or a dense connected division subring of the division ring $\mathfrak{Q}$ of all real quaternions.*

Proof. If the set $A^{\#}$ of nonzero elements of A is not connected then Lemma 4 implies that A is a division ring. On the other hand, if $A^{\#}$ is connected then A is a division ring according to [1; Th. 1] since Lemma 3 implies that A has no generalized zero-divisors other than zero. In either case A is a division ring.

There is an equivalent absolute value for the normed division ring A by Theorem 2. Ostrowski's characterization of connected division rings with absolute value (see for instance [2; Th. 2, p. 131]) may then be applied to obtain the desired result.

COROLLARY. *Let A be a connected normed ring with unity such that the norm for A is commutative and multiplication monotone. Then A is algebraically and topologically isomorphic to $\mathfrak{R}$, to a dense connected subring of $\mathfrak{C}$, or to a dense connected subring of $\mathfrak{Q}$.*

The corollary is obtained by applying the theorem to the completion of A.

REMARK. Another kind of monotonicity condition could be introduced in normed division rings. The norm of a normed division ring can be described as *inversion monotone* provided that whenever $N(x) < N(y)$ for nonzero elements x, y then $N(x^{-1}) \geqq N(y^{-1})$. Theorem 2 remains valid if "multiplication monotone" is replaced by "inversion monotone" in the hypothesis, although some details of the proof must be modified. Similarly, the corollary of Theorem 3 continues to hold if "multiplication monotone" is replaced by "inversion monotone" in the statement of the corollary, provided that it is assumed that the ring is a division ring.

This note evolved from the consideration of some peripheral questions related to a problem which was investigated with the support of the Research Council of Rutgers University; the author wishes to express his appreciation to the Research Council for that support.

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Received April 9, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

NORMED FIELDS WHICH EXTEND NORMED RINGS OF INTEGERS

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It is shown that if the ring of integers is made a normed ring by using a "reasonable" norm, such as the ordinary absolute value or some power thereof, then every normed field which extends such a normed ring is a subfield of the field of complex numbers.

The development of the foundations of analysis involves the construction of the normed field of complex numbers, with the ordinary absolute value as norm, from the normed ring of integers, with the ordinary absolute value as norm, by a process of successive enlargements of algebraic systems. (By a *normed ring* is meant a ring A which is provided with a *norm* function N; that is, N is a real-valued function defined on A such that: (i) $N(0) = 0$ and $N(x) > 0$ for every nonzero x in A, (ii) $N(-x) = N(x)$ for all x in A, (iii) $N(x + y) \leqq N(x) + N(y)$ for all x, y in A, (iv) $N(xy) \leqq N(x)N(y)$ for all x, y in A.) Although some treatments of this construction create only positive numbers in the early stages of the passage from the system of natural numbers to the complex number system, such approaches could easily be modified to retain their basic features while still producing the ring of integers at the outset; thus, all such procedures essentially involve the extension of the normed ring of integers to produce the normed field of complex numbers.

One might ask what normed fields could be produced by enlarging the normed ring of integers, with the ordinary absolute value or some power thereof as norm, if no restriction whatever were placed upon the method of extension. It is shown in Theorem 3 that the only normed fields which can be thus obtained must be (continuously isomorphic to) subfields of the field of complex numbers.

Somewhat similar results are given in §4 for the situation in which the normed field of rational numbers, with a suitably "natural" norm, is enlarged to create a new normed field. For instance, the corollary of Theorem 6 indicates that if the field of rational numbers is provided with a norm which coincides with a power of the ordinary absolute value over a suitable neighborhood of zero, then every normed field which extends this normed field is (continuously isomorphic to) a subfield of the field of complex numbers.

2. Preliminaries. It is useful to recall some of the concepts which are employed in [1] and [2].

A *norm* for a ring A is a real-valued function N defined on A such that: (i) $N(0) = 0$ and $N(x) > 0$ for all nonzero x in A, (ii) $N(-x) = N(x)$ for all x in A, (iii) $N(x + y) \leqq N(x) + N(y)$ for all x, y in A, (iv) $N(xy) \leqq N(x)N(y)$ for all x, y in A. If a norm N for a ring A has the property that $N(xy) = N(x)N(y)$ for all x, y in A then N is called an *absolute value* for A.

By a *normed ring* is meant a ring A, together with a norm N for A; the norm for a normed ring A defines a metric, and therefore a topology, for A.

If N is a norm for a ring A and c is an element of A such that $N(cx) = N(c)N(x)$ for all x in A then N is said to be *homogeneous at* c. A norm N for a ring A is said to be *power multiplicative at* an element c of A provided that $N(c^n) = N(c)^n$ for every natural number n. When a norm N for a ring A is homogeneous (power multiplicative) at every element of a subset C of A then N is said to be *homogeneous* (*power multiplicative*) *on* C.

In case N and N' are norms for a ring A such that $N'(x) \leqq N(x)$ for all x in A then we shall write $N' \leqq N$. The relation $\leqq$ in the set of norms for a ring A constitutes a partial ordering of that set.

An example will serve to illustrate some of these concepts. Let A be the ring of all real functions which are defined and have a continuous derivative on the closed unit interval $[0, 1]$. If $N'(x) = \sup\{|x(t)| \mid 0 \leqq t \leqq 1\}$ and

$$N(x) = \sup\{|x(t)| \mid 0 \leqq t \leqq 1\} + \sup\{|x'(t)| \mid 0 \leqq t \leqq 1\}$$

for all x in A, then N' and N are norms for A, with $N' \leqq N$. It is also easily established that N' is power multiplicative on A and that N is homogeneous at each constant function which belongs to A.

When N is a norm for a field K and c is a nonzero element of K, then for all x in K:

$$N(x) \geqq N(xc)/N(c) \geqq N(xc^2)/N(c)^2 \geqq N(xc^3)/N(c)^3 \geqq \cdots .$$

Thus

$$N_c(x) = \inf\{N(xc^n)/N(c)^n \mid n \text{ a natural number}\} = \lim_{n\to\infty} N(xc^n)/N(c)^n$$

is a well-defined nonnegative real number for all x in A. It can be shown that the function N_c is identically zero on A if and only if N fails to be power multiplicative at c. On the other hand, if N is power multiplicative at c then N_c is a norm for K, with $N_c \leqq N$, as the following lemma indicates. (It is recalled that by a *semigroup* in a ring is meant a nonempty subset of that ring such that the subset is closed under multiplication.)

LEMMA 1. *Let N be a norm for a field K, and let c be a nonzero element of K such that N is power multiplicative at c. Then N_c is a norm for K such that:*

(i) $N_c \leqq N$,

(ii) $N_c(c) = N(c)$,

(iii) *N_c is homogeneous at c,*

(iv) *whenever S is a semigroup in K, with c in S, such that N is power multiplicative on S then N_c is power multiplicative on S.*

It is easily established that N_c possesses properties (ii), (iii), (iv) of a norm, so that the set I of all x in A for which $N_c(x) = 0$ is an ideal in the field K, and N_c is therefore a norm for K. The remaining details of the proof are routine.

The lemma permits us to replace the norm N by a new norm which has properties similar to those of N and is homogeneous at c as well. It is possible to sharpen this result so that the new norm is homogeneous on an entire semigroup on which the original norm is power multiplicative.

THEOREM 1. *Let K be a normed field with norm N, let S be a semigroup in K such that N is power multiplicative on S, and let c be a nonzero element of S. Then there exists a norm N' for K such that:*

(i) $N' \leqq N$,

(ii) $N'(c) = N(c)$,

(iii) N' is homogeneous on S.

Proof. Let $\mathscr{H}$ be the set of all norms N'' for K such that $N'' \leqq N$, $N''(c) = N(c)$, N'' is homogeneous at c, and N'' is power multiplicative on S. Then $\mathscr{H}$ is not empty since it contains N_c; also, $\mathscr{H}$ is partially ordered by the relation $\leqq$. It is easily shown that every totally ordered subset of $\mathscr{H}$ has a lower bound in $\mathscr{H}$, so that Zorn's Lemma implies the existence of a minimal element, N', of $\mathscr{H}$.

If d is a nonzero element of S then Lemma 1 implies that $(N')_d$ belongs to $\mathscr{H}$, with $(N')_d(dx) = (N')_d(d) \cdot (N')_d(x)$ for all x in K. Since N' is a minimal element of $\mathscr{H}$, and since N' and $(N')_d$ both belong to $\mathscr{H}$, with $(N')_d \leqq N'$, it follows that $N' = (N')_d$. Thus, $N'(dx) = N'(d)N'(x)$ for all x in K. We conclude that N' is homogeneous at every element d of S, and the theorem follows.

REMARK. In order to apply Theorem 1 it is useful to have a criterion to determine when a norm for a ring is power multiplicative on a semigroup in that ring. It is easily established that a norm N

for a ring A is power multiplicative on a semigroup S in A if and only if for every element x in S there is an integer $n(x)$, with $n(x) > 1$, such that $N(x^{n(x)}) = N(x)^{n(x)}$. In particular, N is power multiplicative on S if and only if $N(x^2) = N(x)^2$ for all x in S. (Any integer exponent greater than 1 could be used instead of 2 in the preceding statement.)

3. Extensions of the normed ring of integers. We are interested in normed fields which extend the ring of integers when the latter is provided with a norm which is a power of the ordinary absolute value. It will be shown that such fields are (continuously isomorphic to) subfields of the field of complex numbers. First a more general result is obtained which implies that if the ring of integers is given a norm which is power multiplicative and takes a value greater than 1 at least once then any normed field which extends this normed ring must be (continuously isomorphic to) a subfield of the field of complex numbers.

For convenience, whenever n is an integer the symbol n will be used to denote the n-fold of the unit element of the field which is under consideration.

THEOREM 2. *Let K be a normed field for which there is a natural number n_0, with $N(n_0) > 1$, such that $N(n^2) = N(n)^2$ whenever n is a natural number for which $n \geqq n_0$. Then K is continuously algebraically isomorphic to a subfield of the field $\mathfrak{C}$ of complex numbers.*

Proof. If S is the set of all elements n of K such that n is a natural number with $n \geqq n_0$, then S is a semigroup in K such that N is power multiplicative on S. Theorem 1 can be applied to the semigroup S and the element n_0 in order to obtain a norm N' for K such that $N' \leqq N$, $N'(n_0) = N(x_0) > 1$, and N' is homogeneous on S.

If n is an arbitrary natural number greater than 1 then there is a natural number r such that n^r and n^{r+1} both belong to S; the inequality $N'(n^r)N'(n)N'(x) = N'(n^{r+1})N'(x) = N'(n^{r+1}x) \leqq N'(n^r)N'(nx)$ implies that $N'(nx) = N'(n)N'(x)$ for all x in K. From the condition $N'(n_0x) = N'(n_0)N'(x)$ with $x = 1$ we obtain $N'(1) = 1$, and consequently N' is homogeneous at every "integer" in K. Thus N' is homogeneous on the prime field, P, of K. Since $N'(n_0) > 1$, the restriction of N' to P is an archimedean absolute value for P; therefore Ostrowski's results [4] imply that P is algebraically isomorphic to the field of rationals (and can be identified with that field), and there is a real number s, with $0 < s \leqq 1$, such that $N'(x) = |x|^s$ for all x in P.

Let A be the completion of K relative to the norm N', so that

A is a complete commutative normed ring with unity, and there is an obvious continuous isomorphism φ of K into A. We have in fact $N''(\varphi(x)) = N'(x) \leqq N(x)$ for all x in K if N'' is the norm for A. The closure, R, of $\varphi(P)$ in A is the completion of $\varphi(P)$ and can be identified with the completion of P. Therefore R can be identified with the field of real numbers, and we have $N''(y) = |y|^s$ for all y in R.

There is a maximal ideal M in A, and M is closed since the set of invertible elements of a complete normed ring with unity is open. Thus, A/M is a complete normed field and has its norm $\bar{N}$ given by the rule $\bar{N}(X) = \inf\{N''(x) \mid x \in X\}$ for all X in A/M. The natural homomorphism ν of A onto A/M is continuous since $\bar{N}(\nu(y)) \leqq N''(y)$ for all y in A, and $\nu(R)$ is therefore identifiable with the field R. Then A/M may be considered a complete commutative normed division algebra over R, where R is the field of real numbers with a power of the ordinary absolute value as its absolute value. The Gelfand-Mazur Theorem, as it appears in [3; Chap. 6, p. 127, Th. 1], implies that A/M is continuously isomorphic to the field of real numbers or the field of complex numbers, so that there is a continuous isomorphism ψ of A/M into the field $\mathfrak{C}$ of complex numbers.

It is easily seen that the mapping $\psi \circ \nu \circ \varphi$ is a continuous isomorphism of the field K into $\mathfrak{C}$, and the theorem follows.

Note. An alternative means of stating Theorem 2 is that if the ring of integers is given a norm which is power multiplicative at every integer which is sufficiently large, and if the norm takes a value greater than 1 for at least one of those integers, then every normed field which is an extension of this normed ring must be a subfield of $\mathfrak{C}$ with a topology at least as fine as its ordinary relative topology in $\mathfrak{C}$.

The simplest norms which satisfy the hypothesis of Theorem 2 are those which coincide with some power of the ordinary absolute value at all natural numbers which are sufficiently large. We thus obtain the following theorem.

THEOREM 3. *Let K be a normed field for which there exist a natural number n_0 and a positive real number s such that $N(n) = n^s$ whenever n is a natural number with $n \geqq n_0$. Then K is continuously algebraically isomorphic to a subfield of $\mathfrak{C}$.*

It should be noted that s is necessarily less than or equal to 1. A special case of Theorem 3, that in which $s = 1$, has been given in [2; Corollary 2 of Th. 5]. Another result of some interest can be obtained as a corollary of the theorem, and has appeared in [2; Th. 6].

COROLLARY. *Let K be a normed field such that $N(n) = nN(1)$ for infinitely many natural numbers n. Then K is continuously algebraically isomorphic to a subfield of $\mathfrak{C}$.*

The proof involves replacement of the norm N by a new norm N' defined by $N'(x) = \sup\{N(xc)/N(c) \mid c \in K, c \neq 0\}$ for all x in K.

Note. Theorem 3 implies that if the ring of integers is provided with a norm which is a power of the ordinary absolute value (or if the norm merely coincides with some power of the ordinary absolute value at integers which are sufficiently large) then every normed field which extends this normed ring must be a subfield of $\mathfrak{C}$ with a topology at least as fine as its ordinary relative topology.

An interesting consequence of these results concerns normed fields which *satisfy the parallelogram law.*

DEFINITION. A normed ring A is said to *satisfy the parallelogram law* if $N(x + y)^2 + N(x - y)^2 = 2N(x)^2 + 2N(y)^2$ whenever x, y belong to A.

The parallelogram law is characteristic of Euclidean distance and can hold for a normed field only if that field is continuously embeddable in the field of complex numbers.

THEOREM 4. *Let K be a normed field which satisfies the parallelogram law. Then K is continuously algebraically isomorphic to a subfield of $\mathfrak{C}$.*

Proof. The parallelogram law with $x = y$ yields the relation $N(2x) = 2N(x)$ for all x in K. Thus, $N(2^r x) = 2^r N(x)$ for all x in K and for every natural number r. The corollary of the preceding theorem then leads to the desired result.

4. **Extensions of the normed field of rational numbers.** The fields of the preceding section were all necessarily of infinite characteristic although the hypotheses employed in the statements of the results did not explicitly make such an assumption. We now confine our attention to fields of infinite characteristic, and the discussion is simplified by identifying the prime field of each such field with the field of rational numbers. The results of this section then indicate that if the field of rational numbers is given a norm which is "reasonable" in an appropriate sense, then every normed field which extends

such a normed field must be (continuously isomorphic to) a subfield of $\mathfrak{C}$.

We first obtain an analogue of Theorem 2.

THEOREM 5. *Let K be a normed field of infinite characteristic for which there is a natural number n_0, with $N(1/n_0) < 1$, such that $N(1/n^2) = N(1/n)^2$ whenever n is a natural number with $n \geqq n_0$. Then K is continuously algebraically isomorphic to a subfield of $\mathfrak{C}$.*

Proof. If S is the semigroup which consists of the elements $1/n$ of K for which n is a natural number with $n \geqq n_0$, then N is power multiplicative on S and we may apply Theorem 1 to S and the element $1/n_0$. Thus, there is a norm N' for K, with $N' \leqq N$, such that N' is homogeneous on S and $N'(1/n_0) = N(1/n_0) < 1$. We have $N'(n_0) > 1$ since $N'(1/n_0) < 1$. Also, whenever, n is a natural number with $n \geqq n_0$ then $N'(1/n^2)N'(n^2) = 1 = N'(1/n)^2N'(n)^2 = N'(1/n^2)N'(n)^2$, so that $N'(n^2) = N'(n)^2$. Thus, K with the norm N' satisfies the hypothesis of Theorem 2, and the theorem follows since K is continuously algebraically isomorphic to this normed field.

When the norm for a normed field of infinite characteristic coincides with some power of the ordinary absolute value at the reciprocals of all natural numbers which are sufficiently large, we obtain an analogue of Theorem 3.

THEOREM 6. *Let K be a normed field of infinite characteristic for which there exist a natural number n_0 and a positive real number s such that $N(1/n) = 1/n^s$ whenever n is a natural number with $n \geqq n_0$. Then K is continuously algebraically isomorphic to a subfield of $\mathfrak{C}$.*

COROLLARY. *Let K be a normed field of infinite characteristic for which there exist positive real numbers r_0 and s such that $N(r) = r^s$ whenever r is a rational number with $0 < r < r_0$. Then K is continuously algebraically isomorphic to a subfield of $\mathfrak{C}$.*

We note that the corollary implies that if the field of rational numbers is provided with a norm which coincides with some power of the ordinary absolute value over a suitable neighborhood of zero, then every normed field which can be obtained by extending this normed field must be a subfield of $\mathfrak{C}$ with a topology at least as fine as its ordinary relative topology in $\mathfrak{C}$. The special case of this corollary which occurs when $s = 1$ has already been given in [2; Th. 7].

REMARK. Theorems 2, 3, 5, and 6 and their corollaries identify the normed field K with a subfield of the field $\mathfrak{C}$ of complex numbers, but with a topology finer than the ordinary topology inherited from $\mathfrak{C}$. That the topology for K may be strictly finer than the ordinary topology is shown by taking as K the field of complex numbers with the norm N given by $N(x) = \max(|x|, |\sigma(x)|)$ for every complex number x, where σ is a fixed discontinuous automorphism of the field of complex numbers.

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Received April 9, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

INDEFINITE MINKOWSKI SPACES

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The purpose of this article is to characterize Minkowski general G-spaces. The unit sphere K is shown to have at most four components.

Assume the space R is not reducible. If K has one component, R is an ordinary Minkowski G-space. If K has two components they are quadrics and R is nearly pseudoeuclidean. When K has three components, one is a quadric and the other two are strictly convex. The unit sphere has four components only in dimension two.

The axioms of a general G-space have been given in [4] and the interesting two dimensional spaces have been investigated in [1]. We will denote the indefinite distance from x to y by xy. We refer to xy as a metric even though it is not in general a true metric.

DEFINITION 1.1. The general G-space R is called a Minkowski space if R is the real n-dimensional affine space A^n, the family of Arcs A consists of the affine segments and $w = (1/2)(x + y)$ implies $wx = wy = (1/2)xy$.

If L^r is an r-dimensional flat in R, then L^r is an r-dimensional Minkowski space with the induced distance.

Let $e(x, y)$ be an associated euclidean metrization of A^n. Then for each line L in R there is a number $\phi(L)$ such that $xy = \phi(L)e(x, y)$ for all $x, y \in L$. If $\phi(L) = 0$, we call L a null line. The number $\phi(L)$ depends continuously on L and $\phi(L) = \phi(L_1)$ if L_1 is parallel to L, see [1]. It follows that the affine translations preserve the distance xy.

Let z always denote the origin in A^n. We call $C = \{x \mid xz = 0\}$ the *light cone* and $K = \{x \mid xz = 1\}$ the *unit sphere*. If K is given the distance xy is uniquely determined.

For $x \neq y$ let $L(x, y)$ denote the line through x and y and let $\alpha(x, y)$ denote the affine segment from x to y. When $S \subset A^n$ define $-S = \{x \mid -x \in S\}$. If $S = -S$ the set S is called symmetric about z or simply symmetric. The sets C and K are symmetric.

Two general G-spaces R_1 and R_2 are said to be *topologically isometric* if there exists a topological map of R_1 onto R_2 that preserves the indefinite distance xy.

It is easily seen that if R_1 and R_2 are Minkowski spaces defined on A^n with unit spheres K and K^* respectively, then R_1 and R_2 are topologically isometric if and only if there is an affinity mapping K onto K^*.

2. Two dimensional spaces. If R is A^2, then by [4, p. 241] one of the following must hold: (1) no null lines exist in R, (2) there is exactly one null line through each point of R, (3) there are exactly two null lines through each point of R, or (4) all lines in R are null.

In case (1) we call R a *spacelike plane*. By [4, p. 239], a spacelike plane is an ordinary Minkowski G-space with unit sphere a strictly convex closed curve.

In case (2) we call R a *neutral plane*. A neutral plane is topologically isometric to the (s, t) plane with distance from (s_1, t_1) to (s_2, t_2) given by $|t_1 - t_2|$.

When R has exactly two null lines through each point it is called a *doubly timelike* (Minkowski) *plane*, see [1]. The unit sphere has four components each of which is strictly convex and not compact.

If all lines in R are null, we call R a *null plane*.

3. Reducible spaces. Let R be an n-dimensional Minkowski space. Then R is reducible to $R^r \times N^{n-r}$ for $r < n$, provided affine coordinates $x_1, x_2, \cdots, x_n$ may be chosen such that

(1) R^r is given by $x_{r+1} = x_{r+2} = \cdots = x_n = 0$ and N^{n-r} is given by $x_1 = \cdots = x_r = 0$.

(2) The projection of R onto R^r preserves the metric xy.

The maximum possible value of $n - r$ is called the index of reducibility of R. A null plane has index 2 and a neutral plane index 1. Spacelike and doubly timelike planes are not reducible.

Nonreducible spaces often contain reducible subspaces. In the three dimensional Lorentz space any plane tangent to the light cone is neutral and hence reducible.

Given a line N the parallel to N through x will always be denoted by N_x.

DEFINITION 3.1. A line N through z is called a line of reduction of R if $x \in K$ implies $N_x \subset K$.

LEMMA 3.2. *The space R is reducible if and only if R has a line of reduction.*

Proof. If N is a line of reduction of R and L^{n-1} is a hyperplane with $L^{n-1} \cap N = z$, the projection of R onto L^{n-1} along parallels to N preserves the metric.

On the other hand if R is reducible to $R^r \times N^{n-r}$ any line N through z and in N^{n-r} is a line of reduction of R.

4. **The r-flat topology.** If $\{M_m\}$ is a sequence of closed subsets of R, we say M_m converges to the closed set M if $\lim M_m = M$ in the sense of Hausdorff's closed limit, see [2]. This limit induces a topology on the closed subsets of R. If L^r is an r-flat and $W(L^r)$ is a neighborhood of L^r in this topology, let $W_r(L^r)$ denote the r-flats in $W(L^r)$.

LEMMA 4.1. *Let $\{L^2_m\}$ be a sequence of doubly timelike planes, each containing z, such that $\{L^2_m\}$ converges to the two flat L^2. Assume $x^m_i \in K \cap L^2_m$ and $x^m_i \to x_i$ for $i = 1, 2$.*

(1) *Let L^2 be doubly timelike and let x_1, x_2 lie on the same component [opposed components] of K. Then for sufficiently large m the points x^m_1 and x^m_2 always lie on the same component [opposed components] of $K \cap L^2_m$.*

(2) *If L^2 is neutral, then for sufficiently large m the points x^m_1 and x^m_2 are always on the same or else always on opposed components of $K \cap L^2_m$.*

Proof. The proofs are similar and consequently we only consider statement (2) in which L^2 is neutral.

Without loss of generality assume x_1 and x_2 are on the same component of $K \cap L^2$ since if $x^m_1 \to x_1$ then $-x^m_1 \to -x_1$.

If $y \in \alpha(x_1, x_2)$ then $y \in K$ and $zy = 1$. Therefore, there exists an open set V containing the set $\alpha(x_1, x_2)$ such that all $p \in V$ have $zp > 0$. For sufficiently large m all points of $\alpha(x^m_1, x^m_2)$ lie in V and have positive distance from z. It follows that x^m_1 and x^m_2 lie on the same component of $K \cap L^2_m$ for large m.

The components of K are arcwise connected since they are connected and locally arcwise connected.

LEMMA 4.2. *Let x_1 and x_2 lie on the same component of K and let L^2 be a two flat containing z, x_1 and x_2. If S_1 and S_2 are the components of $K \cap L^2$ containing x_1 and x_2 respectively then either $S_1 = S_2$ or else $S_1 = -S_2$.*

Proof. Let $x(t)$ for $0 \leqq t \leqq 1$ be a curve on K connecting x_1 and x_2 with $x(0) = x_1$ and $x(1) = x_2$.

Call the two flat $L^2(t)$ admissible if $z, x_1, x(t) \in L^2(t)$ and $K \cap L^2(t)$ has components S_1 and $S(t)$ containing x_1 and $x(t)$ respectively such that either $S_1 = S(t)$ or else $S_1 = -S(t)$. For sufficiently small t there must exist admissible $L^2(t)$. Set $M = \{t \in [0, 1] \mid \text{there exists an admissible } L^2(t)\}$.

We now show M is closed. If $\{L^2(t_m)\}$ is a sequence of admissible planes and $t_m \to t_0$, then there is a convergent subsequence $\{L^2(t_k)\} \subset \{L^2(t_m)\}$ such that $L^2(t_k) \to L^2_0$. Clearly $z, x_1, x(t_0) \in L^2(t_0)$. Statement (1) of Lemma 4.1 implies L_0^2 cannot be doubly timelike with x_1 and $x(t_0)$ neither on the same nor on opposed components of $K \cap L_0^2$. Therefore, $t_0 \in M$.

To show M is open let $\tau \in M$ and $L^2(\tau)$ be admissible. If $L^2(\tau)$ is spacelike there must exist a neighborhood $W_2(L^2)$ containing only spacelike planes. But this implies the existence of a neighborhood $U(\tau)$ of the number τ with $U(\tau) \subset M$. If $L^2(\tau)$ is a doubly timelike plane statement (1) of Lemma 4.1 implies the existence of a neighborhood $U(\tau) \subset M$. In case $L^2(\tau)$ is a neutral plane first construct a neighborhood $W_2(L^2(\tau))$ in which no null planes exist. If only spacelike and neutral planes exist in $W_2(L^2(\tau))$ there is nothing to show. If there is a sequence of doubly timelike planes $L^2(t_m)$ converging to $L^2(\tau)$, statement (2) of Lemma 4.1 guarantees that for large m the planes $L^2(t_m)$ are admissible. It follows that there is a neighborhood $U(\tau) \subset M$. Therefore, M is open as well as closed. Since $M \neq \phi$, $M = [0, 1]$ and the lemma is established.

THEOREM 4.3. *Let K_1 and K_2 be distinct components of K that are opposed (i.e., $K_2 = -K_1$). Then K_1 and K_2 are convex hypersurfaces.*

Proof. Let $K_1^0 = \{y \mid \alpha(z, y) \cap K_1 \neq \phi\}$. Then K_1^0 has boundary K_1 and $y \in K_1^0$ implies $zy \geqq 1$. If $y_1, y_2 \in K_1^0$ let L^2 be a two flat through z, y_1 and y_2. Then L^2 must either be neutral or doubly timelike. In either case $\alpha(y_1, y_2) \subset K_1^0$ if y_1 and y_2 lie on the same component of $K_1 \cap L^2$. Clearly y_1 and y_2 lie on the same component for L^2 neutral. If L^2 is doubly timelike, then $K_1 \neq K_2$ and Lemma 4.2 imply y_1 and y_2 lie on the same component of $K_1 \cap L^2$. It follows that K_1^0 is convex and that its boundary K_1 is a convex hypersurface. In the same fashion one may show K_2 is a convex hypersurface.

LEMMA 4.4. *Let K have a component K_1 that is symmetric about z. Then for each $x \in K_1$ there is a two flat L^2 through z and x that is spacelike.*

Proof. Assume the statement is false. Any two flat containing $L(z, x)$ is then either neutral or doubly timelike. Orient $L(z, x)$ to get $L^+(z, x)$. If L_1 is a line parallel to $L^+(z, x)$, orient L_1^+ in the same direction. This gives an ordering $<$ on each line parallel to $L(z, x)$.

Let $x(t)$ for $0 \leqq t \leqq 1$ be a curve on K_1 with $x(0) = x$, $x(1) = -x$ and $x(t) \notin L(x, -x)$ for $0 < t < 1$. Let $L^+(t)$ be the oriented line containing $x(t)$ and parallel to $L^+(z, x)$. The line $L^+(t)$ is never a null line.

In the ordering $<$ along $L^+(t)$ let $p(t)$ be the first element in $\{y \mid y \in L^+(t)$ and $zy = 0\}$. Let $f(t)$ be the signed euclidean distance from $x(t)$ to $p(t)$ where $f(t) < 0$ if $x(t) < p(t)$. If $z < x$ then $f(0) < 0$

and $f(1) > 0$.

The function $f(t)$ is continuous at 0 and 1 since $p(t) \to z$ for $t \to 0$ and $t \to 1$. To show $f(t)$ is continuous on $(0, 1)$ let $0 < t_0 < 1$ and $t_m \to t_0$. For $0 < t < 1$ let $L^2(t)$ denote the unique plane containing $L^+(t)$ and z. Clearly if $L(t_0)$ is neutral we have $L(z, p(t_m)) \to L(z, p(t_0))$. If $L^2(t_0)$ is doubly timelike, one can show using (1) of Lemma 4.1 that $L(z, p(t_m)) \to L(z, p(t_0))$. In either case $p(t_m) \to p(t_0)$ and $f(t)$ is continuous. But then $f(\tau) = 0$ for some $0 < \tau < 1$ which implies $x(\tau) = p(\tau)$. This is impossible since $zx(\tau) = 1$ and $zp(\tau) = 0$.

5. **Three dimensional spaces.** In this section we only consider three dimensional Minkowski spaces.

LEMMA 5.1. *Let K have three components K_1, K_2 and K_3 with $K_3 = -K_3$. Then $K_1 = -K_2$ and K_1 (hence also K_2) is strictly convex.*

Proof. By Lemma 4.4 there is a two flat L^2 through z that is spacelike with $L^2 \cap K_3 \neq \phi$. This flat separates A^3 and does not intersect K_2. Hence $K_2 \neq -K_2$. Consequently, $K_1 = -K_2$.

To see that K_1 is strictly convex let $x, y \in K_1$. If L_0^2 is a two flat through x, y and z it must be doubly timelike since $L_0^2 \cap L^2 \neq \phi$. Then $L_0^2 \cap K_1$ is a strictly convex curve. It follows that $u \in \alpha(x, y) - x - y$ implies $zu > 1$. Therefore, K_1 must be strictly convex.

If K_i is a component of K then so is $-K_i$. Consequently, if K has exactly three components there is always one, say K_3, that is symmetric about z.

Extend A^3 to the real three dimensional projective space P^3 by adding a plane L_∞^2 at ∞. The projective lines that the light cone C determine intersect L_∞^2 in a curve C_∞. Let K have exactly three components. Since spacelike planes exist in this case, there is a line $L_0 \subset L_\infty^2$ with $L_0 \cap C_\infty = \phi$. The set $L_\infty^2 - L_0$ is an affine plane with L_0 the line at ∞.

Let $p, q \in C_\infty$ with $p \neq q$. Let L^2 be two flat in P^3 that contains z, p, q. Then $L^2 \cap A^3$ cannot be a null plane, since if it were it would separate A^3 and K_3 could not be symmetric. Consequently, $L^2 \cap A^3$ must be a doubly timelike plane.

It follows that $L^2 \cap (L_\infty^2 - L_0)$ is an affine line in $L_\infty^2 - L_0$ that intersects C_∞ in only the two points p and q. But C_∞ is a closed curve. Hence, C_∞ is a strictly convex curve in $L_\infty^2 - L_0$.

THEOREM 5.2. *Let* $\dim R = 3$. *If K has three components K_1, K_2 and K_3 with $K_3 = -K_3$, then K_3 is a hyperboloid of one sheet.*

Proof. Let $u \in L_\infty^2 - L_0$ and let u be exterior to the convex set in $L^2_\infty - L_0$ whose boundary is C_∞. Then there are lines L_1 and L_2 through u that are supporting lines of C_∞. Let L^2_i be the projective plane containing z and L_i for $i = 1, 2$. Then $L^2_i \cap C_\infty$ is a single point and hence $L^2_i \cap A^3$ is a neutral plane.

The set $L^2_i \cap A^3 \cap K$ consists of two parallel lines which must be on K_3 since K_1 and K_2 are strictly convex. For any $q \in K_3$ let $u = L(z, q) \cap L^2_\infty$ and without loss of generality assume $u \notin L_0$. Then u must be exterior to C_∞. By the above arguments there must be two straight lines on K_3 through q. By [5, p. 272] the set K_3 is a hyperboloid of one sheet.

Notice that the above theorem gives the additional information that C is elliptic and C_∞ is an ellipse in $L^2_\infty - L_0$.

LEMMA 5.3. *K can have at most four components. If K does have four components, R is reducible and no component of K is symmetric about z.*

Proof. Let K_1 be a component of K. Assume $K_1 = -K_1$, then there is a spacelike plane L^2_0 through z with $L^2_0 \cap K_1 \neq \phi$. Take $K_2 \neq K_1$ and $x \in K_2$. Let $L^2(\theta)$ be a two flat containing $L(z, x)$ that revolves continuously in θ and sweeps out A^3 for $0 \leqq \theta \leqq \pi$. Each $L^2(\theta)$ intersects L^2_0 in a line through z so that $L^2(\theta) \cap K_1 \neq \phi$ for all θ. Therefore, each $L^2(\theta)$ is doubly timelike and intersects K in four components. Two of these components lie on K_1, and the other two are subsets of K_2 and $-K_2$. Since this holds for all $\theta \in [0, \pi]$, K can have at most three components. Therefore, $K_1 \neq -K_1$ if K has four components.

By the above, it must be possible to find at least two components K_1 and K_2 of K with $K_1 \neq -K_1$, $K_2 \neq -K_2$ and $K_1 \neq -K_2$. Set $K_3 = -K_1$ and $K_4 = -K_2$. Let $y \in K_1$ and let $L^2(\psi)$ be a two flat through $L(z, y)$ that sweeps out A^3 continuously for $0 \leqq \psi \leqq \pi$. It can be assumed without loss of generality that $L^2(0) \cap K_2 \neq \phi$. Therefore, let x_2 belong to $L^2(0) \cap K_2$. $L^2(\psi)$ cannot be doubly timelike for all ψ or else x_2 and $-x_2$ would be on the same component of K. Therefore, there is a first ψ_0 with $L^2(\psi_0)$ neutral. Let $N \subset L^2(\psi_0)$ be the null line through z. Claim N is a line of reduction of R.

It is clear that if $x \in K_1 \cup K_3$ then $N_x \subset K_1 \cup K_3$ since these are convex surfaces and $N_y \subset K_1$ as well as $N_{-y} \subset K_3$. For $x \in K_2 \cup K_4$ consider the following argument. Let $L^2(\gamma)$ be a plane through $L(z, x_2)$ sweeping out A^3 continuously for $0 \leqq \gamma \leqq \pi$ with $y \in L^2(0)$. By the same reasoning as before, there is a first γ_0 with $L^2(\gamma_0)$ neutral. The above N must be in $L^2(\gamma_0)$ since $N_y \subset K_1$ and K_1 is not flat. This implies $N_x \subset K_2 \cup K_4$ whenever $x \in K_2 \cup K_4$.

It is now possible to show K has at most, four components. If L^2_1 is a two flat containing the above N either L^2_1 is neutral or null.

If it is null, it intersects $L^2(\gamma)$ for $\gamma = 0$ in a null line. If it is neutral, it intersects either K_1 and K_3 or else K_2 and K_4. In any case it cannot contain a point of K not on $K_1 \cup K_2 \cup K_3 \cup K_4$.

An immediate consequence is that if K has four components $R = R^2 \times N'$ where R^2 is a doubly timelike plane.

Consider now the case of K having one component. If R has no null lines, then by [4, p. 239] it is a Minkowski G-space and K must be strictly convex.

LEMMA 5.4. *Let K have one component and not be strictly convex. Then K is a cylinder and $R = R^2 \times N^1$ where R^2 is a spacelike plane.*

Proof. Let K contain a segment α and consider the two flat L_0^2 through z and α. L_0^2 must be neutral, hence the line containing α must lie on K. Let N be the null line in L_0^2 through z. Since K has only one component, there is a spacelike plane L^2 through z. Any two flat L_1^2 containing N must intersect L^2 in a line through z.

The plane L_1^2 cannot be a doubly timelike because of Lemma 4.2 and the fact that K has only one component. Therefore, L_1^2 is neutral and contains two lines on K parallel to N. It follows K must be a cylinder with generators parallel to N.

Projecting R onto L^2 along parallels to N gives $R = R^2 \times N^1$ for R^2 the spacelike plane L^2.

If K has two components K_1 and K_2 in dimension three, then $K_1 = -K_2$ since otherwise there would be a spacelike plane L^2 through z intersecting only one component of K yet separating A^3. Both K_1 and K_2 must be flat since if $x, y \in K_1$ with $x \neq y$, the two flat L_1^2 containing x, y and z would have to be neutral.

It can easily be shown that for K having two components, the space is always topologically isometric to (x_1, x_2, x_3)-space with the distance from (a_1, a_2, a_3) to (b_1, b_2, b_3) given by $|a_1 - b_1|$. K consists of two parallel planes and $R = R^1 \times N^2$ for R^1 the real line.

6. Higher dimensional spaces. The n dimensional situation is now investigated by the use of r-flats.

LEMMA 6.1. *K_1, K_2, K_3 be three distinct components of K, then two are reflections through z of each other.*

Proof. Consider $p_i \in K_i$ for $i = 1, 2, 3$ and let L^3 be a three flat containing z, p_1, p_2, and p_3. Let $S_i = K_i \cap L^3$, then S_1, S_2, and S_3 are disjoint components of $K \cap L^3$. By the last section $K \cap L^3$ has either three or four components, and in any case, any three of the components of $K \cap L^3$ contain a pair that are symmetric to each other. If we assume $S_1 = -S_2$ then clearly $K_1 = -K_2$.

LEMMA 6.2. *K has at most four components. If K does have four components K_1, K_2, K_3 and K_4, without loss of generality, one may assume $K_1 = -K_3$ and $K_2 = -K_4$.*

Proof. Assume K has five components K_1, K_2, K_3, K_4 and K_5. Then lemma 6.1 applied to K_1, K_2 and K_3 allows the assumption $K_3 = -K_1$. Applying Lemma 6.1 to K_1, K_2 and K_4 yields $K_2 = -K_4$.

Let $p_1 \in K_1$, $p_2 \in K_2$ and $p_5 \in K_5$, then let L_3 be a three flat containing p_1, p_2, p_5 and z. $K \cap L^3$ then contains five disjoint components, which is impossible by Lemma 5.3.

LEMMA 6.3. *Let $N_x \subset K$ then if one of the following holds, N_z is a line of reduction.*

(1) *K has exactly one component.*

(2) *K has exactly two components K_1 and K_2 that are symmetric to each other.*

(3) *K has exactly three components K_1, K_2, K_3 with $K_3 = -K_3$ and $N_x \subset K_1 \cup K_2$.*

(4) *K has four components.*

Proof. The proofs of the above four cases all follow the same general pattern. Therefore, the first case is the only one discussed.

If $N_x \subset K$ and K has one component, consider $y \in R$ and let L^3 be a three flat containing z, y and N_x. Either $N_y \subset K$ or else $K \cap L^3$ has three components. If $K \cap L^3$ has three components, there is a two flat $L^2 \subset L^3$ through z that is doubly timelike. But then $K \cap L^2$ has four components, and Lemma 4.2 would imply K had more than one component.

For convenience the following notation is adopted. If $k, p, \cdots, m$ are r distinct integers from the set $1, 2, \cdots, n$ let $L^r_{kp\cdots m}$ be the unique r-flat through the $x_k, x_p, \cdots, x_m$ axes. If L_0 is a line with $L_0 \not\subset L^r_{kp\cdots m}$ let $L^{r+1}_{0kp\cdots m}$ be the $r+1$ flat containing L_0 and $L^r_{kp\cdots m}$. Here we assume $L_0 \cap L^r_{kp\cdots m} \neq \phi$.

Repeated application of the last lemma gives the following partial description of the nonreducible spaces:

THEOREM 6.4. *In all cases K has at most four components. Let R be nonreducible.*

(1) *If K has one component, then R is a Minkowski G-space.*

(2) *If K has two components that are opposed to each other then R is isometric to the real line.*

(3) *If K has three components, then one is symmetric about z*

and the other two are strictly convex.

(4) *If K has four components, then R is a doubly timelike plane.*

The case where K has two components which are not opposed is discussed in Theorem 6.13 and additional information on the case of three components is found in Theorem 6.8.

LEMMA 6.7. *Let $n = 3$ and K have three components. Assume coordinates x_1, x_2, x_3 are chosen such that the light cone is given by $x_1^2 + x_2^2 = x_3^2$. Then the plane $x_3 = 0$ intersects K_3 in a set $x_1^2 + x_2^2 = a^2$ for some $a > 0$.*

Proof. Let p lie on K_3 and in the plane $x_3 = 0$. For some $a > 0$ the point p lies on $x_1^2 + x_2^2 - x_3^2 = a^2$. We claim that the only hyperboloid of one sheet containing p that has C as light cone is $x_1^2 + x_2^2 - x_3^2 = a^2$.

Since p is contained in exactly two planes tangent to C, the two lines on K_3 through p are determined. For any q on one of these two lines, the same argument yields that the two lines on K_3 through q are determined. It follows K_3 is determined by p and C.

Consider now $n > 3$ and extend A^n to P^n by adding a hyperplane L_∞^{n-1} at ∞. Let the projective lines that contain the lines of the light cone C intersect L_∞^{n-1} in a set C_∞.

If R is nonreducible and K has three components, let L_0^{n-1} be a supporting hyperplane to K_1. If L^{n-1} is the hyperplane parallel to L_0^{n-1} through z, then $L^{n-1} \cap C = z$. Otherwize $L^{n-1} \cap C$ would contain a line N. For $p \in L_0^{n-1} \cap K_1$ then the two flat L^2 through p and N would be neutral or doubly timelike. It could not be neutral because of Lemma 6.3. It could not be doubly timelike since then N_p would not be a supporting line of K_1.

Set $L^{n-1} \cap L_\infty^{n-1} = L_\infty^{n-2}$ an $n - 2$ dimensional flat. By taking L_∞^{n-2} as the $n - 2$ flat at ∞ of L_∞^{n-1} the set $L_\infty^{n-1} - L_\infty^{n-2}$ becomes an $n - 1$ dimensional affine space. Let $x, y \in C_\infty$ for $x \neq y$ and let L_1^2 be the two flat containing x, y and z. Then $L_1^2 \cap A^n$ is a doubly timelike plane. In the same manner as the argument after Lemma 5.1, we conclude C_∞ is a strictly convex $n - 2$ dimensional surface in the space $L_\infty^{n-1} - L_\infty^{n-2}$.

LEMMA 6.6. *C_∞ is an ellipsoid in $L_\infty^{n-1} - L_\infty^{n-2}$.*

Proof. Let L_∞^2 be a two flat in L_∞^{n-1} with $L_\infty^2 \cap C_\infty$ containing more than one point. Let L^3 be the three flat containing z and L_∞^2. Then $L^3 \cap A^n$ is an indefinite metric space whose unit sphere has three components. By Theorem 5.2, $L_\infty^2 \cap C_\infty$ is an ellipse and hence by [2, p. 91] C_∞ is an ellipsoid.

Take now coordinates $x_1, x_2, \cdots, x_n$ in A^n such that C has the form

$x_n^2 = x_1^2 + \cdots + x_{n-1}^2$ and let L_1^{n-1} be the hyperplane $x_n = 0$.

LEMMA 6.7. *$L_1^{n-1} \cap K$ has the form $x_1^2 + \cdots + x_{n-1}^2 = a^2$ for $a > 0$.*

Proof. Let L^2 be any two flat in L_1^{n-1} passing through z. Let L^3 be the three flat containing L^2 and the x_n axis. Since $L^3 \cap K$ always has three components, $L^2 \cap K$ is always an ellipse of center z. Therefore, $L_1^{n-1} \cap K$ is an ellipsoid in L_1^{n-1} of center z.

If L^2 contains the x_i and x_j axis Lemma 6.5 implies $L^2 \cap K_3$ has the form $x_i^2 + x_j^2 = a_{ij}^2$. If p_i and p_j are points of $L^2 \cap K_3$ that lie on the i^{th} and j^{th} axes respectively, $|p_i|^2 = |p_j|^2 = a_{ij}^2$. Therefore, a_{ij} is independent of i and j. Setting $a = a_{ij}$ yields the desired result.

THEOREM 6.8. *Let R be nonreducible and K have three components. If K_3 is the components of K symmetric about z it is a quadric. In proper affine coordinates K_3 is given by*

$$x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = a^2 .$$

Proof. Using the same notation as in Lemma 6.9 define

$$S = \{(x_1, x_2, \cdots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = a^2\} .$$

If L^3 contains the x_n axis then $L^3 \cap S = L^3 \cap K_3$. The result follows by letting L^3 sweep out A^n.

In order to investigate nonreducible spaces in which K has two components, we first consider nondegenerate central quadrics that have z as a center. The general form in affine space is

$$\sum_{i,j=1}^{n} a_{ij}x_ix_j = 1 \text{ where } a_{ij} = a_{ji} \text{ and } \det(a_{ij}) \neq 0 .$$

If two such quadrics E_1 and E_2 are given respectively by

$$\sum a_{ij}\, x_ix_j = 1 \text{ and } \sum a_{ij}x_ix_j = -\lambda^2 \text{ for } \lambda > 0 ,$$

they will be called *semiconjugate*. We will refer to E_1 as the λ semiconjugate to E_2. For $\lambda = 1$ the quadrics are conjugate in the usual sense. Notice that one of the quadrics does not have a real locus if the quadric form is definite.

LEMMA 6.9. *Suppose the nonempty sets B_1 and B_2 contained in $\bigcup_{i \neq j} L_{ij}^2$ are such that the locus $B_2 \cap L_{ij}^2$ is always the λ semiconjugate quadric to $B_1 \cap L_{ij}^2$ for fixed λ. Then there are exactly two central quadrics E_1 and E_2 such that $E_1 \cap L_{ij}^2 = B_1 \cap L_{ij}^2$ and $E_2 \cap L_{ij}^2 = B_2 \cap L_{ij}^2$ for all $i \neq j$. Furthermore, E_2 is the λ semiconjugate to E_1.*

LEMMA 6.10. *Let $n = 4$ and K have two components K_1 and K_2*

each symmetric about z. Let L^3 be a three flat through z such that $L^3 \cap K$ has three components. Then $L^3 \cap K$ consists of two semiconjugate quadrics.

Proof. By Theorem 5.2 one component of $L^3 \cap K$ must be a hyperboloid of one sheet. Choose coordinates x_1, x_2, x_3 in L^3 such that $L^3 \cap C$ takes the form $x_1^2 + x_2^2 = x_3^2$. Let $L^3 \cap K$ have components S_1, S_2, S_3 with $S_3 = -S_3$. For some $a > 0$, S_3 is given by $x_1^2 + x_2^2 - x_3^2 = a^2$. Let L_0 be a line through z in L_{12}^2.

In R let L^2 be a spacelike plane containing the x_3 axis, so $L^2 \not\subset L^3$. Choose the x_4 axis in L^2. Assume K has components K_1 and K_2 with $S_3 \subset K_1$, then $L_{034}^3 \cap K_2$ is a hyperboloid of one sheet in L_{034}^3. Consequently, $L_{03}^2 \cap K_2$ is a hyperbola. This hyperbola is determined given only the intersection of K_2 with the x_3 axis and the intersection of L_{03}^2 with the surface $x_1^2 + x_2^2 = x_3^2$ in L^3.

Revolving L_0 in the plane L_{12}^2 shows $L^3 \cap K_2$ consists of a hyperboloid of two sheets that is a semiconjugate of $L^3 \cap K_1$.

LEMMA 6.11. *If $n = 4$ and K has two symmetric components, they are semiconjugate quadrics.*

Proof. Let the notation and coordinates be the same as in the last proof. Set $B_1 = \bigcup_{i \neq j} (L_{ij}^2 \cap K_1)$ and $B_2 = \bigcup_{i \neq j} (L_{ij}^2 \cap K_2)$.

If $L^3 \cap K_2$ is the λ semiconjugate to $L^3 \cap K_1$ in L^3, then $L_{034}^3 \cap K_2$ is the λ semiconjugate to $L_{034}^3 \cap K_1$ in L_{034}^3 for the same λ. This follows since L_{03}^2 is common to both three flats and intersects both components of K. Therefore, B_1 and B_2 satisfy the hypothesis of Lemma 6.9. Let E_1 and E_2 be the semiconjugate quadrics determined by B_1 and B_2.

$L^3 \cap E_1 = L^3 \cap K_1$ since each are quadrics in L^3 determined by $B_1 \cap L^3$ and $B_2 \cap L^3$. By the same reasoning, $L^3 \cap E_2 = L^3 \cap K_2$. Also $L_{124}^3 \cap K_i = L_{124}^3 \cap K_i$ for $i = 1, 2$.

Therefore, $L_{0j}^2 \cap K_i = L_{0j}^2 \cap E_i$ for $i = 1, 2$ and $j = 3, 4$. But then using Lemma 6.11 one last time, we find $L_{034}^3 \cap E_i = L_{034}^3 \cap K_i$. By revolving L_0 in L_{12}^2 it follows $E_i = K_i$ for $i = 1, 2$.

LEMMA 6.12. *Let $n = 5$ and K have two components K_1 and K_2 symmetric about z. If R is not reducible, K_1 and K_2 are semiconjugate quadrics.*

Proof. Two cases are considered.

Case 1. Let there exist a three flat L^3 through z such that $L^3 \cap K$ has one component. Assume $L^3 \cap K_2 \neq \phi$. Choose coordinates x_1, x_2, x_3 in L^3. We may assume that $L_{12}^2, L_{13}^2, L_{23}^2$ are spacelike planes. Choose coordinates x_4, x_5 such that L^2_{45} is spacelike and intersects K_1. By arguments as in Lemma 6.10 and Lemma 6.11, it is possible to show $L^2_{ij} \cap K_1$ and $L^2_{ij} \cap K_2$ are always semiconjugate quadrics for fixed λ. Therefore, $B_1 = \bigcup_{i \neq j} (L^2_{ij} \cap K_1)$ and $B = \bigcup_{i \neq j} (L^2_{ij} \cap K_2)$ satisfy the hypothesis of Lemma 6.9.

Let E_1 and E_2 be the quadrics determined by B_1 and B_2. Let L_0 be a line through z in L^2_{12}. Since $L^2_{12j} \cap E_i = L^3_{12j} \cap K_i$, clearly $L^2_{0j} \cap E_i = L^2_{0j} \cap K_1$ for $i = 1, 2$ and $j = 3, 4, 5$. Therefore $L^4_{0345} \cap E_i = L^4_{0345} \cap K_i$. By revolving L_0 in L^2_{12} it follows that $E_i = K_i$.

Case 2. Assume no L^3 through z exists with $L^3 \cap K$ having only one component. We will show this leads to a contradiction.

Choose coordinates x_1, x_2, x_3, x_4, x_5 such that L^2_{12} and L^2_{34} are spacelike planes intersecting respectively K_1 and K_2. By Theorem 6.8, the set $K \cap L^4_{2345}$ cannot have exactly three components. Consequently, $L^3_{2345} \cap K$ consists of two symmetric components. The same must also be true of $L^4_{1235} \cap K$.

By Lemma 6.11 the sets $L^4_{1234} \cap K$, $L^4_{2345} \cap K$ and $L^4_{1235} \cap K$ each consists of two quadrics. In each of the three sets one quadric is the semiconjugate of the other for some fixed λ. Define

$$B_1 = \bigcup_{i \neq j} (L^2_{ij} \cap K_1) \quad \text{and} \quad B_2 = \bigcup_{i \neq j} (L^2_{ij} \cap K_2) \, .$$

Let E_1 and E_2 be the quadrics determined.

Let L_0 be a line through z in L^2_{12}. Then $L^2_{0j} \cap K_i = L^2_{0j} \cap E_i$ for $j = 3, 4, 5$ and $i = 1, 2$. Therefore, $L^4_{0345} \cap E_i = L^4_{0345} \cap K_i$ and revolving L_0 in L^2_{12} gives $E_i = K_i$ for $i = 1, 2$.

Then in proper affine coordinates y_1, y_2, y_3, y_4, y_5 the components of K are given by $y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 = -\lambda^2$. This contradicts the assumption of Case 2.

The n dimensional case now follows using induction.

THEOREM 6.13. *If R is not reducible and K has two components which are not opposed, then $n \geqq 4$ and the components are semiconjugate quadrics.*

Proof. Assume $n \geqq 6$. Take L^{n-1} to be a hyperplane containing L^2_1 and L^2_2, which are spacelike two flats through z with $L^2_i \cap K_i \neq \phi$. Then $L^{n-1} \cap K$ has exactly two symmetric components. Because of Lemma 6.12, there exists an L^3 through z and contained in L^{n-1} with $L^3 \cap K$ having one component. Take the x_1, x_2, x_3 affine coordinates in L^3 and $x_1, x_2, \cdots, x_{n-1}$ affine coordinates in L^{n-1}. For $p \in K - L^{n-1}$ let the x_n axis be $L(z, p)$. Take L_0 to be a line through z in L^2_{12}. By induction $L^{n-1}_{0345\cdots n} \cap K_i$ must consist of two semiconjugate quadrics. The argument is the same as before, letting L_0 revolve in L^2_{12}.

An interesting result of this section is the following.

COROLLARY 6.14. *If R is a nonreducible Minkowski space and not a G-space, then any spacelike plane in R is euclidean.*

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Received August 25, 1969. This paper is part of a Ph. D. dissertation written under the direction of Professor H. Busemann at the University of Southern California.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

TRAJECTORY INTEGRALS OF SET VALUED FUNCTIONS

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Let I be a compact interval of the real line and for each t in I, let $F(t)$ denote a nonvoid subset of euclidean n-space E^n. Let $\mathscr{F}_I(F)$ be the collection of all Lebesgue summable functions $u: I \to E^n$ having the property that $u(t) \in F(t)$ almost everywhere on I. Following the lead of Kudo and Richter, Aumann defines the integral of F over I by

$$\int_I F(t)dt = \left\{\int_I f(t)dt \mid f \in \mathscr{F}_I(F)\right\}$$

and, in addition to other results, establishes a dominated convergence theorem for such integrals. Hermes has pursued Aumann's line of thought to obtain results concerning something akin to a "derivative" for set valued functions.

It is certainly also valid (and for control theoretic applications essential) to define the *trajectory integral* of F to be the set $\mathscr{S}_I(F)$ of all *functions* which vanish at the left endpoint of I and have derivatives in $\mathscr{F}_I(F)$. The purpose of this paper is taken to be the study of the trajectory integrals of nonvoid, compact set valued functions. A primary goal is the extension of the results of Aumann to include the trajectory integral. A secondary goal is the provision of an intuitively meaningful definition of "derivative" for set valued functions.

Whereas $\int_I F(t)dt$ is a subset of E^n, $\mathscr{S}_I(F)$ is a subset of a space of functions on I to E^n. Taking note of the relation

$$(1) \qquad \int_{[0,t]} F(\tau)d\tau = \{\mu(t) \mid \mu \in \mathscr{S}_I(F)\},\ t \in I\ ,$$

the validity of which is obvious when $\mathscr{F}_I(F)$ is nonvoid, it is clear that the distinction between $\mathscr{S}_I(F)$ and $\int_{[0,t]} F(\tau)d\tau$ is essentially that between "function" and "value of a function". In view of this distinction, one necessarily anticipates that a study of the trajectory integral would, in some sense, subsume that of the integral defined by Aumann.[1] Concrete justification for this point of view already exists in control theory [4].

Further motivation for the study of the trajectory integral arises in connection with the existence theory of the generalized differential equation

[1] The work of Kudo, Richter, Aumann and Hermes cited previously is to be found in references [**13**], [**18**], [**1**] and [**11**] respectively.

$$\dot{x} \in R(t, x),\ x(t_0) = x_0\ , \tag{2}$$

in the case in which the set valued function satisfies, in particular, a condition of measurability in its first argument. Here one anticipates that a suitably formulated dominated convergence theorem for the trajectory integral would provide the means for a constructive proof of existence, along classical lines, thereby providing at same time a method of approximation to solutions. This is a question of no little importance inasmuch as the general existence theorem of Pliś [17] and Castaing [5] has been established by nonconstructive methods.

The goals of this paper are achieved in the following way. After developing, in § 1, the pertinent algebraic and topological properties of the space Ω^n of nonvoid compact subsets of E^n, in § 2 we establish several fundamental structural properties of Lebesgue measurable functions on E^1 to Ω^n. The concept of Lebesgue measurability for functions on E^1 to Ω^n is due to Pliś [16] and is a natural generalization of the concept of measurability of functions with range in E^n. As Hermes has pointed out [11], Aumann's "Borel measurability" implies measurability in the sense defined by Pliś. Some of the theorems of § 2 have already been stated, without proof and in a somewhat less general form, by Filippov [9]. The central result of § 2 is Theorem 2.3 which is the counterpart of the theorem for point valued functions which asserts that almost every point in the domain of a summable function is a Lebesgue point of the function. This theorem plays an essential role in the proofs of two of the major results of the paper: Theorems 3.1 and 5.1.

Theorems 3.1 and 3.2 are the principal results of interest in § 3. In the former, conditions are stated—the chief one of which is measurability of F—under which $\mathscr{S}_I(F)$ is a nonvoid compact subset of each of two linear topological (function) spaces. One of these compactness properties, together with Hermes' refinement [12, Lemma 1.2] of Filippov's "measurable selection" lemma [8], permits a short proof of the dominated convergence theorem (Theorem 3.2) in a form suited to the proof of the existence theorem (Theorem 4.1) for (2). In § 3 we also devote some attention to the relationship between Aumann's results and our own.

Finally, in § 5, we define a derivative for an element of a certain function space which, owing to its obvious relationship to Huygen's principle of wave propagation, we have styled "the Huygens derivative". The principal result (Theorem 5.1) of this section asserts, loosely speaking, that the Huygens derivative of the trajectory integral of a measurable function F is almost everywhere the convex hull of $F(t)$. As easy corollaries to this theorem we obtain generalizations of some of the results of Hermes [11] mentioned previously.

1. Algebraic and topological preliminaries. In this paper we shall need the following Banach spaces.

E^n: euclidean n-space, with the scalar product of $a, b \in E^n$ denoted by $a \circ b$ and with norm denoted by $\|x\| \equiv (x \circ x)^{1/2}$;

$\mathscr{C}^n(I)$: space of continuous functions on I to E^n, with supremum norm $\langle x \rangle = \max \{\|x(t)\| \mid t \in I\}$;

$\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$: space of absolutely continuous functions on I to E^n, vanishing at the left endpoint of I, with norm $\hat{x} = \int_I \|\dot{x}(t)\| \, dt$;

$\mathscr{L}_1^n(I)$: space of Lebesgue summable functions on I to E^n, with norm $\langle\!\langle x \rangle\!\rangle = \int_I \|x(t)\| \, dt$.

In each instance, I denotes a nondegenerate compact interval of E^1. Throughout this paper the symbol ϕ will be used to denote the null set. We shall also need the following classes of subsets of E^n and $\mathscr{C}^n(I)$:

Ω^n: class of nonvoid, compact subsets of E^n;

Γ^n: class of nonvoid, compact, convex subsets of E^n;

$\mathscr{H}^n(I)$: class of nonvoid, compact subsets of $\mathscr{C}^n(I)$;

$\mathscr{K}^n(I)$: class of nonvoid, compact, convex subsets of $\mathscr{C}^n(I)$.

DEFINITION 1.1. Given a field, Φ, of scalars and a set, K, of vectors, together with functions $+ : K \times K \to K$ and $\times : \Phi \times K \to K$, K is called a *quasilinear space over* Φ if and only if all the axioms for a linear space obtain except (i) the distributivity of $\times$ over scalar addition and (ii) the existence of an inverse under $+$.

DEFINITION 1.2. For $\alpha \in E^1$, $A, B \in \Omega^n$,

$$A + B = \{a + b \mid a \in A; b \in B\} ,$$
$$\alpha A = \{\alpha a \mid a \in A\} .$$

The following result is easy to verify.

LEMMA 1.1. *With the foregoing definition (Definition 1.2) of addition and scalar multiplication, Ω^n and Γ^n are quasilinear spaces over the real field.*

DEFINITION 1.3. Let $A, B \in \Omega^n$, $Y, Z \in \mathscr{H}^n(I)$ and $x \in E^n$, $y \in \mathscr{C}^n(I)$; then we may define:

$$\alpha(x, A) = \min\{\|x - a\| \mid a \in A\}$$
$$\beta(y, Z) = \min\{\langle y - z\rangle \mid z \in Z\}$$
$$\bar{\rho}(B, A) = \max\{\alpha(x, A) \mid x \in B\}$$
$$\bar{\sigma}(Y, Z) = \max\{\beta(y, Z) \mid y \in Y\}$$
$$\rho(A, B) = \max\{\bar{\rho}(A, B), \bar{\rho}(B, A)\}$$
$$\sigma(Y, Z) = \max\{\bar{\sigma}(Y, Z), \bar{\sigma}(Z, Y)\}$$
$$\nu(A, p) = \max\{p \circ \sigma \mid \sigma \in A\}$$
$$\|A\| = \rho(A, \{0\})$$
$$\bar{\Delta}(A, B) = \max\{\nu(A, p) - \nu(B, p) \mid \|p\| = 1\}$$
$$A_\eta = \{x \in E^n \mid \alpha(x, A) \leqq \eta\}$$
$$\Delta(A, B) = \max\{\bar{\Delta}(A, B), \bar{\Delta}(B, A)\}$$
$$S(x, p) = \{\xi \in E^n \mid \|\xi - x\| \leqq p\},\ p \geqq 0\ .$$

LEMMA 1.2. (i) $\{\Omega^n, \rho\}$, $\{\Gamma^n, \rho\}$, $\{\mathscr{H}^n(I), \sigma\}$ *and* $\{\mathscr{K}^n(I), \sigma\}$ *are metric spaces.*

(ii) *If* $A \in \Omega^n (\in \Gamma^n)$ *then* $A_\eta \in \Omega^n (\in \Gamma^n)$ *for all* $\eta > 0$ *and* $A_\eta = A + S(0, \eta)$.

(iii) *If* $A, B \in \Gamma^n$ *then* $\bar{\rho}(A, B) = \bar{\Delta}(A, B)$ *and*

$$\Delta(A, B) = \max\{|\nu(A, p) - \nu(B, p)| \mid \|p\| = 1\}\ .$$

(iv) *If* $A, B, C \in \Gamma^n$ *then* $\bar{\rho}(A + B, A + C) = \bar{\rho}(B, C)$.

Proof. The proofs of (i), (ii) and (iii) are to be found in [4]. For (iv), we have, by virtue of (iii),

$$\begin{aligned}\bar{\rho}(A + B, A + C) &= \max\{\nu(A + B, p) - \nu(A + C, p) \mid \|p\| = 1\}\\ &= \max\{\nu(A, p) + \nu(B, p) - \nu(A, p) - \nu(C, p) \mid \|p\| = 1\}\\ &= \bar{\rho}(B, C).\end{aligned}$$

Henceforth we shall use $\Omega^n, \Gamma^n, \mathscr{H}^n(I), \mathscr{K}^n(I)$ to denote the metric spaces obtained by virtue of Definition 1.3 and Lemma 1.2 (i) and in the cases of Ω^n, Γ^n we shall suppose that the algebraic structure of Definition 1.2 has been imposed. For a point $A \in \Omega^n$ we shall denote by A^* the convex hull of A; it is well known that $A^* \in \Gamma^n$. Moreover, if $\eta \in E^1$ and $A, B \in \Omega^n (\in \Gamma^n)$ then ηA and $A + B$ are in Ω^n (in Γ^n) [6, V. 1.4].

LEMMA 1.3. (i) *If* $\eta \in E^1$ *and* $A, B \in \Omega^n$ *then* $\bar{\rho}(\eta A, \eta B) = |\eta|\bar{\rho}(A, B)$.

(ii) *If* $A, B, C \in \Omega^n$ *then* $\bar{\rho}(B^*, C^*) \leqq \bar{\rho}(A + B, A + C) \leqq \bar{\rho}(B, C)$.

(iii) *If* $A, B, C, D \in \Omega^n$ *then* $\bar{\rho}(A + B, C + D) \leqq \bar{\rho}(A, C) + \bar{\rho}(B, D)$.

Proof. The proof of (i) is trivial. Part (iii) is an easy consequence of (ii) and the "relaxed" triangle law [4, Lemma 1.1]. The second inequality of (ii) follows readily from the definitions and only the first inequality remains to be proved. By [6, V. 2.4]

$$\bar{\rho}(A^* + B^*, A^* + C^*) = \bar{\rho}((A + B)^*, (A + C)^*)$$

and then by Lemma 1.2 (iv)

$$\bar{\rho}(B^*, C^*) = \bar{\rho}((A + B)^*, (A + C)^*) .$$

Now for $D, E \in \Omega^n$ we have $D \subset E + S(0, \gamma)$, where $\gamma = \bar{\rho}(D, E)$; hence $D^* \subset E^* + S(0, \gamma)$ or $D^* \subset (E^*)_\gamma$ by Lemma 1.2 (ii) from which we conclude $\bar{\rho}(D^*, E^*) \leqq \bar{\rho}(D, E)$. Setting $D = A + B$, $E = A + C$, the first inequality of (ii) follows from this result and the last formula line.

COROLLARY 1.1. *Let* $\eta, \gamma \in E^1$, $A, B \in \Omega^n$; *then*
(i) $\| \eta A \| = | \eta | \, \| A \|$;
(ii) $\| A \| \geqq 0$ *and* $\| A \| = 0$ *if and only if* $A = \{0\}$;
(iii) $\| A + B \| \leqq \| A \| + \| B \|$;
(iv) $| \, \| A \| - \| B \| \, | \leqq \rho(A, B) \leqq \| A \| + \| B \|$;
(v) $\bar{\rho}(\eta A, \gamma A) \leqq | \eta - \gamma | \, \| A \|$.

Proof. (i) through (iv) follow easily from the definitions and Lemma 1.3. For (v) we have from Lemma 1.3 (i), (ii)

$$\begin{aligned} \bar{\rho}(\eta A, \gamma A) &= | \eta - \gamma | \bar{\rho}\Big(\Big(1 + \frac{\gamma}{\eta - \gamma}\Big)A, \Big(\frac{\gamma}{\eta - \gamma}\Big)A\Big) \\ &\leqq | \eta - \gamma | \bar{\rho}(A, \{0\}) = | \eta - \gamma | \, \| A \| . \end{aligned}$$

DEFINITION 1.4. (*Kuratowski.*) Let $\mathscr{M}$ denote a metric space and let $\mathscr{M}^*$ denote the space of all nonvoid, compact subsets of $\mathscr{M}$, metrized by the Hausdorff metric, ρ (cf. Definition 1.3). For a sequence $\{A_i\} \subset \mathscr{M}^*$, $\underline{\lim}_{i \to \infty} A_i$ is the set of all $x \in \mathscr{M}$ having the property that each neighborhood of x intersects all but a finite number of the A_i, whereas $\overline{\lim}_{i \to \infty} A_i$ is the set of all $x \in \mathscr{M}$ having the property that each neighborhood of x intersects infinitely many A_i. If $\underline{\lim}_{i \to \infty} A_i = \overline{\lim}_{i \to \infty} A_i$, the common value will be denoted by $\lim_{i \to \infty} A_i$.

LEMMA 1.4. ([14, p. 248]) *If* $\{A_i\} \subset \mathscr{M}^*$ *and* $A \in \mathscr{M}^*$, *with* $\lim_{i \to \infty} \rho(A_i, A) = 0$, *then* $\lim_{i \to \infty} A_i = A$.

LEMMA 1.5. *Let* $\{A_i\} \subset \mathscr{M}^*$ *and let* $\bar{A} \in \mathscr{M}^*$ *be a cluster point* (*in the Hausdorff metric topology*) *of* $\{A_i\}$; *then*

$$\underline{\lim}_{i \to \infty} A_i \subset \bar{A} \subset \overline{\lim}_{i \to \infty} A_i .$$

Proof. Let $\{A_{i_k}\}$ satisfy $\lim_{k\to\infty}\rho(A_{i_k}, \bar{A}) = 0$. By [14, pp. 242-243]

$$\varliminf_{i\to\infty} A_i \subset \varliminf_{k\to\infty} A_{i_k} \subset \varlimsup_{k\to\infty} A_{i_k} \subset \varlimsup_{i\to\infty} A_i \ ;$$

but by Lemma 1.4, $\bar{A} = \lim_{k\to\infty} A_{i_k}$.

COROLLARY 1.2. *Let $\{A_i\}\subset\Gamma^n$ satisfy $\|A_i\| \leqq \lambda$, for some $\lambda \geqq 0$; if $\bar{A} = \lim_{i\to\infty} A_i$ then $\bar{A}\in\Gamma^n$ and $\lim_{i\to\infty}\rho(A_i, \bar{A}) = 0$.*

Proof. By Blaschke's Auswahlsatz, the set $U=\{A\cap S(0, \lambda) \mid A\in\Gamma^n\}$ is a compact subset of Γ^n so that $\{A_i\}$ has a cluster point in U. By hypothesis and Lemma 1.5, $\bar{A}$ is the only cluster point of $\{A_i\}$ and then $\bar{A}\in\Gamma^n$. Again since U is compact, the assertion of the lemma follows.

LEMMA 1.6. *Let $\{A_i\}\subset\Omega^n$ satisfy, for some $\lambda \geqq 0$, $\|A_i\| \leqq \lambda$; if $A = \lim A_i$ and $A \neq \phi$ then $A\in\Omega^n$ and $\lim A_i^* = A^*\in\Gamma^n$.*

Proof. Since [14, pp. 242-243] A is closed, the fact that $A\in\Omega^n$ follows easily from the hypotheses. We shall prove that

$$A^* \equiv (\varliminf A_i)^* \subset \varliminf A_i^* \subset \varlimsup A_i^* \subset (\varlimsup A_i)^* \equiv A^* \ ,$$

the second inequality being trivial. For the proof of the first inequality, let $x\in A^*$; by Carathéodory's theorem [7, p. 35] there exist $x^k\in A$, $k = 1, \cdots, n+1$, such that $x = \sum_{k=1}^{n+1}\alpha_k x^k$,

$$\sum \alpha_k = 1,\ \alpha_k \geqq 0,\ k = 1, \cdots, n+1 \ .$$

Despite Lemma 1.1, it is trivial to establish that

$$\{x\}_\eta \equiv \{x\} + S(0, \eta) = \sum_{k=1}^{n+1}\alpha_k[\{x^k\} + S(0, \eta)] \equiv \sum_{k=1}^{n+1}\alpha_k\{x^k\}_\eta \ .$$

It is easy to see that there exists $K\geqq 0$, independent of $k = 1, \cdots, n+1$, such that $\{x^k\}_\eta\cap A_i \neq \phi$ for all $i\geqq K$. Letting $a_i^k\in\{x^k\}_\eta\cap A_i$ there follows $\sum_{k=1}^{n+1}\alpha_k a_i^k\in\{x\}_\eta$ for all $i\geqq K$; but clearly $\sum_{k=1}^{n+1}\alpha_k a_i^k\in A_i^*$ and we conclude that $x\in\varliminf A_i^*$.

For the proof of the third inequality, let $\bar{x}\in\varlimsup A_i^*$; then by [14, p. 243] there exists a subsequence $\{A_{i_k}^*\}$ and a sequence $\{x_k\}$ satisfying $x_k\in A_{i_k}^*$ and $\lim x_k = \bar{x}$. Now for each index k, there exist vectors $\xi_k^j\in A_{i_k}$, $j = 1, \cdots, n+1$ and numbers $\alpha_j^k\geqq 0$, $j = 1, \cdots, k+1$, satisfying $\sum_{j=1}^{n+1}\alpha_j^k = 1$ and $x_k = \sum_{j=1}^{n+1}\alpha_j^k\xi_k^j$. Setting $X_k = (\xi_k^1, \cdots, \xi_k^{n+1})$ and $\alpha_k = (\alpha_1^k, \cdots, \alpha_{n+1}^k)^T$, the superscript denoting transpose, we may write $x_k = X_k\alpha_k$. By virtue of the fact that $\|A_{i_k}\| \leqq \lambda$ for all k, it is clear that $\{X_k\}$ is contained in a compact subset of the cartesian product ($n+1$ factors) $E^n \times \cdots \times E^n$. Moreover, the compact set $\sum = \{p \in E^{n+1} \mid p^i \geqq 0,\ \ i = 1, \cdots, n+1;\ \sum_{i=1}^{n+1} p^i = 1\}$ contains $\{\alpha_k\}$. Hence $\{X_k\}$ and $\{\alpha_k\}$ have cluster points $\bar{X}$, $\bar{\alpha}$ respectively with $\bar{\alpha} \in \Sigma$, and now there follows readily $\bar{x} = \bar{X}\bar{\alpha}$. Writing $\bar{X} = (\bar{\xi}^1, \cdots, \bar{\xi}^{n+1})$, it is clear that $\bar{\xi}^j \in A$, $j = 1, \cdots, n$, so that $\bar{x} \in A^*$ and the proof is complete.

2. Lebesgue measurable functions on I to Ω^n.

DEFINITION 2.1 (Pliś [16].) A function $F: I \to \Omega^n$ is *measurable* if and only if the set $E(F, D) = \{t \in I \mid F(t) \cap D \neq \phi\}$ is Lebesgue measurable for each open set $D \subset E^n$.

Filippov [9] has stated without proof the following easily established result.

LEMMA 2.1. *Let $\mathscr{D}$ be the class of all open balls in E^n having positive rational radii and centers with rational coordinates; then a function $F: I \to \Omega^n$ is measurable if and only if the set $E(F, D)$ is measurable for every $D \in \mathscr{D}$.*

LEMMA 2.2. *If P is a closed subset of I and $F: P \to \Omega^n$ is continuous then there exists $\Phi: I \to \Omega^n$ having the following properties:*

(i) *Φ is continuous on I;*
(ii) *$\Phi(t) = F(t)$ on P;*
(iii) *for $t \in I$, $\|\Phi(t)\| \leqq \sup\{\|F(\tau)\| \mid \tau \in P\}$;*
(iv) *if the range of F is in Γ^n, so is that of Φ.*

Proof. Define Φ on P by setting $\Phi(t) = F(t)$ there; without loss of generality one many assume that P is properly contained in I and that I is the smallest interval containing P. If (t_0, t_1) is one of the at most countably many complementary intervals of P, define Φ on (t_0, t_1) by

$$\Phi(t) = \left(\frac{t - t_0}{t_1 - t_0}\right)F(t_1) + \left(\frac{t_1 - t}{t_1 - t_0}\right)F(t_0)\,.$$

For any points τ, τ_0 in $[t_0, t_1]$ there follows

$$\begin{aligned}\rho(\Phi(\tau), \Phi(\tau_0)) &\leqq (t_1 - t_0)^{-1}\rho(\tau(F(t_1) - F(t_0)), \tau_0(F(t_1) - F(t_0))) \\ &\leqq \frac{|\tau - \tau_0|}{t_1 - t_0}\|F(t_1) - F(t_0)\|\end{aligned}$$

the last inequality being a consequence of Corollary 1.1(v). The availability of this estimate makes possible the proof that Φ is continuous on I by means of an argument like that of Natanson [15, pp. 102–104].

LEMMA 2.3. (*Pliś* [16].) *If* $F: I \to \Omega^n$ *is continuous it is measurable.*

Filippov [9] has stated the next theorem, without proof, again for *bounded* functions.

THEOREM 2.1. *If* $F_k: I \to \Omega^n$, $k = 1, 2, 3, \cdots$, *are measurable and if* $\lim \rho(F_k(t), F(t)) = 0$ *almost everywhere (a.e.) on I, where* $F: I \to \Omega^n$, *then F is measurable.*

Proof. (After Natanson [15, Th. 2, p. 94].) Let a, r be fixed and such $S^0(a, r) \in \mathscr{D}$, the class defined in Lemma 2.1, where the superscript denotes interior. For positive integers m satisfying $mr > 1$ define

$$T_m^k = E(F_k, S^0(a, r - m^{-1})), \ k = 1, 2, 3, \cdots ,$$
$$Z_m^n = \bigcap_{k \geqq n} T_m^k, \ n = 1, 2, 3, \cdots .$$

We shall prove that

$$E(F, S^0(a, r)) = \bigcup_{n,m} Z_m^n . \tag{3}$$

Certainly T_m^k is measurable by hypothesis and Lemma 2.1; thus Z_m^n and the right member of (3) are measurable. Then by Lemma 2.1, (3) implies the measurability of F.

Let $t_0 \in E(F, S^0(a, r))$; then $F(t_0) \cap S^0(a, r) \neq \phi$ and there exists an integer $m_0, m_0 r > 2$, such that $F(t_0) \cap S^0(a, r - 2m_0^{-1}) \neq \phi$. Since $\bar{\rho}(F(t_0), F_k(t_0)) \to 0$, it follows that $\bar{\rho}(F(t_0) \cap S(a, r - 2m_0^{-1}), F_k(t_0)) \to 0$. Consequently there exists $n_0 = n_0(m_0)$ such that if $k \geqq n_0$ then $F_k(t_0) \cap S^0(a, r - m_0^{-1}) \neq \phi$. Hence $t_0 \in T_{m_0}^k$ for $k \geqq n_0$ which implies $t_0 \in Z_{m_0}^{n_0}$ and then of course $t_0 \in \bigcup_{n,m} Z_m^n$.

Now let $t_0 \in \bigcup_{n,m} Z_m^n$; then there exist n_0, m_0 such that $t_0 \in Z_{m_0}^{n_0}$. Hence $t_0 \in T_{m_0}^k$ for $k \geqq n_0$; i.e., $F_k(t_0) \cap S^0(a, r - m_0^{-1}) \neq \phi$ for $k \geqq n_0$. Now since $\bar{\rho}(F_k(t_0), F(t_0)) \to 0$ it follows that

$$\bar{\rho}(F_k(t_0) \cap S(a, r - m_0^{-1}), F(t_0)) \to 0 .$$

This in turn implies that $S(a, r - m_0^{-1}) \cap F(t_0) \neq \phi$ so that certainly $F(t_0) \cap S^0(a, r) \neq \phi$. Thus $t_0 \in E(F, S^0(a, r))$ and (3) follows.

The necessity of the condition of the next theorem (generalized Lusin theorem) was established, for *bounded*, measurable F, by Pliś

[16]. The entire theorem, again restricted to bounded functions, was stated without proof by Filippov [9]. For a measurable set $B \subset I$, let $\mu(B)$ denote its Lebesgue measure.

THEOREM 2.2. *A function* $F: I \to \Omega^n$ *is measurable if and only if for each* $\eta > 0$ *there exists* $E_\eta \subset I$ *which is closed,* $\mu(I - E_\eta) < \eta$ *and the restriction of* F *to* E_η *is continuous.*

Proof. (Necessity, using a device of Natanson [15, p. 10].) Let $T_k = E(F, S^\sim(0, k))$, where k is a positive integer and the tilde denotes complementation. Now $\bigcap T_k = \phi$ for otherwise, if $t_0 \in \bigcap T_k$,

$$F(t_0) \bigcap S^\sim(0, k) \neq \phi$$

for all k, contradicting the assumption that $F(t_0) \in \Omega^n$. Hence $\mu(\bigcap T_k) = 0$ and since $T_i \subset T_j$ for $i > j$ it follows that $\lim \mu(T_k) = 0$. Thus for $\eta > 0$ there exists k_0 such that $\mu(T_{k_0}) < \eta/4$; moreover, there exists open $T^* \supset T_{k_0}$ such that

$$\mu(T^*) < \mu(T_{k_0}) + \eta/4 < \eta/2 .$$

Defining $F^*: I \to \Omega^n$ by

$$F^*(t) = F(t),\ t \in I - T^* ,$$
$$F^*(t) = \{0\},\ t \in T^* ,$$

the measurability of F^* follows from that of F; in addition $\| F^*(t) \| \leqq k_0$ for all $t \in I$. Hence, by the aforementioned theorem of Pliś [16], there exists closed $E_\eta^* \subset I$ such that the restriction of F^* to E_η^* is continuous and $\mu(I - E_\eta^*) < \eta/2$. Consequently, the restriction of F to the set $E_\eta \equiv (I - T^*) \bigcap E_\eta^*$ is continuous and E_η is certainly closed. Moreover,

$$\mu(I - E_\eta) = \mu(T^* \bigcup (I - E_\eta^*)) \leqq \mu(T^*) + \mu(I - E_\eta^*) < \eta ,$$

and the argument is complete.

(Sufficiency.) For each $\eta > 0$, denote by $\Phi(\circ, \eta)$ the continuous extension of F, from E_η to I, guaranteed by Lemma 2.2. Let $\eta_m = 2^{-m}$, $m = 1, 2, 3, \cdots$; then setting

$$S_m = I - E_{\eta_m}$$

it follows that $\mu(S_m) < 2^{-m}$. Define

$$M_i = \bigcup_{k \geqq i} S_k;\ Q = \bigcap_{i \geqq 1} M_i .$$

Now $M_1 \supset M_2 \supset \cdots$ so that $\lim \mu(M_i) = \mu(Q)$; but since $\mu(M_i) < \sum_{k=i}^{\infty} 2^{-k}$ there follows $\mu(Q) = 0$. Let $t_0 \in I - Q$; then $t_0 \in \bigcup_{i \geqq 1} (I - M_i)$ so that

$t_0 \in I - M_{i_0}$ for some i_0. But then $t_0 \in I - S_k$ for all $k \geqq i_0$; i.e., $\rho(F(t_0), \Phi(t_0, \eta_k)) = 0$ for all $k \geqq i_0$ and this in turn implies

$$\lim \rho(F(t_0), \Phi(t_0, \eta_k)) = 0 .$$

By Lemma 2.3, $\Phi(\circ, \eta_k)$ is measurable for each k so that by Theorem 2.1 and the result just obtained, F is measurable.

COROLLARY 2.1. *If* $F: I \rightarrow \Omega^n$ *is continuous (measurable) then the function* $F^*: I \rightarrow \Gamma^n$ *defined by* $F^*(t) = (F(t))^*$ *is continuous (measurable).*

Proof. The assertion concerning continuity is immediate from Lemma 1.3 (ii). Now suppose F is measurable; by Theorem 2.2, for $\eta > 0$ there exists closed $E_\eta \subset I$ such that $\mu(I - E_\eta) < \eta$ and the restriction of F to E_η is continuous. But by Lemma 1.3 (ii), the restriction of F^* to E_η is continuous. Another application of Theorem 2.2 yields the measurability of F*.

The next two lemmas were originally stated for bounded functions; an examination of their proofs (*vide* [12]) reveals, in the light of Theorem 2.2, that this boundedness restriction is superfluous.

LEMMA 2.4. (*Hermes-Filippov.*) *Let* $g: E^n \rightarrow E^k$ *be continuous and let* $H: I \rightarrow \Omega^n$ *be measurable. If* $r: I \rightarrow E^n$ *is measurable and* $r(t) \in g(H(t))$ *on* I *then there exists measurable* $\nu: I \rightarrow E^n$ *satisfying* $\nu(t) \in H(t)$ *and* $r(t) = g(\nu(t))$ *on* I.

LEMMA 2.5. (*Hermes.*) *Let* $R: I \rightarrow \Omega^n$ *be measurable and let* $w: I \rightarrow E^n$ *be measurable; then there exists measurable* $r: I \rightarrow E^n$ *satisfying* $r(t) \in R(t)$ *and* $\| w(t) - r(t) \| = \alpha(w(t), R(t))$ *on* I.

The next lemma was originally stated by Hermes [11, Lemma 1.1] for bounded functions; again by virtue of Theorem 2.2, the boundedness restriction is superfluous. A function $F: I \rightarrow \Omega^n$ is *approximately continuous* at $t \in I$ if and only if there exists a measurable set $B \subset I$ for which t is a point of density and such that the restriction of F to B is continuous at t.

LEMMA 2.6. *If* $F: I \rightarrow \Omega^n$ *is measurable then* F *is approximately continuous a.e. on* I.

DEFINITION 2.2. (i) Let $F: I \rightarrow \Omega^n$; if there exists a Lebesgue summable function $h: I \rightarrow E^1$ such that $\| F(t) \| \leqq h(t)$ on I then F is *integrably bounded.*

(ii) Let A be an index set and let $F_\gamma: I \to \Omega^n$ for all $\gamma \in A$; if there exists a Lebesgue summable function $h: I \to E^1$ such that $\| F_\gamma(t) \| \leqq h(t)$ for all $t \in I$ and all $\gamma \in A$ then $\{F_\gamma \mid \gamma \in A\}$ is *uniformly integrably bounded.*

The next lemma has an easy proof which will be omitted.

LEMMA 2.7. (i) *If $F: I \to \Omega^n$ is continuous it is integrably bounded.*

(ii) *If $F: I \to \Omega^n$ is integrably bounded then the function F^* defined in Corollary 2.1 has the same integrable bound as F.*

DEFINITION 2.3. Let $F: I \to \Omega^n$ be such that for each $t \in I$ the function $\rho(F(\circ), F(t))$ is summable on I. A point $t \in I$ for which

$$\lim_{\eta \to 0} \eta^{-1} \int_t^{t+\eta} \rho(F(\tau), F(t)) d\tau = 0$$

is called a *Lebesgue point of F.*

THEOREM 2.3. *If $F: I \to \Omega^n$ is measurable and integrably bounded then almost all $t \in I$ are Lebesgue points of F.*

Proof. Theorem 2.2 and the continuity of $\rho(\circ, \circ)$, together with Lusin's theorem for real valued functions, implies that $\rho(F(\circ), F(t))$ is measurable for each $t \in I$. Let h be an integrable bound for F; without loss of generality one may suppose that $h(t) > 0$ on I. By Corollary 1.1 (iv), $\rho(F(\tau), F(t)) \leqq h(\tau) + h(t)$ for all $\tau, t \in I$. Hence $\rho(F(\circ), F(t))$ is summable on I for each $t \in I$. Now by Lemma 2.6 and [15, Th. 5, p. 255] almost all points of I are, at once, points of approximate continuity of F and Lebesgue points of h. Let t be such a point and let $B \subset I$ be a measurable set for which t is a point of density and such that the restriction of F to B is continuous at t. For $\eta > 0$, set

$$B_1(\eta) = [t, t + \eta] \cap B ,$$
$$B_2(\eta) = [t, t + \eta] \cap (I - B) .$$

Then, given $\varepsilon > 0$, one may choose $\eta = \eta(\varepsilon, t) > 0$ sufficiently small that the following three conditions are satisfied:

(i) for $\tau \in B_1(\eta)$, $\rho(F(\tau), F(t)) < \varepsilon/6$;

(ii) $\mu(B_2(\eta)) < \varepsilon\eta/6h(t)$;

(iii) $\int_t^{t+\eta} | h(\tau) - h(t) | d\tau < \eta\varepsilon/3$.

By virtue of (i), (ii), (iii) and Corollary 1.1 (iv) there follows

$$\begin{aligned}\eta^{-1}\int_t^{t+\eta}\rho(F(\tau), F(t))d\tau &= \eta^{-1}\int_{B_1(\eta)}\rho(F(\tau), F(t))d\tau + \eta^{-1}\int_{B_2(\eta)}\rho(F(\tau), F(t))d\tau \\ &< \varepsilon/3 + \eta^{-1}\int_{B_2(\eta)}[\|F(\tau)\| + \|F(t)\|]d\tau \\ &< \varepsilon/3 + \eta^{-1}\int_t^{t+\eta}|h(\tau) - h(t)|\,d\tau + 2h(t)\eta^{-1}\mu(B_2(\eta)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\,.\end{aligned}$$

Thus $\lim_{\eta\to 0+}\eta^{-1}\int_t^{t+\eta}\rho(F(\tau), F(t))d\tau = 0$, and a similar argument shows that the left hand limit is also zero.

We close this section with the following important lemma on the measurability of composite functions.

LEMMA 2.8. *Let D be a nonvoid, open subset of $E^1 \times E^n$ and let $R\colon E^1 \times E^n \to \Omega^n$ satisfy:*

(i) for each t in the projection of D on E^1, $R(t, \circ)$ is continuous on the set $D_t = \{x \in E^n \mid (t, x) \in D\}$;

(ii) for each x in the projection of D on E^n and each compact interval $I \subset E^1$ for which $I \times \{x\} \subset D$, $R(\circ, x)$ is measurable on I;

(iii) for each compact $C \subset D$ there exists a Lebesgue summable function $h_c\colon E^1 \to E^1$ such that $\|R(t, x)\| \leqq h_c(t)$ on C.

If I is a compact interval in E^1 and S is a compact ball in E^n satisfying $I \times S \subset D$ then for each continuous function $x\colon I \to S$ the function $R(\circ, x(\circ))$ is integrably bounded and measurable on I.

Proof. If the assertion of the lemma is true with "continuous" replaced by "step" as the restriction on $x\colon I \to S$ then the validity of the original statement, insofar as measurability is concerned, follows by virtue of (i) and Theorem 2.1 since a continuous function $x\colon I \to S$ may be uniformly approximated by step functions. Hence suppose that for $c_k \in S$, $k = 1, \cdots, m$, $x^*\colon I \to S$ is defined by

$$x*(t) = c_k,\ t \in I_k,\ k = 1, \cdots, m\,,$$

where $I = \bigcup I_k$, $I_j \cap I_k = \phi$ for $j \neq k$ and each I_k is an interval. Then for an open set $K \subset E^n$, $E(R(\circ, x^*(\circ)), K) = \bigcup M_j$,

$$M_j = \{t \in I_j \mid R(t, c_j) \bigcap K \neq \phi\},\ j = 1, \cdots, m\,.$$

But by (ii), each M_j is measurable so that $E(R(\circ, x^*(\circ)), K)$ is measurable. Integrable boundedness of $R(\circ, x(\circ))$ is an easy consequence of (iii).

3. Trajectory integrals of measurable functions. In this

section we set $I = [0, 1]$ without loss of generality and suppose that $F: I \to \Omega^n$ is a given function. As in the introduction we denote by $\mathscr{F}_I(F)$ the set of all Lebesgue summable functions $u: I \to E^n$ having the property that $u(t) \in F(t)$ a.e. on I. Let $\mathscr{T}$ on $\mathscr{L}_1^n(I)$ be defined by

$$(\mathscr{T} q)(t) = \int_0^t q(\tau)d\tau, \ t \in I ,$$

and define

$$\mathscr{S}_I(F) = \mathscr{T} \mathscr{F}_I(F) .$$

$\mathscr{S}_I(F)$ may be considered as a subset of any of a number of Banach spaces but the ones we shall be primarily concerned with here are $\mathscr{C}^n(I)$ and $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$.

LEMMA 3.1. (i) *If* $F: I \to \Omega^n$ *is measurable and integrably bounded then* $\mathscr{F}_I(F) \neq \phi$.

(ii) *If* $F: I \to \Gamma^n$ *then* $\mathscr{F}_I(F)$ *is a convex subset of* $\mathscr{L}_1^n(I)$.

Proof. That there exists a measurable $\nu: I \to E^n$ satisfying $\nu(t) \in F(t)$ a.e. on I follows from Lemma 2.4 by taking $g = 0$, $r = 0$, and $H = F$. The assertion of (i) then follows by the integrable boundedness of F. The proof of (ii) is trivial.

THEOREM 3.1. *If* $F: I \to \Gamma^n$ *is measurable and integrably bounded then* $\mathscr{S}_I(F) \in \mathscr{K}^n(I)$; *moreover*, $\mathscr{S}_I(F)$ *is a weakly compact subset of* $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$.

Proof. From Lemma 3.1 and the linearity of $\mathscr{T}$ follow the facts that $\mathscr{S}_I(F)$ is nonvoid and convex; that $\mathscr{S}_I(F)$ is conditionally compact follows readily from the integrable boundedness of F together with the Arzelà-Ascoli theorem. The first assertion of the theorem will be established if we show that $\mathscr{S}_I(F)$ is closed in $\mathscr{C}^n(I)$. To this end let $w \in \overline{\mathscr{S}_I(F)}$ and let $\{w_m\} \subset \mathscr{S}_I(F)$ satisfy $\lim \langle w_m - w \rangle = 0$. Now $\dot{w}_m(t) \in F(t)$ a.e. on I so that with h denoting the integrable bound on F we obtain

$$\begin{aligned} \| w(t_2) - w(t_1) \| &\leqq \| w(t_2) - w_m(t_2) \| + \| w(t_1) - w_m(t_1) \| \\ &\quad + \| w_m(t_2) - w_m(t_1) \| < \varepsilon + \left| \int_{t_1}^{t_2} h(\tau)d\tau \right| \end{aligned}$$

for $\varepsilon > 0$ and m sufficiently large. Thus w is absolutely continuous on I and it is easy to see that there exists measurable $U \subset I$, $\mu(I - U) = 0$, having the following properties:

(i) $\dot{w}(t)$ exists on U;

(ii) each $t \in U$ is a Lebesgue point of F.

The validity of (ii) is of course a consequence of Theorem 2.3. With ν being the function defined in Definition 1.3, by virtue of Theorem 2.2, the Lusin theorem for real valued functions and the continuity of $\nu(\circ, \circ)$ on $\Gamma^n \times E^n$ [3, Lemma 1] there follows the fact that $\nu(F(\circ), p)$ is measurable for each $p \in E^n$. By virtue of Lemma 1.2 (iii) and Corollary 1.1 (iv) there obtains $|\nu(F(t), p)| \leqq h(t)$ for all $(t, p) \in I \times E^n$ and thus $\nu(F(\circ), p)$ is summable for $p \in E^n$. Moreover, there exists measurable $V \subset I$, $\mu(I - V) = 0$, such that for all $(t, p) \in V \times E^n$ and all m,

$$\dot{w}_m(t) \circ p \leqq \nu(F(t), p) .$$

Thus for all m, all $p \in E^n$ and all $t_1, t_2 \in I$,

$$[w_m(t_2) - w_m(t_1)] \circ p \leqq \int_{t_1}^{t_2} \nu(F(\tau), p) d\tau ;$$

in particular for $t \in U$, $\eta > 0$, all m and all p such that $\|p\| = 1$,

$$\begin{aligned} \eta^{-1}[w_m(t + \eta) - w_m(t)] \circ p &\leqq \eta^{-1} \int_t^{t+\eta} \nu(F(\tau), p) d\tau \\ &\leqq \nu(F(t), p) + \eta^{-1} \int_t^{t+\eta} \rho(F(\tau), F(t)) d\tau , \end{aligned}$$

the final inequality being a consequence of Lemma 1.2 (iii). For all $\eta > 0$ such that $t + \eta \in I$, the convergence of w_m to w implies that

$$\eta^{-1}[w(t + \eta) - w(t)] = \lim_{m \to \infty} \eta^{-1}[w_m(t + \eta) - w_m(t)] .$$

This and the last formula line imply that for $\|p\| = 1$, $t \in U$, $\eta > 0$ and $t + \eta \in I$,

$$\eta^{-1}[w(t + \eta) - w(t)] \circ p \leqq \nu(F(t), p) + \eta^{-1} \int_t^{t+\eta} \rho(F(\tau), F(t)) d\tau .$$

Letting $\eta \to 0+$ in this inequality yields, for $\|p\| = 1$,

$$\dot{w}(t) \circ p \leqq \nu(F(t), p)$$

and in turn this implies [19, Th. 5.3] that $\dot{w}(t) \in F(t)$. Thus is $\mathscr{S}_I(F)$ closed.

For the proof of the second assertion of the theorem, let x be a weak limit point (i.e., a limit point relative to the weak topology in $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$) of $\mathscr{S}_I(F)$. By [6, IV. 13.31] there exists a sequence $\{x_m\} \subset \mathscr{S}_I(F)$ which converges pointwise to x on I. But by the first assertion of the theorem, there is a subsequence $\{x_{m_k}\}$ which converges in $\mathscr{C}^n(I)$ to x so that necessarily $x \in \mathscr{S}_I(F)$. Thus is $\mathscr{S}_I(F)$ weakly closed. Now $\left\| \int_E q(\tau) d\tau \right\| \leqq \int_E h(\tau) d\tau$ for all $q \in \mathscr{F}_I(F)$ and all measurable $E \subset I$; hence by [6, IV. 8.11] and the absolute continuity of the set function $\int_E h(\tau)d\tau$, $\mathscr{F}_I(F)$ is weakly sequentially compact in $\mathscr{L}_1^n(I)$. Since $\mathscr{T}$ is linear and continuous with respect to the metric topologies in $\mathscr{L}_1^n(I)$ and $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$, by [6, V. 3.15] $\mathscr{S}_I(F)$ is weakly sequentially compact in $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$. Now the weak compactness of $\mathscr{S}_I(F)$ is a consequence of [6, V. 6.1].

THEOREM 3.2. *Let $F, F_k: I \to \Gamma^n$, $k = 1, 2, 3, \cdots$, satisfy*

$$\lim \rho(F_k(t), F(t)) = 0$$

on I; if $\{F_k\}$ is uniformly integrably bounded and each F_k is measurable then $\mathscr{S}_I(F_k)$ and $\mathscr{S}_I(F)$ are in $\mathscr{K}^n(I)$ and $\lim \sigma(\mathscr{S}_I(F_k), \mathscr{S}_I(F)) = 0$.

Proof. That $\mathscr{S}_I(F_k) \in \mathscr{K}^n(I)$ is a consequence of Theorem 3.1. That F is measurable is implied by Theorem 2.1. Let h be a uniform integrable bound for $\{F_k\}$ and let $t \in I$ be fixed; by hypothesis and Corollary 1.1 (iv) we find that, given $\varepsilon > 0$, there exists $K = K(\varepsilon, t)$ such that for $k > K$, $\|F(t)\| < \varepsilon + \|F_k(t)\| \leqq \varepsilon + h(t)$. Thus F is integrably bounded by h and from Theorem 3.1 there follows $\mathscr{S}_I(F) \in \mathscr{K}^n(I)$. Now there exists $w_k \in \mathscr{S}_I(F_k)$ such that $\beta(w_k, \mathscr{S}_I(F)) = \bar{\sigma}(\mathscr{S}_I(F_k), \mathscr{S}_I(F))$. Let $q_k \in \mathscr{F}_I(F_k)$ be such that $w_k = \mathscr{T}q_k$ and, by Lemma 2.5, let $u_k \in \mathscr{F}_I(F)$ satisfy $\|u_k(t) - q_k(t)\| = \alpha(q_k(t), F(t)) \leqq \bar{\rho}(F_k(t), F(t))$ on I. Then $\bar{\sigma}(\mathscr{S}_I(F_k), \mathscr{S}_I(F)) \leqq \langle w_k - \mathscr{T}u_k \rangle$; but

$$\langle w_k - \mathscr{T}u_k \rangle \leqq \int_0^1 \| q_k(\tau) - u_k(\tau) \| \, d\tau = \int_0^1 \alpha(q_k(\tau), F(\tau)) d\tau$$

and since $\alpha(q_k(t), F(t)) \to 0$ on I and $\alpha(q_k(t), F(t)) \leqq 2h(t)$ on I it follows from [6, III. 6.16] that $\lim \langle w_k - \mathscr{T}u_k \rangle = 0$. Hence

$$\lim \bar{\sigma}(\mathscr{S}_I(F_k), \mathscr{S}_I(F)) = 0 \, .$$

There also exists $y_k \in \mathscr{S}_I(F)$ such that $\beta(y_k, \mathscr{S}_I(F_k)) = \bar{\sigma}(\mathscr{S}_I(F), \mathscr{S}_I(F_k))$. Let $u_k \in \mathscr{F}(F)$ satisfy $y_k = \mathscr{T}u_k$ and, by Lemma 2.5, let $q_k \in \mathscr{F}_I(F_k)$ satisfy $\| \mathrm{u}_k(t) - q_k(t) \| = \alpha(u_k(t), F_k(t)) \leqq \bar{\rho}(F(t), F_k(t))$ on I. Then $\bar{\sigma}(\mathscr{S}_I(F), \mathscr{S}_I(F_k)) \leqq \langle y_k - \mathscr{T}q_k \rangle$; but

$$\langle y_k - \mathscr{T}q_k \rangle \leqq \int_0^1 \| u_k(\tau) - q_k(\tau) \| \, d\tau = \int_0^1 \alpha(u_k(\tau), F_k(\tau)) d\tau \, .$$

Arguing as in the preceding part of the proof we conclude

$$\lim \bar{\sigma}(\mathscr{S}_I(F), \mathscr{S}_I(F_k)) = 0$$

and the proof is complete.

DEFINITION 3.1. Let $\mathscr{S}$ be a set of functions on I to E^n; then

$$G(t; \mathscr{S}) = \{\varphi(t) \mid \varphi \in S\}, \ t \in I \, .$$

LEMMA 3.2. *If either of the following conditions is satisfied then for all* $t \in I$, $G(t; \mathscr{S}) \in \Gamma^n$:

(i) $\mathscr{S} \in \mathscr{K}^n(I)$;

(ii) $\mathscr{S}$ *is a nonvoid, convex, weakly compact subset of* $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$.

Proof. (i) is an immediate consequence of [4, Th. 1.4]. For (ii) we observe first of all that by [6, IV. 12.3] there is a unique nonvoid, convex, weakly compact subset $\mathscr{F} \subset \mathscr{L}_1^n(I)$ such that $\mathscr{S} = \mathscr{T}\mathscr{F}$. By virtue of [6, V. 6.1], $\mathscr{F}$ is weakly sequentially compact; from [6, IV. 8.8] it then follows that F is bounded. The function $\mathscr{T}_t: \mathscr{L}_1^n(I) \to E^n$ defined for each fixed $t \in I$ by

$$\mathscr{T}_t q = \int_0^t q(\tau)d\tau$$

is linear and continuous with respect to the metric topologies in $\mathscr{L}_1^n(I)$, E^n; hence by [6, V. 3.15] it is continuous with respect to the weak topologies in these spaces. Consequently $\mathscr{T}_t\mathscr{F}$ is bounded, convex and weakly compact, hence, by [6, V. 3.13], closed. We conclude that $G(t; \mathscr{S}) \equiv \mathscr{T}_t\mathscr{F} \in \Gamma^n$.

The next lemma generalizes a result due to Hermes [12, Th. 1.2].

LEMMA 3.3. *If* $F: I \to \Omega^n$ *is measurable and integrably bounded then* $G(t; \mathscr{S}_I(F)) = G(t; \mathscr{S}_I(F^*)) \in \Gamma^n$ *for all* $t \in I$.

Proof. By Corollary 2.1, Lemma 2.7 (ii), Theorem 3.1 and Lemma 3.2, $G(t; \mathscr{S}_I(F^*)) \in \Gamma^n$. Certainly $G(t; \mathscr{S}_I(F)) \subset G(t; \mathscr{S}_I(F^*))$ and the remainder of the proof coincides with the second part of Hermes' proof for [12, Th. 1.2].

Hermes [11] has observed that: *if* $F: I \to \Omega^n$ *is Borel measurable* [1] *then it is measurable.* Our next result is the combined assertion of Theorems 1 through 4 of [1] for Borel measurable, integrably bounded $F: I \to \Omega^n$. It is an immediate consequence of Lemma 3.3 and Hermes' observation.

COROLLARY 3.1. *If* $F: I \to \Omega^n$ *is Borel measurable and integrably bounded then for each* $t \in I$, $G(t; \mathscr{S}_I(F)) \in \Gamma^n$.

Lemma 3.3 provides the instrument for establishing the following corollaries to Theorem 3.2.

COROLLARY 3.2. *Let* $F, F_k: I \to \Omega^n$, $k = 1, 2, 3, \cdots$, *satisfy*

$$\lim \rho(F_k(t), F(t)) = 0$$

on I; if $\{F_k\}$ is uniformly integrably bounded and each F_k is measurable then for each $t \in I$, $G(t; \mathscr{S}_I(F_k))$ and $G(t; \mathscr{S}_I(F))$ are in Γ^n and

$$\lim \rho(G(t; \mathscr{S}_I(F_k)), G(t; \mathscr{S}_I(F))) = 0 ,$$

uniformly on I.

Proof. By Corollary 2.1 and Lemma 2.7, each F_k^* is measurable and $\{F_k^*\}$ has the same uniform integrable bound as $\{F_k\}$. By Theorem 2.1, F is measurable and, by an argument like that used in Theorem 3.2, F is integrably bounded. Thus by Corollary 2.1 and Lemma 2.7, F^* is measurable and integrably bounded and, by hypothesis and Lemma 1.3 (ii), $\lim \rho(F_k^*(t), F^*(t)) = 0$. From Theorem 3.2 there follows $\lim \sigma(\mathscr{S}_I(F_k^*), \mathscr{S}_I(F^*)) = 0$ and this result together with [4, Th. 1.5] implies

$$\lim \rho(G(t; \mathscr{S}_I(F_k^*)), G(t; \mathscr{S}_I(F^*))) = 0 ,$$

uniformly for $t \in I$. The proof is completed by application of Lemma 3.3.

COROLLARY 3.3. *Let $F_k: I \to \Omega^n$, $k = 1, 2, 3, \cdots$, satisfy the following conditions:*

(i) *$\{F_k\}$ is uniformly integrably bounded;*

(ii) *for each k, F_k is Borel measurable;*

(iii) *$F(t) = \lim F_k(t)$ exists and is nonvoid for each $t \in I$. Then $F: I \to \Omega^n$ and, for each $t \in I$,*

$$\lim G(t; \mathscr{S}_I(F_k)) = G(t; \mathscr{S}_I(F)) \in \Gamma^n .$$

Proof. By virtue of (i), (iii) and Lemma 1.6, $F: I \to \Omega^n$ and $\lim F_k^*(t) = F^*(t)$. Lemma 2.7 implies that $\{F_k^*\}$ has the same uniform integrable bound as $\{F_k\}$ so that Corollary 1.2 yields $\lim \rho(F_k^*(t), F^*(t)) = 0$. The observation of Hermes quoted above, together with (ii) and Corollary 2.1, yields the measurability of F_k^*. Now Corollary 3.2 and Lemma 1.4 permit the assertion

$$\lim G(t; \mathscr{S}_I(F_k^*)) = G(t; \mathscr{S}_I(F^*)) \in \Gamma^n ;$$

hence Lemma 3.3 yields

$$\lim G(t; \mathscr{S}_I(F_k)) = G(t; \mathscr{S}_I(F^*)) \in \Gamma^n . \qquad (\tilde{\ })$$

But the assertion of [1, Th. 5] is that the left member of this equation is equal to $G(t; \mathscr{S}_I(F))$; the proof is complete.

Discussion. It is easy to see that in Corollary 3.3, the requirement that F_k be nonvoid, compact valued for each k can be replaced by the requirement that it be nonvoid, closed valued for each k. The corresponding replacement can be made in Corollary 3.1. It is noteworthy that Corollary 3.1 bears out the anticipation, expressed in the introduction that a study of $\mathscr{S}_I(F)$ subsumes, in an obvious sense, a study of Aumann's integral. Corollary 3.3 shows that our expectations in this direction cannot be too high; indeed, under hypotheses of this corollary, ($\tilde{\ }$) appears to be the strongest result we can obtain within the confines of the theory developed in this paper. The utilization of [1, Th. 5] in this corollary could be supplanted by a counterpart of Theorem 2.1 in which Hausdorff convergence is replaced by Kuratowski convergence. However, we have not been successful in obtaining such a counterpart of Theorem 2.1; moreover, in view of the proof of Theorem 2.1 it does not appear likely that such a counterpart is valid. It is also noteworthy that the lack of such a counterpart for Theorem 2.1 prevents the inference from [1, Th. 5] alone that $G(t; \mathscr{S}_I(F)) \neq \phi$ for some $t \in I$ even under the hypotheses of Corollary 3.3.

The weak compactness of $\mathscr{S}_I(F)$ in $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$ may be shown to follow directly from the hypotheses of Theorem 3.1; the device of using the compactness of $\mathscr{S}_I(F)$ in $\mathscr{C}^n(I)$ to establish weak compactness of $\mathscr{S}_I(F)$ was a matter of convenience in the proof of that theorem. Taking this observation into account, it is not difficult to see that Corollary 3.2 may be established independently by means of an argument which depends only on weak compactness of $\mathscr{S}_I(F)$, Lemma 3.2 (ii), Lemma 3.3 and Lemma 2.5. Thus Corollaries 3.1, 3.2 constitute a theory which is a direct counterpart of Aumann's theory, the major distinction between the two theories being that between Hausdorff and Kuratowski convergence. The discussion of the preceding paragraph indicates that whereas these theories are supplementary, neither implies the other.

The proof of [12, Corollary 1.1] applies with trivial modification, taking into account Lemma 3.3, to yield

LEMMA 3.4. *Let $F: I \rightarrow \Omega^n$ be measurable and integrably bounded, and let $y \in \mathscr{S}_I(F^*)$; then for each $\eta > 0$ there exists $z_\eta \in \mathscr{S}_I(F)$ satisfying $\langle y - z_\eta \rangle < \eta$.*

This lemma has the following immediate consequence.

COROLLARY 3.4. *If $F: I \rightarrow \Omega^n$ is measurable and integrably bounded then $\mathscr{S}_I(F^*)$ is the closure of $\mathscr{S}_I(F)$ in $\mathscr{C}^n(I)$.*

REMARK 3.1. [12, Example 2.3.] shows that with the hypotheses of Corollary 3.4 $\mathscr{S}_I(F)$ need not be closed in $\mathscr{C}^n(I)$; there thus appears to be no possibility of generalizing Theorem 3.2 by requiring that F, F_k have values in Ω^n.

Let us denote by $\mathscr{S}_I^*(F)$ the closed (in $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$) convex hull of $\mathscr{S}_I(F)$ and by $\mathscr{S}_I^\sharp(F)$, the weak closure of $\mathscr{S}_I(F)$ in $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$.

THEOREM 3.3. *If $F: I \to \Omega^n$ is measurable and integrably bounded then*

$$\mathscr{S}_I^\sharp(F) = \mathscr{S}_I^*(F) = \mathscr{S}_I(F^*) .$$

Proof. By means of an argument like that for the second assertion of Theorem 3.1 it may be inferred that $\mathscr{S}_I(F)$ is weakly sequentially compact. Now there follows from [6, V. 3.13, 3.14] and Theorem 3.1,

$$\mathscr{S}_I^\sharp(F) \subset \mathscr{S}_I^*(F) \subset \mathscr{S}_I(F^*) .$$

But from these inclusions, Lemma 3.4 and [6, IV. 13.31], the theorem follows.

REMARK 3.2. It is easy to see that $\mathscr{S}_I^*(F) = \mathscr{T}\mathscr{F}_I^*(F)$, where $\mathscr{F}_I{}^*(F)$ is the closed convex hull of $\mathscr{F}_I(F)$.

Arguing again as in the proof of the second assertion of Theorem 3.1, it follows that if $F: I \to \Omega^n$ is measurable and integrably bounded and if $\mathscr{S}_I(F)$ is closed in $\mathscr{C}^n(I)$ then $\mathscr{S}_I(F)$ is weakly closed in $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$.

In view of this result, Theorem 3.3 yields

COROLLARY 3.5. *If $F: I \to \Omega^n$ is measurable and integrably bounded then $\mathscr{S}_I(F) \in \mathscr{K}^n(I)$ only if $\mathscr{S}_I(F) = \mathscr{S}_I(F^*)$.*

The final result of this section provides a marked strengthening of Theorem 3.1 and of the assertion of Remark 3.1.

THEOREM 3.4. *Let $F: I \to \Omega^n$ be measurable and integrably bounded; then the following statements are equivalent*:

(i) *$\mathscr{S}_I(F) \in \mathscr{K}^n(I)$.*

(ii) *$\mathscr{S}_I(F)$ is a nonvoid, weakly compact subset of $\mathscr{N}\mathscr{A}\mathscr{C}^n(I)$.*

(iii) *$F(t)$ is convex a.e. on I.*

Proof. That (iii) implies both (i) and (ii) is an easy consequence of Theorem 3.1. For the remainder of the proof, consider the function $\rho(F^*(\circ), F(\circ))$. By virtue of Corollary 2.1, an argument similar to that of the first part of the proof of Theorem 2.3 permits the assertion that this function is measurable on I. Hence the set

$$M \equiv \{t \in I \mid \bar{\rho}(F^*(t), F(t)) > 0\} = \{t \in I \mid \rho(F^*(t), F(t)) > 0\}$$

is measurable. We need prove only that if $\mu(M) > 0$ then $\mathscr{S}_I(F)$ is a proper subset of $\mathscr{S}_I(F^*)$. Indeed, in this event it follows from Corollary 3.5 that $\mathscr{S}_I(F) \notin \mathscr{K}^n(I)$ and, from Theorem 3.3, that $\mathscr{S}_I(F)$ is not weakly compact. Now we observe that minor modification of Hermes' proof [12] of Lemma 2.4 produces the following result: *there exists a measurable function* $w: I \to E^n$ *satisfying* $w(t) \in F^*(t)$ *and* $\alpha(w(t), F(t)) = \rho(F^*(t), F(t))$ *for all* $t \in I$. A function w so determined thus satisfies $\alpha(w(t), F(t)) > 0$ on M. Hence, if $\mu(M) > 0$ it follows that $\mathscr{F}_I(F)$ is a proper subset of $\mathscr{F}_I(F^*)$ and this in turn implies that $\mathscr{S}_I(F)$ is a proper subset of $\mathscr{S}_I(F^*)$ and the proof is complete.

4. An existence theorem.

THEOREM 4.1. *Let D be a nonvoid open subset of $E^1 \times E^n$ and let $R: E^1 \times E^n \to \Gamma^n$ satisfy conditions* (i), (ii), (iii) *of Lemma* 2.8 *on D; then for each $(t_0, x_0) \in D$ there exists a solution*[2] *of*

$$\dot{x} \in R(t, x),\ x(t_0) = x_0\ , \tag{2}$$

and every solution of (2) *may be continued to the boundary of D.*

Proof. There is no loss of generality in assuming that $(0, 0) \in D$ and proving the theorem in the case $(t_0, x_0) = (0, 0)$. The proof is based on that of Hartman [10, Th. 2.1, p. 10]. Let $a, b > 0$ be sufficiently small that $C \subset D$, where

$$C = \{(t, x) \in E^1 \times E^n \mid 0 \leqq t \leqq a; \|x\| \leqq b\}\ .$$

Define $\alpha = \max\left\{t \in [0, a] \,\middle|\, \int_0^t h_c(\tau)d\tau \leqq b\right\}$; evidently $\alpha \in (0, a]$. Let $\eta \in (0, \alpha]$ be fixed; then on $[0, \eta]$ the function whose value is $R(t, 0)$ is measurable and integrably bounded. By Theorem 3.1 there exists $w_1 \in \mathscr{S}_{[0,\eta]}(R(\circ, 0))$ and we define a function χ_η on $[0, \eta]$ by

$$\chi_\eta(t) = w_1(t),\ t \in [0, \eta]\ .$$

There follows easily

$$\|\chi_\eta(t)\| \leqq \int_0^\eta h_c(\tau)d\tau < b,\ t \in [0, \eta]\ ; \tag{4a}$$

[2] I.e., an absolutely continuous function satisfying $\dot{x}(t) \in R(t, x(t))$ a.e. on an interval containing t_0 in its relative interior and satisfying $x(t_0) = x_0$.

(4b) $$\| \chi_\eta(t_2) - \chi_\eta(t_1) \| \leqq \left| \int_{t_1}^{t_2} h_c(\tau) d\tau \right| , \; t_1, t_2 \in [0, \eta] .$$

If $\eta < \alpha$, let $\eta^1 = \min\{\alpha, 2\eta\}$; then by Lemma 2.8 the function whose value is $R(t, \chi_\eta(t - \eta))$ is measurable and integrably bounded on $[\eta, \eta^1]$. Hence by Theorem 3.1 there exists $w_2 \in \mathscr{S}_{[\eta,\eta^1]}(R(\circ, \chi_\eta(\circ - \eta)))$. We extend χ_η to $[\eta, \eta^1]$ by defining

$$\chi_\eta(t) = \chi_\eta(\eta) + w_2(t), \; t \in [\eta, \eta^1] ;$$

it is easy to see that χ_η satisfies (4) on $[\eta, \eta^1]$, hence on $[0, \eta^1]$. If $\eta^1 < \alpha$ the foregoing process may be iterated at most a finite number of steps to extend the definition of χ_η to $[0, \alpha]$ in such a way that the following property obtains:

(*) $\chi_\eta \in \mathscr{S}_{[0,\alpha]}(R^\eta(\circ))$, where $R^\eta: [0, \alpha] \to \Gamma^n$ is defined by

$$R^\eta(t) = R(t, 0), \; t \in [0, \eta] ,$$
$$R^\eta(t) = R(t, \chi_\eta(t - \eta)), \; t \in (\eta, \alpha]$$

with the family $\{R^\eta \mid \eta \in (0, \alpha]\}$ being uniformly integrably bounded and each member of the family measurable on $[0, \alpha]$.

Now let $\{\eta_m\}$ be a monotone null sequence of points in $[0, \alpha]$; then by property (*) and the Arzela-Ascoli theorem $\{\chi_{\eta_m}\}$ contains a subsequence (assume it is the original) which converges uniformly on $[0, \alpha]$ to a limit function, χ, which is easily shown to be absolutely continuous (cf. the proof of Theorem 3.1). Equicontinuity of $\{\chi_{\eta m}\}$ implies

$$\lim \chi_{\eta_m}(t - \eta_m) = \chi(t), \; t \in [0, \alpha] ,$$

so that by condition (i)

(5) $$\lim \rho(R^{\eta_m}(t), R(t, \chi(t))) = 0, \; t \in [0, \alpha] .$$

Thus from (*), (5) and Theorem 3.2 there follows

(6) $$\lim \sigma(\mathscr{S}_{[0,\alpha]}(R^{\eta_m}), \mathscr{S}_{[0,\alpha]}(R(\circ, \chi(\circ)))) = 0 .$$

Since $\chi_{\eta_m} \to \chi$ and $\mathscr{S}_{[0,\alpha]}(R(\circ, \chi(\circ)))$ is compact, (*) and (6) imply that

(7) $$\chi \in \mathscr{S}_{[0,\alpha]}(R(\circ, \chi(\circ))) .$$

But (7) is equivalent to the assertion that $\chi(0) = 0$ and, a.e. on $[0, \alpha]$,

$$\dot{\chi}(t) \in R(t, \chi(t))$$

and the proof of existence is complete. The continuability assertion follows in a straightforward way from [2, Th. 4].

COROLLARY 4.1. *If in the statement of Theorem 4.1 conditions* (i), (ii), (iii) *of Lemma* 2.8 *are replaced by* (iv) R *is continuous on* D, *then the conclusion of that theorem remains valid.*

Proof. That (iv) implies (i) is obvious; that (iv) implies (ii) is a consequence of Lemma 2.3. Finally, (iii) follows from (iv) by setting

$$h_c(t) = \max\{\max\{\|\xi\| \mid \xi \in R(\tau, x)\} \mid (\tau, x) \in C\},\ t \in E^1 .$$

REMARK 4.1. The demonstration that all solutions of (2) may be continued over the interval $[0, \alpha]$, defined in the proof of Theorem 4.1, is exactly like the corresponding proof for ordinary differential equations. The compactness of the solution family as a subset of $\mathscr{C}^n([0, \alpha])$ is then an easy consequence of Theorem 3.2; this again is a parallel to the corresponding argument for ordinary differential equations. Invoking [5, Th. 1] and Corollary 2.1, only slight modification of the proof of Theorem 4.1 is needed to establish the more general Pliś-Castaing existence theorem [17], [5].

5. The Huygens derivative.

DEFINITION 5.1. Let $\mathscr{S} \in \mathscr{H}^n(I)$; given $t \in I$, if there exists $S(t) \in \Gamma^n$ such that

$$\lim_{\eta \to 0+} \eta^{-1}\rho(G(t + \eta; \mathscr{S}), G(t; \mathscr{S}) + \eta S(t)) = 0$$

then $S(t)$ is called a *right hand* (*Huygens*) *derivative of* $\mathscr{S}$ *at* t. If there exists $V(t) \in \Gamma^n$ such that

$$\lim_{\eta \to 0+} \eta^{-1}\rho(G(t - \eta; \mathscr{S}) + \eta V(t), G(t; \mathscr{S})) = 0$$

the $V(t)$ is called a *left hand* (*Huygens*) *derivative of* $\mathscr{S}$ *at* t.

LEMMA 5.1. *The one-sided Huygens derivatives of* $\mathscr{S} \in \mathscr{H}^n(I)$ *are unique.*

Proof. We give the proof for right hand derivatives, the proof for left hand derivatives being similar. Let $R(t)$, $S(t)$ be right hand derivatives of $\mathscr{S}$ at t; then for $\eta > 0$ it follows from Lemma 1.3 and the triangle law that

$$\begin{aligned}\rho(R(t), S(t)) &= \eta^{-1}\rho(\eta R(t), \eta S(t)) = \eta^{-1}\rho(G(t; \mathscr{S}) + \eta R(t), G(t; \mathscr{S}) + \eta S(t)) \\ &\leqq \eta^{-1}\rho(G(t+\eta; \mathscr{S}), G(t; \mathscr{S}) + \eta R(t)) \\ &\qquad + \eta^{-1}\rho(G(t + \eta; \mathscr{S}), G(t; \mathscr{S}) + \eta S(t)) .\end{aligned}$$

By hypothesis, the limit, as $\eta \to 0+$, of the rightmost member is zero so that $\rho(R(t), S(t)) = 0$.

DEFINITION 5.2. When these exist, the right hand and left hand derivatives at t of $\mathscr{S} \in H^n(I)$ will be denoted by $(D^+\mathscr{S})(t)$ and $(D^-\mathscr{S})(t)$ respectively. If the one-sided derivatives of $\mathscr{S}$ at t both exist and are equal, their common value is called the *Huygens derivative of* $\mathscr{S}$ *at* t and is denoted by $(D\mathscr{S})(t)$.

LEMMA 5.2. *If* $F: I \to \Gamma^n$ *is measurable and integrably bounded then*

$$\nu(G(t; \mathscr{S}_I(F)), p) = \int_0^t \nu(F(\tau), p)d\tau,\ t \in I,\ p \in E^n .$$

Proof. Let us condense notation by defining

$$\begin{aligned} \omega(t, p) &= \nu(G(t; \mathscr{S}_I(F)), p) , \\ \lambda(t, p) &= \nu(F(t), p) . \end{aligned}$$

Then the assertion of the lemma is that $\omega(t, p) = \int_0^t \lambda(\tau, p)d\tau,\ t \in I,$ $p \in E^n$. By an argument similar to that for Theorem 3.1 it follows that $\lambda(\circ, p)$ is summable for each $p \in E^n$ so that $\int_0^t \lambda(\tau, p)d\tau$ is well defined. If $\sigma \in G(t; \mathscr{S}_I(F))$ then there exists $u^* \in \mathscr{F}_I(F)$ such that $\sigma = \int_0^t u^*(\tau)d\tau$; hence

$$\sigma \circ p = \int_0^t u^*(\tau) \circ p d\tau \leqq \int_0^t \lambda(\tau, p)d\tau,\ \ t \in I,\ \ p \in E^n .$$

We infer that $\omega(t, p) \leqq \int_0^t \lambda(\tau, p)d\tau$. For the proof of the reverse inequality let h be the integrable bound on F; then for $\eta > 0$ and $\| p \| = 1$, $(h(t) + \eta) \notin F(t)$ on I. For suppose the contrary; then

$$h(t) < h(t) + \eta = \| (h(t) + \eta)p \| \leqq h(t) ,$$

which is absurd. Let $q(t, \eta, p)$ be the unique point in the boundary of $F(t)$ nearest $(h(t) + \eta)p$; then by virtue of Lemma 2.5, $q(\circ, \eta, p)$ is summable and

$$\int_0^t \lambda(\tau, p)d\tau = \int_0^t q(\tau, \eta, p) \circ p d\tau = \left(\int_0^t q(\tau, \eta, p)d\tau\right) \circ p \leqq \omega(t, p) .$$

This completes the proof.

THEOREM 5.1. *If* $F: I \to \Omega^n$ *is measurable and integrably bounded then a.e. on* I, $(D\overline{\mathscr{S}_I(F)})(t) = F^*(t)$.

Proof. By virtue of Corollary 3.4, $(D\overline{\mathscr{S}_I(F)})(t)$ exists if and only if $(D\mathscr{S}_I(F^*))(t)$ exists; moreover, the two have the same value. It is thus sufficient to show that $(D\mathscr{S}_I(F^*))(t) = F^*(t)$ a.e. on I; we shall carry out the proof for D^+, the proof for D^- being similar. For $\eta > 0$ we find that with ω, λ being as defined in the proof of Lemma 5.2,

$$\begin{aligned}
&\eta^{-1}\rho(G(t+\eta; \mathscr{S}_I(F^*)), G(t; \mathscr{S}_I(F^*)) + \eta F^*(t)) \\
&= \eta^{-1}\max\{|\,\omega(t+\eta, p) - [\omega(t, p) + \eta\lambda(t, p)]\,| \mid \|p\| = 1\} \\
&= \eta^{-1}\max\left\{\left|\int_t^{t+\eta}\lambda(\tau, p)d\tau - \eta\lambda t, p)\right| \,\middle|\, \|p\| = 1\right\} \quad \text{(by Lemma 5.2)} \\
&= \eta^{-1}\max\left\{\left|\int_t^{t+\eta}[\lambda(\tau, p) - \lambda(t, p)]d\tau\right| \,\middle|\, \|p\| = 1\right\} \\
&\leqq \eta^{-1}\int_t^{t+\eta}\Delta(F^*(\tau), F^*(t))d\tau \leqq \eta^{-1}\int_t^{t+\eta}\rho(F(\tau), F(t))d\tau \\
&\hspace{20em}\text{(by Lemma 1.3 (ii))}.
\end{aligned}$$

The proof is completed by invoking Theorem 2.3.

COROLLARY 5.1. *If $F_i\colon I \to \Omega^n$, $i = 1, 2$, are measurable and integrably bounded, a necessary and sufficient condition that the closures of $\mathscr{S}_I(F_1)$ and $\mathscr{S}_I(F_2)$ be equal is that $F_1^*(t) = F_2^*(t)$ a.e. on I.*

Proof. (*Sufficiency.*) Evidently $\mathscr{S}_I(F_1^*) = \mathscr{S}_I(F_2^*)$ and the assertion follows from Corollary 3.4.

(Necessity.) By hypothesis, Corollary 3.4 and Theorem 5.1, a.e. on I we have

$$F_1^*(t) = (D\overline{\mathscr{S}_I(F_1)})(t) = (D\overline{\mathscr{S}_I(F_2)})(t) = F_2^*(t)\ .$$

For $t_1, t_2 \in I$, let us set

$$\int_{t_1}^{t_2}F(\tau)d\tau = \left\{\int_{t_1}^{t_2}q(\tau)d\tau \mid q \in \mathscr{F}_I(F)\right\}$$

where $F\colon I \to \Omega^n$. It is not difficult to verify that for $\eta > 0$

$$G(t+\eta; \mathscr{S}_I(F)) = G(t; \mathscr{S}_I(F)) + \int_\tau^{t+\eta}F(\tau)d\tau,\ t,\ t+\eta \in I\ ,$$

and

$$G(t-\eta; \mathscr{S}_I(F)) + \int_{t-\eta}^{t}F(\tau)d\tau = G(t; \mathscr{S}_I(F)),\ t,\ t-\eta \in I\ .$$

Thus if $F\colon I \to \Omega^n$ is measurable and integrably bounded there follow from Lemma 3.3, Lemma 1.3 and the foregoing identities, both

$$\eta^{-1}\rho(G(t+\eta; \mathscr{S}_I(F)), G(t; \mathscr{S}_I(F)) + \eta F^*(t)) = \rho\left(\eta^{-1}\int_t^{t+\eta}F(\tau)d\tau, F^*(t)\right)$$

and

$$\eta^{-1}\rho(G(t-\eta; \mathscr{S}_I(F)) + \eta F^*(t), G(t; \mathscr{S}_I(F))) = \rho\Big(\eta^{-1}\int_{t-\eta}^{t} F(\tau)d\tau, F^*(t)\Big)$$

when $\eta > 0$. Together with Theorem 5.1, these last formulae establish the following generalization of [**11**, Lemmas 1.2, 1.3].

COROLLARY 5.2. *If $F: I \to \Omega^n$ is measurable and integrably bounded then, a.e. on I,*

$$\lim_{\eta\to 0} \rho\Big(\eta^{-1}\int_{t}^{t+\eta} F(\tau)d\tau, F^*(t)\Big) = 0 .$$

REMARK 5.1. Note that now Corollary 5.1 appears as a generalization of [**11**, Th. 1.1].

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Received January 3, 1969, and in revised form September 11, 1969. This research was sponsored by the National Science Foundation under grant GP-8921.

DREXEL UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

A GENERALIZED HAUSDORFF DIMENSION FOR FUNCTIONS AND SETS

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A generalization of the Hausdorff dimension of sets is given by restricting the lengths of the intervals in the covering family. The dependence of this dimension on the choice of covering family is studied by considering the set of points in the countable unit cube I^ω whose coordinates are the values of the dimensions of some set for a fixed, countable collection of covering families. General conditions are given in order that two families yield the same dimension on each set, and that a covering family give the ordinary Hausdorff dimension.

In 1919, Hausdorff [3] introduced a notion of dimension for subsets of the unit interval. For any set E, this dimension is $H(E) = \sup\{\gamma: \lambda_\gamma(E) > 0\}$, where $\lambda_\gamma(E) = \inf\{\Sigma(l(I_j))^\gamma: \cup I_j \supseteqq E\}$; and it can take any value between 0 and 1, being 1 in the case that E has positive Lebesgue outer measure. This notion of dimension can be generalized in various directions and the approach taken here follows Billingsley [1]. In particular, consider the dimension $H'(E)$ given by the outer measure $\lambda'_\gamma(E) = \inf\{\Sigma(m(C_i))^\gamma: \cup C_i \supseteqq E \ \&\ C_i \in \mathscr{J}\}$, where m denotes Lebesgue measure and $\mathscr{J}$ is any collection of m-measurable sets containing sets of arbitrarily small measure. If $\mathscr{J}$ contains the intervals and their finite unions, then $H'(E)$ assumes only the values 0 and 1, as $m(E) = 0$ or not. Thus for the study of sets of Lebesgue measure zero, it appears that $\mathscr{J}$ cannot be too large with respect to the family of all intervals. Accordingly, the dimension $H'(E)$ is studied only where $\mathscr{J}$ is any collection of intervals containing intervals of arbitrarily small length and where $\mathscr{J}$ is closed under translations, i.e., where $\mathscr{J}$ is completely determined by the length of its members. Rather than use the set of these lengths to describe $\mathscr{J}$, it is more convenient to use the set S of their negative logarithms, which is unbounded in $(0, \infty)$. The dimension then becomes a function $S(E)$ of the set E and the unbounded set S. In § 2, dimension is defined for a certain family $\mathscr{F}$ of nondecreasing functions, c.f. [2], [4], [5], which greatly facilitates the study.

The principal results concern the dependence of $S(E)$ on the choice of the covering set determined by S, and are obtained by considering the set $\mathscr{R}(S, T)$ of points in the unit square whose coordinates are respectively $S(E)$ and $T(E)$, for some set E. If Ω denotes the product of the closed unit interval with itself countably many times, Theorem 5 shows that the set of points in Ω, whose coordinates are

$S_k(E)$ for some E and fixed sequence of unbounded sets $\{S_k\}$, is precisely the intersection of all cylinders in Ω determined by the sets $\mathscr{R}(S_j, S_k)$, $j < k$. A characterization of $\mathscr{R}(S, T)$ directly in terms of the relative gaps in the sets S and T is given by Theorem 6. The set $\mathscr{R}(S, T)$ is closed and star-shaped with respect to the diagonal $0 \leqq x = y \leqq 1$ and Theorem 7 shows that these are characteristic properties. Theorem 9 gives an especially simple necessary and sufficient condition on S and T for the equivalence: $S(E) = T(E)$ for all sets E. The remaining theorems of § 4 show that for this equivalence, an unbounded set S may be replaced by an increasing sequence $\{s_n\}$ and that $\lim s_{n+1}/s_n = 1$ is a necessary and sufficient condition that $\{s_n\}$ give the ordinary Hausdorff dimension for all sets E.

1. Preliminaries. Let $\mathscr{F}$ be the collection of all real-valued functions f, defined on $(-\infty, \infty)$ with the property that $x \leqq y \rightarrow 0 \leqq f(y) - f(x) \leqq y - x$. The following elementary properties of $\mathscr{F}$ will be continually used without mention:

$$f \in \mathscr{F} \rightarrow f + \alpha \in \mathscr{F}, \alpha \text{ any constant };$$
$$f \in \mathscr{F} \text{ and } 0 \leqq \alpha \leqq 1 \rightarrow \alpha f \in \mathscr{F};$$
$$f, g \in \mathscr{F} \text{ and } 0 \leqq \alpha, \beta \leqq 1, \alpha + \beta \leqq 1 \rightarrow \alpha f + \beta g \in \mathscr{F};$$
$$\bigvee f_a \in \mathscr{F} \text{ for } f_a \in \mathscr{F}, \text{ if } \bigvee f_a(x_0) < \infty \text{ for some } x_0 ;$$
$$\bigwedge f_a \in \mathscr{F} \text{ for } f_a \in \mathscr{F}, \text{ if } \bigwedge f_a(x_0) > -\infty \text{ for some } x_0 .$$

Let S, T, etc., denote unbounded sets in $(0, \infty)$ and let $f \in \mathscr{F}$. Define $S(f) = \liminf f(x)/x$, over $x \rightarrow \infty, x \in S$. For $f \in \mathscr{F}$, $S(f)$ satisfies: $0 \leqq S(f) \leqq 1$. The number $S(f)$ is called the Hausdorff dimension of f with respect to S. The following properties are immediate consequences of the definition:

$$S(\bigwedge f_a) = \bigwedge S(f_a) \text{ over finite collections } \{f_a\} ;$$
$$S(f + \alpha) = S(f) ;$$
$$S(\alpha f + \beta x) = \alpha S(f) + \beta ;$$
$$S(f \vee \beta x) = S(f) \vee \beta .$$

LEMMA 1. *Given* $\varepsilon > 0, f \in \mathscr{F}$, *and unbounded sets* $S_1, \cdots, S_p$, *there is* $g \in \mathscr{F}$ *such that* (i) $g(0) \geqq 0$, $g(x) \geqq (S_k(f) - \varepsilon)x$, *for* $x \in S_k$, $k = 1, 2, \cdots, p$; *and* (ii) $S(g) = S(f)$ *for all unbounded* S.

Proof. Choose $x_0 > 0$ large enough so that $f(x) \geqq (S_k(f) - \varepsilon)x$ for $x \geqq x_0$, $x \in S_k$, $k = 1, \cdots, p$. Write $g(x) = (f(x) \vee 0) + x_0$. Then $g \in \mathscr{F}$ and $g(0) \geqq 0$. Moreover, if $0 \leqq x \leqq x_0$, then $g(x) \geqq x \geqq (S_k(f) - \varepsilon)x$. For $x \geqq x_0$, and $x \in S_k$, $g(x) \geqq f(x) \geqq (S_k(f) - \varepsilon)x$, which proves (i).

Finally, from the construction of $g(x)$ it is clear that $S(g) = S(f)$ for all unbounded S.

LEMMA 2. *Let $f_n \in \mathscr{F}$, $n = 1, 2, \cdots$ and unbounded sets $S_1, S_2, \cdots$ be given. There is $f \in \mathscr{F}$ such that $S_k(f) = \liminf S_k(f_n)$ as $n \to \infty$, for each $k = 1, 2, \cdots$.*

Proof. By Lemma 1, it can be assumed that for each $n, f_n(0) \geqq 0$ and $f_n(x) \geqq (S_k(f_n) - \varepsilon_n)x$, for $x \in S_k$, $k \leqq n$ and $\varepsilon_n \to 0$ as $n \to \infty$. For each k and n choose $x_{n,k} \in S_k$ such that $x_{n,k} \to \infty$ as $n \to \infty$ and $f_n(x_{n,k}) \leqq (S_k(f_n) + \varepsilon_n)x_{n,k}$. Let $C_n = \bigvee_{k=1}^{n} (x_{n,k} - f_n(x_{n,k}))$ and put $g_n(x) = f_n(x) \vee (x - C_n)$. Finally write $f = \bigwedge g_n$. Since $g_n(0) \geqq 0$, it follows that $f \in \mathscr{F}$. Moreover, $S_k(g_n) = 1$ for each k and n implies $S_k(f) = S_k(\bigwedge_{n \geqq m} g_n)$ for all m. If $k \leqq m$, then $\bigwedge g_n \geqq \bigwedge (S_k(f_n) - \varepsilon_n)x$ over $n \geqq m$, so that $S_k(f) \geqq \liminf S_k(f_n)$ as $n \to \infty$. On the other hand, from the construction of C_n, it follows that for $k \leqq n$, $f(x_{n,k}) \leqq (S_k(f_n) + \varepsilon_n)x_{n,k}$. Since $x_{n,k} \to \infty$ as $n \to \infty$, $S_k(f) \leqq \liminf S_k(f_n)$ as $n \to \infty$.

2. **The Hausdorff dimension of sets.** Let $\mathscr{M}$ be the set of all continuous, real-valued, nondecreasing functions μ defined on $[0, \infty)$ such that $\mu(0) = 0$ and $\mu(x) = 1$, for $x \geqq 1$. Let $\mathscr{M}_{sa}$ be the subset of $\mathscr{M}$ consisting of those μ in $\mathscr{M}$ which are sub-additive, i.e., $\mu(x + y) \leqq \mu(x) + \mu(y)$. Finally, given a subset E of $[0, 1]$, let $\mathscr{M}(E)$ be the subset of $\mathscr{M}$ consisting of all functions μ in $\mathscr{M}$ supported by E, i.e., $(a, b) \cap E = \phi$ implies $\mu(a) = \mu(\mathrm{b})$. The set $\mathscr{M}(E)$ may be void. The operator Δ, defined on $\mathscr{M}$ by $\Delta\mu(x) = \sup\,(\mu(y + x) - \mu(y))$ over all $y \geqq 0$, is clearly a projection of $\mathscr{M}$ onto $\mathscr{M}_{sa}$. The properties of subadditive functions needed here are given by

LEMMA 3. *If $\mu \in \mathscr{M}_{sa}$, then* (i) $\mu(tx) \geqq \mu(x)t/(t + 1)$ *for* $t, x \geqq 0$; *and* (ii) $\mu(x) > 0$ *for* $x > 0$.

Proof. If $t = 0$, (i) is obvious. Otherwise

$$\mu(x) = \mu(txt^{-1}) \leqq (tx(1 + [t^{-1}])) \leqq (1 + t^{-1})\mu(tx) ,$$

where $[z]$ denotes the greatest integer $\leqq z$. This shows (i). Part (ii) follows from (i), since $\mu(t) \geqq t/(t + 1)$.

Corresponding to each μ in $\mathscr{M}$, there is $f_\mu \in \mathscr{F}$ defined by $f_\mu(x) = \bigvee (x - y - \log \Delta\mu(e^{-y}))$ over $y \geqq x$. The following estimates for $f_\mu(x)$ will be needed:

LEMMA 4. *For $\mu \in \mathscr{M}$, $-\log \Delta\mu(e^{-x}) \leqq f_\mu(x) \leqq \log 2 - \log \Delta\mu(e^{-x})$.*

Proof. The first inequality is trivial. By Lemma 3 $\Delta\mu(e^{-x}) \leqq 2e^{y-x}\Delta\mu(e^{-y})$, which establishes the second inequality.

Using the correspondence $\mu \to f_\mu$, the Hausdorff dimension of functions $\mu \in \mathscr{M}$ can be defined by writing $S(\mu) = S(f_\mu)$, for each unbounded set S. Given any set $E \subseteqq [0, 1]$, the Hausdorff dimension of E with respect to S is defined to be the number:

$$S(E) = \sup\{S(\mu): \mu \in \mathscr{M}(E)\} ,$$

taking $S(E) = 0$ in the case that $\mathscr{M}(E) = \varnothing$. The connection between $S(E)$ and the classical Hausdorff dimension of E is given by

THEOREM 1. ([2], [4]) $S(E) = \sup\{\gamma: \lambda_{S,\gamma}(E) > 0\}$, *where* $\lambda_{S,\gamma}(E) = \inf\{\Sigma(l(I_j))^\gamma: \cup I_j \supseteqq E \ \& \ -\log l(I_j) \in S\}$.

Proof. Let $\beta < S(E)$ and $\{I_k\}$ be a covering of E by intervals such that $-\log\ l(I_j) \in S$. By Lemmas 1 and 4, $2e^{-\beta s} \geqq \Delta\mu(e^{-s})$ for $s \in S$ and some $\mu \in \mathscr{M}(E)$, so in particular

$$\Sigma(l(I_k))^\beta \geqq 1/2\ \Sigma\Delta\mu(l(I_k)) \geqq 1/2 .$$

It follows that $\lambda_{S,\lambda}(E) > 0$, and hence $S(E) \leqq \sup\{\gamma: \lambda_{S,\gamma}(E) > 0\}$. To show the reverse inequality, $\lambda_{S,\gamma}(E) > 0$ implies that

$$\mu(x) = (\lambda_{S,\gamma}(E))^{-1}\lambda_{S,\gamma}(E \cap [0, x])$$

belongs to $\mathscr{M}(E)$. Moreover $\mu(x + e^{-s}) - \mu(x) \leqq (\lambda_{S,\gamma}(E))^{-1}e^{-\gamma s}$ for all x, so that by Lemma 3, $f_\mu(s)/s - (\log(\lambda_{S,\gamma}(E))/s \geqq \gamma$ for all $s \in S$; and it follows that $S(E) \geqq \sup\{\gamma: \lambda_{S,\gamma}(E) > 0\}$.

The fact that $\lambda_{S,\gamma}$ is a sub-additive and monotone set function implies

THEOREM 2. *Given any countable collection* $\{E_n\}$ *of subsets of* $[0, 1]$, $S(\cup E_n) = \bigvee S(E_n)$ *for all unbounded sets* S.

Let $\mathscr{C}$ be the collection of all sets E of the form: $E = \{\xi: \xi = \Sigma\varepsilon_k\xi_k,\ \varepsilon_k = 0 \text{ or } 1\}$ for some positive, nonincreasing sequence $\{\xi_k\}$ with $\Sigma\xi_k \leqq 1$. For such sets E, the function μ_E, defined on $[0, \infty)$ by $\mu_E(x) = \sup\{\Sigma\varepsilon_k 2^{-k}: x \geqq \Sigma\varepsilon_k\xi_k\}$, belongs to $\mathscr{M}(E)$ and is sub-additive.

THEOREM 3. *If* $E \in \mathscr{C}$, *then* $S(E) = S(\mu_E)$ *for all unbounded sets* S.

Proof. Let $\lambda \in \mathscr{M}(E)$ and consider $s \in S$ such that $\xi_{k+1} \leqq e^{-s} \leqq$

ξ_k. Since E is contained in the union of the 2^{k+1} intervals:

$$I(\varepsilon_1, \cdots, \varepsilon_{k+1}) = [\Sigma_{j=1}^{k+1}\varepsilon_j\xi_j, \Sigma_{j=1}^{k+1}\varepsilon_j\xi_j + \xi_{k+1}] ,$$

and any two of these intervals intersect in at most one point, it follows that $\Delta\lambda(e^{-s}) \geqq 2^{-k-1} \geqq \Delta\mu(e^{-s})/2$. By Lemma 4, $f_\lambda(s) \leqq \log 4 + f_\mu(s)$ for $s \in S$, so that $S(\lambda) \leqq S(\mu_E)$.

Since $S(\mu) = S(f_\mu)$, Theorem 3 shows that for $E \in \mathscr{C}$, there is $f \in \mathscr{F}$ such that $S(E) = S(f)$ for all S. The converse is also true.

THEOREM 4. *For each $f \in \mathscr{F}$, there is $E_f \in \mathscr{C}$ such that $S(f) = S(E_f)$ for all unbounded sets S.*

Proof. If f is bounded, then $S(f) = 0$ and E_f can be taken to be void. Thus assume $f(x) \to \infty$ as $x \to \infty$ and without loss of generality, $f(0) = 0$. Select a positive, nonincreasing sequence ξ_k satisfying $f(-\log \xi_k) = k \log 2$. Such sequences exist since f is continuous nondecreasing and tends to ∞ as $x \to \infty$. Moreover, since $f(x) - x$ is nonincreasing, $\xi_1 \leqq 1/2$ and $\xi_{k+1} \leqq \xi_k/2$, which implies $\Sigma\xi_k \leqq 1$. Let $E = E_f$ be the set $\{\xi : \xi = \Sigma\varepsilon_k\xi_k, \varepsilon_k = 0 \text{ or } 1\}$, and let $\mu = \mu_E$. For $s \in S$ and $\xi_{k+1} \leqq e^{-s} \leqq \xi_k$, $\log \mu(e^{-s}) \geqq -\log 2 - f(s)$, so that $f(s) \geqq -\log 4 + f_\mu(s)$ by Lemma 4. Also $\log \mu(e^{-s}) \leqq \log 2 - f(s)$, which shows $f(s) \leqq \log 2 + f_\mu(s)$. Since these inequalities hold for all $s \in S$, this proves $S(f) = S(E)$.

If $\mathscr{H}_{\mathscr{C}} = \{(\alpha_S): \text{for some } E \in \mathscr{C}, \alpha_S = S(E) \text{ for all } S\}$, and if $\mathscr{R}_{\mathscr{F}} = \{(\beta_S): \text{for some } f \in \mathscr{F}, \beta_S = S(f) \text{ for all } S\}$, then Theorems 3 and 4 show $\mathscr{H}_{\mathscr{C}} = \mathscr{R}_{\mathscr{F}}$. The situation for arbitrary subsets of $[0, 1]$ is more difficult and the results are restricted to countable collections $\{S_k\}$ of unbounded sets.

For any pair of unbounded sets S and T, let $\mathscr{R}(S, T) = \{(\alpha, \beta): \alpha = S(f), \beta = T(f) \text{ for some } f \in \mathscr{F}\}$. From the properties of $\mathscr{F}$ and $S(f)$ for $f \in \mathscr{F}$ listed in § 1, it is clear that $\mathscr{R}(S, T)$ is star-shaped with respect to each point (α, α), $0 \leqq \alpha \leqq 1$. Moreover, Lemma 2 implies that $\mathscr{R}(S, T)$ is always closed. Let

$$\Omega = \{(x_r): 0 \leqq x_r \leqq 1, r = 1, 2, \cdots\} .$$

For each pair of natural numbers j, k with $j < k$, let $A_{j,k}$ be the cylinder in Ω: $A_{j,k} = \{(x_r): (x_j, x_k) \in \mathscr{R}(S_j, S_k)\}$. Finally, let $\mathscr{H}[\{S_k\}] = \{(\alpha_k): \text{for some } E \subseteqq [0, 1], \alpha_k = S_k(E), k = 1, 2, \cdots\}$.

THEOREM 5. *Given any countable collection of unbounded sets $\{S_k\}$, $\mathscr{H}[\{S_k\}] = \cap A_{j,k}$ over $j < k$.*

Proof. Suppose $(\alpha_r) \in \mathscr{H}[\{S_k\}]$. Let $j < k$ and $E \subseteqq [0, 1]$ such that $\alpha_j = S_j(E)$, $\alpha_k = S_k(E)$. If $\alpha_j = \alpha_k$ then $(\alpha_j, \alpha_k) \in \mathscr{R}(S_j, S_k)$ so $(\alpha_r) \in A_{j,k}$. Thus assume $\alpha_j \neq \alpha_k$ and by symmetry, consider only the case $\alpha_j < \alpha_k$. Then given any $\varepsilon > 0$, there is $f \in \mathscr{F}$ such that

$$S_k(E) - \varepsilon < S_k(f) \leqq S_k(E) \quad \text{and} \quad S_j(f) \leqq S_j(E) .$$

The function $g = f \vee S_j(E)x$ belongs to $\mathscr{F}$ and

$$S_j(g) = S_j(E), S_k(E) - \varepsilon < S_k(g) \leqq S_k(E) .$$

Since $\mathscr{R}(S_j, S_k)$ is closed, this shows $(\alpha_r) \in A_{j,k}$, and hence $\mathscr{H}[\{S_k\}] \subseteqq \cap A_{j,k}$ over $j < k$. Now suppose $(x_r) \in \cap A_{j,k}$. Then for every pair $j < k$, there is $f_{j,k} \in \mathscr{F}$ with $x_j = S_j(f_{j,k})$ and $x_k = S_k(f_{j,k})$. For each pair of natural numbers p, n, write

$$g_{p,n} = \bigwedge\{f_{j,k}\colon k = p \text{ or } j = p, k + j \leqq p + n\} .$$

By Lemma 2, for each p, there is $g_p \in \mathscr{F}$ such that $S_k(g_p) = \liminf S_k(g_{p,n})$ as $n \to \infty$, for each $k = 1, 2, \cdots$. Now write $E = \cup E_{g_p}$ over $p = 1, 2, \cdots$. By Theorems 2 and 4, for each k, $S_k(E) = \bigvee S_k(g_p) \geqq \lim_n \inf S_k(g_{k,n}) = x_k$. On the other hand, if $p \neq k$, then either $g_{p,n} \leqq f_{k,p}$ or $g_{p,n} \leqq f_{p,k}$ for $n \geqq k$, depending whether $p < k$ or $p > k$. Thus $S_k(E) = S_k(g_k) \vee \bigvee_{p \neq k} \lim_n \inf S_k(g_{p,n}) \leqq x_k$, for each k, which shows $(x_r) \in \mathscr{H}[\{S_k\}]$.

In general, if the sequence $\{S_k\}$ contains more than two terms, the set $\mathscr{H}[\{S_k\}]$ properly contains the set $\{(x_k)$: for some $f \in \mathscr{F}, x_k = S_k(f), k = 1, 2, \cdots\}$.

3. The set $\mathscr{R}(S, T)$. The results of § 2 show that the set $\mathscr{H}[\{S_k\}]$ is determined by the sets $\mathscr{R}(S_j, S_k), j < k$. This section lists a few of the properties of $\mathscr{R}(S, T)$. The first of these is a characterization of $\mathscr{R}(S, T)$ solely in terms of the sets S and T.

For each x, let $A(x, S, T)$ consist of all pairs (α, β) with $1 > \alpha \geqq \beta > 0$ and $(x\beta/\alpha, x(1-\beta)/(1-\alpha)) \cap S = \varnothing$. Let $B(x, S, T)$ be the set of all pairs (α, β) with $(\beta, \alpha) \in A(x, T, S)$. Finally let $\mathscr{A}(S, T) = \limsup A(t, S, T)$ as $t \to \infty, t \in T$, and $\mathscr{B}(S, T) = \limsup B(s, S, T)$ as $s \to \infty, s \in S$.

THEOREM 6. *For every pair of unbounded sets S and T,*

$$\mathscr{R}(S, T) = \mathrm{Cl}\,(\mathscr{A}(S, T) \cup \mathscr{B}(S, T)) .$$

Proof. Suppose $(\alpha, \beta) \in \mathscr{A}(S, T)$. If $\alpha = \beta$, then $(\alpha, \beta) \in \mathscr{R}(S, T)$. Thus assume $\beta < \alpha$. Then for some unbounded subset T_0 of T, the intervals $I_t = (t\beta/\alpha, t(1-\beta)/(1-\alpha))$ do not intersect S for $t \in T_0$. Define a function f in $\mathscr{F}$ by

$$f(x) = \begin{cases} \beta t \vee (x - (1 - \beta)t), & \text{if } x \in I_t, t \in T_0 \\ \alpha x, \text{ otherwise}. \end{cases}$$

Then $S(f) = \alpha$ and $f(t)/t = \beta$ for $t \in T_0$, and so $T(f) \leqq \beta$. It follows that $\beta S(f) \geqq \alpha T(f)$ and $(1 - \beta)(1 - S(f)) \leqq (1 - \alpha)(1 - T(f))$. Since $\mathscr{R}(S, T)$ is closed and star-shaped with respect to $(0, 0)$ and $(1, 1)$ it follows that $\mathrm{Cl}\,(\mathscr{A}(S, T)) \subseteqq \mathscr{R}(S, T)$. A similar argument shows $\mathrm{Cl}\,(\mathscr{B}(S, T)) \subseteqq \mathscr{R}(S, T)$. On the other hand, let f belong to $\mathscr{F}$. If $S(f) = T(f)$, then $(S(f), T(f))$, belongs to $\mathrm{Cl}\,(\mathscr{A}(S, T) \cup \mathscr{B}(S, T))$. Thus assume $S(f) \neq T(f)$ and by symmetry in S and T, assume $S(f) > T(f)$. It suffices to show that $S(f) > \alpha > \beta > T(f)$ implies $(\alpha, \beta) \in \mathscr{A}(S, T)$. In this case, it can be assumed by Lemma 1, that $f(s) > \alpha s$ for all $s \in S$ and that there is an unbounded subset T_0 of T on which $f(t) < \beta t$. Since $f \in \mathscr{F}, f(s) \leqq ((s - t) \vee 0) + f(t)$ for all pairs s and t. If $t \in T_0$ and $s \leqq t$, this implies $\alpha s < \beta t$. If $s \geqq t$, then $\alpha s < s - t + \beta t$. These last two inequalities imply $(t\beta/\alpha, t(1 - \beta)/(1 - \alpha)) \cap S = \emptyset$ or $(\alpha, \beta) \in A(t, S, T)$ for each $t \in T_0$. It follows that $(\alpha, \beta) \in \mathscr{A}(S, T)$.

As was noted before $\mathscr{R}(S, T)$ is always closed and star-shaped with respect to all points $(\alpha, \alpha), 0 \leqq \alpha \leqq 1$. These two properties actually characterize the shape of $\mathscr{R}(S, T)$ as is seen by

THEOREM 7. *Let $\mathscr{R}$ be a closed set in the unit square, $0 \leqq \alpha, \beta \leqq 1$, star-shaped with respect to $(0, 0)$ and $(1, 1)$. There are unbounded sets S and T such that $\mathscr{R} = \mathscr{R}(S, T)$.*

Proof. The theorem is obvious if $\mathscr{R}$ is the diagonal $0 \leqq \alpha = \beta \leqq 1$, since for $S = T, \mathscr{R}(S, T)$ is this diagonal. Otherwise, there is a sequence $(\alpha_n, \beta_n), 0 < \alpha_n, \beta_n < 1, \alpha_n \neq \beta_n$ which is everywhere dense in $\mathscr{R}$. Select a sequence of intervals (a_n, b_n) such that $a_n \to \infty$ as $n \to \infty$, $b_n \leqq a_{n+1}$ and

$$b_n/a_n = (\alpha_n^{-1} - 1)/(\beta_n^{-1} - 1), \text{ if } \alpha_n < \beta_n$$
$$b_n/a_n = (\beta_n^{-1} - 1)/(\alpha_n^{-1} - 1), \text{ if } \alpha_n > \beta_n .$$

If $\alpha_n < \beta_n$, the interval (a_n, b_n) is called an interval of type I. If $\alpha_n > \beta_n$, the interval (a_n, b_n) is said to be of type II. In each interval of type I, let $s_n = a_n\beta_n/\alpha_n$, and in each interval of type II, let $t_n = a_n\alpha_n/\beta_n$. In either case the constructed point belongs to (a_n, b_n). Let S consist of all the points a_n, b_n and the points s_n. Let T consist of all the points a_n, b_n and the points t_n. Assume first that $(\alpha, \beta) \in \mathscr{R}$. If $\alpha = \beta$, then $(\alpha, \beta) \in \mathscr{R}(S, T)$. Thus suppose $\alpha \neq \beta$ and by symmetry in S and T assume $\alpha > \beta$. Select a sequence of intervals $I_n =$

(a_n, b_n) of type II, such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$. Define f in $\mathscr{F}$ by

$$f(x) = \begin{cases} \alpha a_n \vee (x - (1-\alpha)b_n), \text{ if } x \in I_n, n = 1, 2, \cdots \\ \alpha x, \text{ otherwise} . \end{cases}$$

Then $S(f) = \alpha$ and for $t_n \in I_n$, $f(t_n)/t_n = \alpha\beta_n/\alpha_n \vee (1 - (1-\alpha)(1-\beta_n)/(1-\alpha_n))$ which tends to β as $n \to \infty$. Thus $T(f) = \beta$, which shows $\mathscr{R} \subseteqq \mathscr{R}(S, T)$. To show the reverse containment it is sufficient, by Theorem 6, to show $\mathscr{A}(S, T) \subseteqq \mathscr{R}$. If $(\alpha, \beta) \in \mathscr{A}(S, T)$, then for a subsequence t_k of $\{t_n\}$, $(t_k\beta/\alpha, t_k(1-\beta)/(1-\alpha)) \cap S = \emptyset$. This implies $\beta_k/\alpha_k \leqq \beta/\alpha$ and $(1-\beta)/(1-\alpha) \leqq (1-\beta_k)/(1-\alpha_k)$. Since $\mathscr{R}$ is star-shaped with respect to $(0, 0)$ and $(1, 1)$, this shows $(\alpha, \beta) \in \mathscr{R}$.

4. Equivalence of unbounded sets. By Theorem 5 of § 2 the statement, $S(E) = T(E)$ for all $E \subseteqq [0, 1]$, is the same as, $S(f) = T(f)$ for all $f \in \mathscr{F}$. The induced equivalence relation, $S \equiv T$, deserves some study.

THEOREM 8. *For all unbounded sets* S, $S \equiv \mathrm{Cl}(S)$.

Proof. Since $S \subseteqq \mathrm{Cl}(S)$, it is clear that $S(f) \geqq \mathrm{Cl}(S)(f)$ for all $f \in \mathscr{F}$. On the other hand, there is a map $\psi: \mathrm{Cl}(S) \to S$ such that $|1 - x/\psi(x)| \leqq 1/x$ for each $x \in \mathrm{Cl}(S)$. If $f \in \mathscr{F}$, then

$$f(s) \leqq [(s - x) \vee 0] + f(x)$$

for every pair s, x. Hence $f(\psi(x))/\psi(x) \leqq 1/x + (1 + 1/x)f(x)/x$ for all $x \in \mathrm{Cl}(S)$. It follows that $S(f) \leqq \mathrm{Cl}(S)(f)$ for $f \in \mathscr{F}$ and so $S \equiv \mathrm{Cl}(S)$.

The related partial ordering: $S \leqq T$, if and only if, $S(f) \leqq T(f)$ for all $f \in \mathscr{F}$, again equivalent to $S(E) \leqq T(E)$ for all $E \subseteqq [0, 1]$, has the following characterization.

THEOREM 9. *A necessary and sufficient condition that* $S \leqq T$, *is that there exist a function* $\varphi: T \to S$ *such that* $\lim t/\varphi(t) = 1$, *as* $t \to \infty$, $t \in T$.

Proof. If $\varphi: T \to S$ and $t/\varphi(t) \to 1$ as $t \to \infty$, $t \in T$, then for $f \in \mathscr{F}$, $f(\varphi(t)) \leqq [(\varphi(t) - t) \vee 0] + f(t)$, which implies

$$f(\varphi(t))/\varphi(t) \leqq |1 - t/\varphi(t)| + (t/\varphi(t))(f(t)/t) .$$

Hence $S(f) \leqq \varphi(T)(f) \leqq T(f)$. On the other hand, assume $S(f) \leqq T(f)$ for all $f \in \mathscr{F}$. In particular this is true for $g(x) = \bigvee(s/2 \wedge (x - s/2))$ over $s \in \mathrm{Cl}(S)$. Here, $S(g) = 1/2 \leqq T(g)$. For each $t \in T$, let $s(t) = \sup\{s: s \in S, s \leqq t\}$ and $s'(t) = \inf\{s: s \in S, s \geqq t\}$. Then $s(t)$ and $s'(t)$

belong to $\mathrm{Cl}(S)$ and it is easy to see that $g(t) = s(t)/2 \vee (t - s'(t)/2)$. Now let $\theta\colon T \to \mathrm{Cl}(S)$ be defined by

$$\theta(t) = \begin{cases} s(t), & \text{if } t/s'(t) \leqq s(t)/t \\ s'(t), & \text{otherwise}. \end{cases}$$

If $0 < \varepsilon < 1/2$, then for $t \in T$, t sufficiently large, $1/2 - \varepsilon \leqq g(t)/t$, which means $1 - 2\varepsilon \leqq s(t)/t$ or $s'(t)/t \leqq 1 + 2\varepsilon$. Since θ satisfies: $1 \leqq t/\theta(t) \leqq s'(t)/t$ or $1 \geqq t/\theta(t) \geqq s(t)/t$, it follows that $|1 - t/\theta(t)| \leqq 2\varepsilon$ and so $t/\theta(t) \to 1$ as $t \to \infty, t \in T$. If $\psi\colon \mathrm{Cl}(S) \to S$ is the mapping introduced in the proof of Theorem 8, then the composition, $\varphi = \psi\theta$, satisfies the required property, i.e., $t/\varphi(t) \to 1$ as $t \to \infty, t \in T$.

Given any unbounded S, let $I_k = [n_k, n_k + 1)$, for n_k nonnegative integers, be a sequence of intervals such that $S \subset \cup I_k$ and $I_k \cap S$ is nonempty. Let $s_k = \inf\{s\colon s \in S \cap I_k\}$. Then $\{s_k\} \subseteqq \mathrm{Cl}(S)$ and so $\{s_k\} \geqq S$. On the other hand the map $\varphi\colon S \to \{s_k\}$ defined by $\varphi(s) = s_k$, if $s \in S \cap I_k$, clearly satisfies the condition of Theorem 9. This proves

THEOREM 10. *Given any unbounded S, there is an increasing sequence $\{s_k\}$ such that $S \equiv \{s_k\}$.*

The final result concerns the classical Hausdorff dimension $H(f)$, where $H = (0, \infty)$.

THEOREM 11. *If $S = \{s_n\}$ and $s_n \leqq s_{n+1}$, then $S \equiv H$, if and only if, $\lim s_{n+1}/s_n = 1$, as $n \to \infty$.*

Proof. If $s_n \leqq x \leqq s_{n+1}$, then for $f \in \mathscr{F}, f(s_{n+1}) \leqq s_{n+1} - x + f(x)$, so that $f(s_{n+1})/s_{n+1} \leqq s_{n+1}/s_n - 1 + f(x)/x$. In the case that $s_{n+1}/s_n \to 1$ as $n \to \infty$, it follows that $S(f) \leqq H(f)$ for all $f \in \mathscr{F}$. Since $S \subseteqq H$, this shows $S \equiv H$. Conversely, if $S \leqq H$, then for $g = \bigvee(\alpha s \wedge (x - (1-\alpha)s)$ over $s \in S$, $H(g) \geqq S(f) = \alpha$, for a fixed $\alpha, 0 < \alpha < 1$. Thus, in particular for the points

$$x_n = \alpha s_n + (1 - \alpha)s_{n+1}, \liminf g(x_n)/x_n = \liminf \alpha/(\alpha + (1 - \alpha)s_{n+1}/s_n) \geqq \alpha \text{ as } n \to \infty. \quad \text{Thus } s_{n+1}/s_n \to 1 \text{ as } n \to \infty.$$

5. Connection with other dimension functions. Dimension can be defined for more general classes of intervals, $\mathscr{J}$ cf. [1], i.e., where $\mathscr{J}$ need not be closed under translations. It is known that if $\mathscr{J}$ is the class of r-adic intervals, then the dimension $H'(E)$ determined by $\mathscr{J}$ coincides with the usual Hausdorff dimension $H(E)$, as an easy application of Theorem 11 shows, taking

$$S = \{-\log \mathscr{L}(I) \colon I \in \mathscr{J}\} .$$

For which classes $\mathscr{J}$, does the dimension $S(E)$, where

$$S = \{-\log \mathscr{L}(I) \colon I \in \mathscr{J}\} ,$$

coincide with that determined by $\mathscr{J}$? More generally, for which $\mathscr{J}$, do there exist unbounded sets S, such that $S(E)$ coincides with $H'(E)$ determined by $\mathscr{J}$? In general, the solution of these problems is not known. Notice that for such classes $\mathscr{J}$, the dimension $H'(E)$ is necessarily a translation invariant dimension, so that one might ask if this property is also sufficient.

The author is indebted to Professor F. Bohnenblust for his advice and guidance during the preparation of this paper, which formed a part of the author's Doctoral dissertation submitted to the California Institute of Technology.

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Received January 3, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

A CHARACTERIZATION OF PERFECT RINGS

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J. P. Jans has shown that if a ring R is right perfect, then a certain torsion in the category $\mathrm{Mod}\, R$ of left R-modules is closed under taking direct products. Extending his method, J. S. Alin and E. P. Armendariz showed later that this is true for every (hereditary) torsion in $\mathrm{Mod}\, R$. Here, we offer a very simple proof of this result. However, the main purpose of this paper is to present a characterization of perfect rings along these lines: A ring R is right perfect if and only if every (hereditary) torsion in $\mathrm{Mod}\, R$ is fundamental (i.e., derived from "prime" torsions) and closed under taking direct products; in fact, then there is a finite number of torsions, namely 2^n for a natural number n. Finally, examples of rings illustrating that the above characterization cannot be strengthened are provided. Thus, an example of a ring R_1 is given which is not perfect, although there are only fundamental torsions in $\mathrm{Mod}\, R_1$, and only $4 = 2^2$ of these. Furthermore, an example of a ring R_{2*} is given which is not perfect and which, at the same time, has the property that there is only a finite number (namely, 3) of (hereditary) torsions in $\mathrm{Mod}\, R_{2*}$ all of which are closed under taking direct products. Moreover, the ideals of R_{2*} form a chain (under inclusion) and $\mathrm{Rad}\, R_{2*}$ is a nil idempotent ideal; all the other proper ideals are nilpotent and R_{2*} can be chosen to have a (unique) minimal ideal.

In what follows, R stands always for a ring with unity, $\mathscr{L}$ for the set of all left ideals of R and $\mathrm{Mod}\, R$ for the category of all (unital) left R-modules and R-homomorphisms. Given $L \in \mathscr{L}$ and $\rho \in R$, $L: \rho$ denotes the (right) ideal-quotient of L by ρ, i.e., the left ideal of all $\chi \in R$ such that $\chi\rho \in L$. We shall call a subset $\mathscr{K}$ of $\mathscr{L}$ a *Q-set* if it is closed with respect to this operation, i.e., if $K \in \mathscr{K}$ and $\rho \in R$ implies $K: \rho \in \mathscr{K}$; obviously, $\mathscr{L}$ and $\{R\}$ are the greatest and the least Q-sets, respectively. Thus, a topologizing idempotent filter (briefly, a *T-set*) of left ideals of P. Gabriel [4] is a Q-set $\mathscr{K}$ satisfying, in addition to the filter properties, also the following "radical" condition: If L is a left ideal of R such that $L: \kappa \in \mathscr{K}$ for every element κ of $K \in \mathscr{K}$, then $L \in \mathscr{K}$, as well.

By a *torsion* T *in* $\mathrm{Mod}\, R$ we shall always understand a hereditary torsion; thus, a torsion T in $\mathrm{Mod}\, R$ is a full subcategory of $\mathrm{Mod}\, R$ such that

(a) T is closed under taking submodules,

(b) for every $M \in \mathrm{Mod}\, R$, there is the greatest submodule (the T-torsion part) $T(M)$ of M belonging to T and

(c) $T(M/T(M)) = 0$ for every $M \in \operatorname{Mod} R$.

As a consequence, every torsion in $\operatorname{Mod} R$ is closed under taking quotients, direct sums and inductive limits. There is a one-to-one correspondence between the torsions in $\operatorname{Mod} R$ and the T-sets of left ideals of R:

If $\mathscr{K}$ is a T-set, then the class $T(\mathscr{K})$ of all R-modules whose elements have orders from $\mathscr{K}$ is a torsion in $\operatorname{Mod} R$; on the other hand, if T is a torsion in $\operatorname{Mod} R$, then the T-set $\mathscr{K}(T) = \{L \mid L \in \mathscr{L}$ and $R \bmod L \in T\}$ satisfies $T = T[\mathscr{K}(T)]$. Given an R-module M, let us always denote the T-torsion part of it by $T(M)$.

Thus, given a torsion T, we can define the following two-sided ideals I_T and $J_T \supseteqq I_T$ of R:

$$I_T = \bigcap_{L \in \mathscr{K}(T)} L$$

and

$$J_T/I_T = T(R/I_T) .$$

Using this notation, we can prove easily

PROPOSITION 1. *The following four statements are equivalent*:

(i) *A torsion T in* $\operatorname{Mod} R$ *is closed under taking direct products.*

(ii) *For every subset $\mathscr{S}$ of $\mathscr{K}(T)$,*

$$\bigcap_{L \in \mathscr{S}} L \in \mathscr{K}(T) .$$

(iii) $$I_T \in \mathscr{K}(T)$$

(iv) $$J_R = R .$$

Proof. The equivalence of (ii), (iii), and (iv) is trivial. Also the implication (ii) $\longrightarrow$ (i) follows easily; for, the order of an element of a direct product is evidently the intersection of the orders of its components. Finally, in order to show that (i) $\longrightarrow$ (iv), we consider the monogenic submodule of the direct product

$$\prod_{L \in \mathscr{K}(T)} R \bmod L$$

generated by the element whose components are generators of $R \bmod L$; it is obviously R-isomorphic to R/I_T.

PROPOSITION 2. *Let every proper (i.e., $\neq R$) two-sided ideal J of R satisfy the following condition*: *There is $\kappa \notin J$ such that, for every $\rho \in R$ with $\rho\kappa \notin J$, there exists $\sigma \in R$ with $\sigma\rho\kappa = \kappa$. Then every torsion in* $\operatorname{Mod} R$ *is closed under taking direct products.*

Proof. Let T be a torsion and J_T the two-sided ideal defined above. Assume that $J_T \neq R$. Thus, there exists $\kappa \notin J_T$ with the properties stated in our assumption. Since

$$\bigcap_{L \in \mathscr{K}(T)} L = I_T \subseteqq J_T ,$$

there is $L_0 \in \mathscr{K}(T)$ such that $\kappa \notin L_0$. Hence

$$L_0 \colon \kappa = (R\kappa \cap L_0) \colon \kappa \subseteqq J_T \colon \kappa ,$$

and therefore $J_T \colon \kappa \in \mathscr{K}(T)$, in contradiction to the fact that R/J_T has no nonzero element of order belonging to $\mathscr{K}(T)$. Consequently, $J_T = R$ and Proposition 2 follows in view of Proposition 1.

THEOREM A. *If a ring R satisfies the minimum condition on principal left ideals, i.e., if R is right perfect* (*cf. H. Bass* [2]), *then every torsion in* Mod R *is closed undertaking direct products.*

Proof. Given an ideal $J \neq R$, consider the (nonempty) set of all principal left ideals which are not contained in J; take a minimal element K of this set, $\kappa \in K \backslash J$ and apply Proposition 2.

REMARK 1. We can see easily that if R satisfies the minimum condition on principal left ideals, then every R-module M has a nonzero socle; the latter property is, in turn, obviously equivalent to either of the following two statements:

(i) Every monogenic R-module has a nonzero socle.

(ii) For every proper left ideal L of R, there is $\rho \in R \backslash L$ such that $L \colon \rho \neq R$ is maximal in R.

Before we proceed to establish the characterization of perfect rings, left us introduced the following convenient notation and terminology. Denote by $\mathscr{W} \subseteqq \mathscr{L}$ the Q-set of all maximal left ideals of R (R itself including). Obviously, for every $W \in \mathscr{W}$, $W \neq R$, the subset

$$\{W \colon \rho \mid \rho \in R\}$$

is a minimal Q-set contained in $\mathscr{W}$. Denoting by $\mathscr{W}_\omega$, $\omega \in \Omega$, all such (distinct) minimal Q-sets, it is easy to see that $\{\mathscr{W}_\omega \mid \omega \in \Omega\}$ is a covering of $\mathscr{W}$, i.e.,

$$\mathscr{W} = \bigcup_{\omega \in \Omega} \mathscr{W}_\omega \quad \text{and} \quad \mathscr{W}_{\omega_1} \cap \mathscr{W}_{\omega_2} = \{R\} \quad \text{for} \quad \omega_1 \neq \omega_2 .$$

Furthermore, for every $\Omega_1 \subseteqq \Omega$, put

$$\mathscr{W}_{\Omega_1} = \bigcap_{\omega \in \Omega_1} \mathscr{W}_\omega ;$$

of course, $\mathscr{W} = \mathscr{W}_\Omega$ and $\mathscr{W}_\omega = \mathscr{W}_{\{\Omega\}}$ for each $\omega \in \Omega$. Now, for every $\Omega_1 \subseteqq \Omega$, denote the smallest T-set containing $\mathscr{W}_{\Omega_1}$ by $\mathscr{W}_{\Omega_1}^*$. It can be easily shown (cf. [3]) that $\mathscr{W}_{\Omega_1}^*$ is the unique T-set $\sim$-equivalent to $\mathscr{W}_{\Omega_1}$ in the sense that, for every proper left ideal $L \in \mathscr{W}_{\Omega_1}^*$,

$$\{L\colon \rho \mid \rho \in R\} \cap \mathscr{W}_{\Omega_1} \neq \{R\} .$$

As a consequence,

$$\mathscr{W}_{\Omega_1}^* \cap \mathscr{W} = \mathscr{W}_{\Omega_1} .$$

Let us call the torsions $T(\mathscr{W}_\omega^*)$, $\omega \in \Omega$, the *prime torsions* in Mod R and, more generally, torsions $T(\mathscr{W}_{\Omega_1}^*)$ corresponding to the subsets Ω_1 of Ω, the *fundamental torsions* (i.e., derived from prime ones) in Mod R.

On the basis of the above characterization of the T-sets $\mathscr{W}_{\Omega_1}^*$, one can derive very easily the following well-known

PROPOSITION 3. *For any ring R, all the fundamental torsions $T(\mathscr{W}_{\Omega_1}^*)$ in* Mod R *are distinct and form a lattice ideal of the complete lattice of all torsions in* Mod R, *which is isomorphic to the lattice 2^Ω of all subsets of Ω.*

Proof. In order to complete the proof we need only to show that every torsion T in Mod R contained in $T(\mathscr{W}^*)$ is fundamental. But this follows from the fact that the T-set $\mathscr{K}(T) \subseteqq \mathscr{W}^*$ is evidently $\sim$-equivalent to $\mathscr{K}(T) \cap \mathscr{W}$ and since $\mathscr{K}(T) \cap \mathscr{W} = \mathscr{W}_{\Omega_0}$ for a suitable $\Omega_0 \subseteqq \Omega$, we have, in view of the fact that there is unique T-set $\sim$-equivalent to $\mathscr{W}_{\Omega_0}$,

$$\mathscr{K}(T) = \mathscr{W}_{\Omega_0}^* ,$$

as required.

REMARK 2. We can see easily that the assertion that every torsion in Mod R is fundamental is equivalent to the assertion that $\mathscr{W}^* = \mathscr{L}$, which in turn is equivalent to any of the statements of the previous Remark 1 (for, $\mathscr{W}^* \sim \mathscr{W}$).

Now, let us formulate the following

THEOREM B. *Let R be a ring such that every fundamental torsion in* Mod R *is closed under taking direct products. Then $R/\mathrm{Rad}\, R$ is semisimple* (*i.e., artinian*); *in particular, Ω is finite.*

Proof. For each $\omega \in \Omega$, put

$$W_\omega^0 = \bigcap_{W \in \mathscr{W}} W$$

and notice that the intersection

$$\operatorname{Rad} R = \bigcap_{\omega \in \Omega} W_\omega^0$$

is, according to Proposition 3, irredundant. For, $\mathscr{W}_\omega^*$ (for each $\omega \in \Omega$) and $\mathscr{W}^*$ are the smallest T-sets containing W_ω^0 and $\operatorname{Rad} R$, respectively.

In order to prove our theorem, it is sufficient to show that the socle of $R/\operatorname{Rad} R$ is the whole quotient ring $R/\operatorname{Rad} R$; for, $R/\operatorname{Rad} R$ is a ring with unity. First, observe that, in view of the fact that $\operatorname{Rad} R \in \mathscr{W}^*$, the socle of $R/\operatorname{Rad} R$ is essential in $R/\operatorname{Rad} R$ in the sense that it intersect every nonzero left ideal of $R/\operatorname{Rad} R$ nontrivially. Write

$$S/\operatorname{Rad} R = \operatorname{Socle}(R/\operatorname{Rad} R)$$

with the two-sided ideal $S \supseteqq \operatorname{Rad} R$ of R and assume

$$S \neq R .$$

Then, there is a (proper) maximal left ideal W of R such that

$$S \subseteqq W \subset R ;$$

and, $W \in \mathscr{W}_{\omega_1}$ for a suitable $\omega_1 \in \Omega$. Moreover, clearly

$$S \subseteqq W_{\omega_1}^0 .$$

Hence, since $\bigcap_{\omega \in \Omega} W_\omega^0$ is irredundant,

$$\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_\omega^0 \neq \Big(\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_{\omega_1}^0\Big) \cap W_{\omega_1}^0 = \operatorname{Rad} R ;$$

on the other hand, since $\operatorname{Rad} R \subseteqq S \subseteqq W_{\omega_1}^0$,

$$\Big(\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_\omega^0\Big) \cap S = \operatorname{Rad} R ,$$

and thus

$$\bigcap_{\substack{\omega \in \Omega \\ \omega \neq \omega_1}} W_\omega^0 = \operatorname{Rad} R ,$$

a contradiction.

The proof of the theorem is completed.

Now, the main result of the present paper, namely the characterization of perfect rings, follows straight forward from Theorem A, Remarks 1 and 2, Theorem B and the fact that a (right) perfect rings can be characterized as a ring R with unity such that every (left) R-module has a nonzero socle and that $R/\operatorname{Rad} R$ is artinian (H. Bass [2]):

COROLLARY. *A ring R is right perfect if and only if all torsions in* Mod R *are fundamental and are closed under taking direct products.*

In conclusion, let us remark that the above characterization cannot be strengthened, even if we take into account the additional condition that there is a finite number of fundamental torsions in Mod R (the fact which is a consequence of our characterization). To show this, we present the following two examples of rings (which can easily be generalized):

EXAMPLE 1. Let N be the set of all natural numbers, F a field. Denote by $R_1 = R_1(\aleph_0, F)$ the ring of all countable "bounded" matrices over F, i.e., the ring of all functions $f: N \times N \to F$ satisfying the condition that there is a natural number n_f such that

$$f(i, j) = 0 \qquad \text{for } i \neq j,\ i > n_f \text{ or } j > n_f$$

and

$$f(i, i) = f(n_f + 1, n_f + 1) \qquad \text{for all } i > n_f\ ,$$

with matrix addition and multiplication. It is easy to verify that, for every $n \in N$,

$$C_n = \{f \mid f \in R_1 \text{ and } f(i, j) = 0 \text{ for } j \neq n\}$$

are minimal left ideals in R_1 and that the socle

$$S = \bigoplus_{n \in N} C_n$$

of R_1 is a (two-sided) maximal ideal in R_1; obviously, $R_1/S \cong F$. Furthermore, $\mathscr{W}_1' = \{S, R_1\}$ is a minimal Q-set of left ideals of R_1. Also, for every $n \in N$, the left ideals

$$W_n = \{f \mid f \in R_1 \text{ and } f(i, n) = 0\}$$

are maximal in R_1 and belong to the same minimal Q-set $\mathscr{W}_2'$. It is easy to see that the set of all maximal left ideals of R_1

$$\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$$

and that there are 4 torsions in Mod R, all of them fundamental, namely

$$0 = T(\{R\}),\ T(\mathscr{W}_1^*),\ T(\mathscr{W}_2^*) \text{ and } \operatorname{Mod} R = T(\mathscr{W}^*)\ .$$

Only $T(\mathscr{W}_2^*)$ is not closed under taking direct products. Of course, R_1 is not perfect.

EXAMPLE 2. Denote by Q^+ the set of all nonnegative rational numbers endowed with the usual order $\leqq$. Let F be a field. Denote by $R_2 = R(Q^+, F)$ the ring of all functions $f: Q^+ \rightarrow F$ such that the support

$$\operatorname{Sup} f = \{r \mid r \in Q^+ \text{ and } f(r) \neq 0\}$$

is contained in a well-ordered (with respect to $\leqq$) subset of Q^+ which has no finite limit point, with the addition and multiplication defined by

$$(f_1 + f_2)(r) = f_1(r) + f_2(r)$$

and

$$(f_1 * f_2)(r) = \sum_{\substack{t \in Q^+ \\ t \leqq r}} f_1(t) \cdot f_2(r - t) ,$$

respectively.

It is a matter of routine to verify that R_2 is a (commutative) ring. Now, for every $f \in R_2$, denote by r_f the least nonzero rational number such that $f(r_f) \neq 0$. Moreover, for every $t \in Q^+$, denote by $f^{(t)}$ the function of R_2 defined by

$$f^{(t)}(r) = \begin{cases} 1 & \text{for } r = t , \\ 0 & \text{otherwise} . \end{cases}$$

Now, we can see easily that, for every $f \in R_2$,

$$f = f^{(r_f)} * \bar{f} ,$$

where $\bar{f}(r) = f(r + r_f)$ for $r \in Q^+$ (and thus, $r_{\bar{f}} = 0$). First, we are going to prove the following

LEMMA. *If $\bar{f} \in R_2$ such that $r_{\bar{f}} = 0$, then there is $\bar{g} \in R_2$ satisfying*

$$\bar{f} * \bar{g} = f^{(0)} \ (= \textit{unity of } R_2) .$$

Proof. In order to ease the technical difficulties of the proof, observe first that having a well-ordered subset S of Q^+ with no finite limit point, we can consider the subsemigroup $\bar{S}$ of Q^+ generated by S: $\bar{S}$ is again well-ordered and has no finite limit point. Hence, we may consider, for a moment, that our function $\bar{f}$ is defined on a well-ordered subsemigroup $\bar{S}$ of Q^+ with no limit point and try to find $\bar{g}$ defined on the same set $\bar{S}$, i.e., with $\operatorname{Sup} \bar{g} \subseteqq \bar{S}$. Write

$$S = \{r_i\}_{i=0}^{\infty} \text{ with } 0 = r_0 < r_1 < r_2 < \cdots < r_n < \cdots .$$

Let us proceed by induction: Denoting by $\bar{g}_1$ the function defined by

$$\bar{g}_1(0) = [\bar{f}(0)]^{-1},\ \bar{g}_1(r_1) = -[\bar{f}(0)]^{-2}\cdot f(r_1) \text{ and } \bar{g}_1(r) = 0 \text{ otherwise,}$$

we can see easily that

$$\bar{f}*\bar{g}_1 = f^{(0)} + h_1 ,$$

where

$$\operatorname{Sup} \bar{g}_1 \subseteqq \{r_i\}_{i=0}^{1} \text{ and } \operatorname{Sup} h_1 \subseteqq \{r_i\}_{i=2}^{\infty} .$$

Assuming that, for a natural $n \geqq 1$, we have $\bar{g}_n \in R_2$ and $h_n \in R_2$ with

$$\operatorname{Sup} \bar{g}_n \subseteqq \{r_i\}_{i=0}^{n} \text{ and } \operatorname{Sup} h_n \subseteqq \{r_i\}_{i=n+1}^{\infty}$$

such that

$$\bar{f}*\bar{g}_n = f^{(0)} + h_n ,$$

let us define

$$\bar{g}_{n+1} = \bar{g}_n + g_{n+1} ,$$

where

$$g_{n+1}(r_{n+1}) = -[\bar{f}(0)]^{-1}h_n(r_{n+1}) \text{ and } g_{n+1}(r) = 0 \text{ otherwise .}$$

Then,

$$\bar{f}*\bar{g}_{n+1} = \bar{f}*\bar{g}_n + \bar{f}*g_{n+1} = f^{(0)} + h_n + \bar{f}*g_{n+1}$$

and, writing

$$h_{n+1} = h_n + \bar{f}*g_{n+1} ,$$

we can easily check that

$$\operatorname{Sup} h_{n+1} \subseteqq \{r_i\}_{i=n+2}^{\infty} .$$

For,

$$h_{n+1}(r) = (\bar{f}*g_{n+1})(r) = \sum_{0 \leqq t \leqq r} \bar{f}(t) g_{n+1}(r - t) = 0 \quad \text{for } r < r_{n+1}$$

and

$$\begin{aligned} h_{n+1}(r_{n+1}) &= h_n(r_{n+1}) + \sum_{0 \leqq t \leqq r_{n+1}} \bar{f}(t) g_{n+1}(r_{n+1} - t) \\ &= h_n(r_{n+1}) + \bar{f}(0) g_{n+1}(r_{n+1}) = h_n(r_{n+1}) - h_n(r_{n+1}) = 0 , \end{aligned}$$

as required.

Finally, to complete the proof of our lemma, denote by $\bar{g}$ the function defined by

$$\bar{g}(r) = \begin{cases} g_i(r_i) & \text{for } r = r_i,\ i = 0, 1, \cdots \\ 0 & \text{elsewhere .} \end{cases}$$

Then,

$$\bar{f}*\bar{g} = f^{(0)} ;$$

for, if $i = 1, 2, \cdots$

$$\begin{aligned}(\bar{f}*\bar{g})(r_i) &= (\bar{f}*[\bar{g}_i + (\bar{g} - \bar{g}_i)])(r_i) \\ &= (\bar{f}*\bar{g}_i)(r_i) + [\bar{f}*(\bar{g} - \bar{g}_i)](r_i) \\ &= (f^{(0)} + h_i)(r_i) + [\bar{f}*(\bar{g} - \bar{g}_i)](r_i) \\ &= 0 + \sum_{0 \leqq t \leqq r_i} \bar{f}(t)(\bar{g} - \bar{g}_i)(r_i - t) \\ &= 0 .\end{aligned}$$

As a consequence, $f \in R_2$ is a unit in R_2 if and only if $r_f = 0$. Moreover, for every $r \in Q^+$, there exist two ideals

$$\bar{I}_r = \{f \mid f \in R_2 \text{ and } r_f \geqq r\}$$

and

$$I_r = \{f \mid f \in R_2 \text{ and } r_f > r\} ;$$

these are all ideals of R_2. Notice that,

$$I_r \subset \bar{I}_r$$

and that

$$r_1 < r_2 \quad \text{implies} \quad I_{r_1} \supset \bar{I}_{r_2} ;$$

in particular,

$$\bar{I}_0 = R_2 \quad \text{and} \quad I_0 = \operatorname{Rad} R_2 .$$

It is also easy to see that there are no divisors of zero in R_2 and that

$$(\operatorname{Rad} R_2)^2 = \operatorname{Rad} R_2 .$$

For, if $f \in \operatorname{Rad} R_2$, then $r_f > 0$ and obviously,

$$f = f^{((1/2)r_f)} * g ,$$

where

$$g(r) = f\left(r + \frac{1}{2} r_f\right) \qquad \text{for } r \in Q^+ ;$$

here, both $f^{((1/2)r_f)}$ and g evidently belong to $\operatorname{Rad} R_2$.

Finally, given a positive rational number q, define

$$R_{2q} = R_2 / I_q$$

(similarly, we can consider $\bar{R}_{2q} = R_2/\bar{I}_q$). It is easy to see that

$$\text{Rad}\, R_{2q} \cong I_0/I_q$$

satisfies again

$$(\text{Rad}\, R_{2q})^2 = \text{Rad}\, R_{2q}\ ,$$

but that every other proper ideal (which is isomorphic to either I_r/I_q or I_r/I_q for $r \geqq q$) is nilpotent; besides,

$$\text{Socle}\,(R_{2q}) \cong \bar{I}_q/I_q\ .$$

Thus, there are only three torsions in Mod R_{2q}, namely

$$0 = T(\{R\}),\ T(\{R_{2q}, \text{Rad}\, R_{2q}\}) \quad \text{and} \quad \text{Mod}\, R_{2q} = T(\mathscr{L}^{R_{2q}})\ .$$

All of them are evidently closed under taking direct products; but, only the first two are fundamental. And, R_{2q} is not perfect.

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Received April 4, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

SOME EXAMPLES IN FIXED POINT THEORY

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It is known that the fixed point property (f.p.p.) is not invariant under suspension and join in the category of simply connected polyhedra. In this paper we exhibit examples to show that f.p.p. is not invariant under suspension and join in the category of simply connected polyhedra satisfying the Shi condition and more strongly, in the category of simply connected compact manifolds. We also exhibit a simply connected polyhedron X such that the smash product $X \wedge X$ fails to have f.p.p. if one choice of base point is used to form $X \wedge X$, while $X \wedge X$ has f.p.p. using another choice of base point. In the last section we prove that f.p.p. is invariant under Cartesian products in very special circumstances.

It is known that the fixed point property (f.p.p.) in the category of *simply connected polyhedra* is not an invariant under cartesian products, smash products, suspension, join or homotopy type (Lopez [3] and [1]). In all cases the counterexamples are based upon polyhedra which fail to satisfy the Shi condition, namely that for each vertex v, ∂Stv (boundary of the star of v) be connected and the dimension is $\geqq 3$. It is therefore natural to consider the behavior of f.p.p. in more restrictive categories. As suggested in [1], one should look at f.p.p. in the following categories:

$\mathscr{S}$: Polyhedra satisfying the Shi condition.
$\mathscr{S}_0$: Simply connected polyhedra in $\mathscr{S}$.
$\mathscr{M}$: Compact topological manifolds, dimension $\geqq 3$.
$\mathscr{M}_0$: Simply connected manifolds in $\mathscr{M}$.

In the categories $\mathscr{S}$ and $\mathscr{M}$ f.p.p. is a homotopy type invariant. In fact, if X is any compact ANR *dominated* by Y, where Y is in $\mathscr{S}$ or $\mathscr{M}$, then Y f.p.p. implies X f.p.p. [1]. Thus the result, Y f.p.p. implies $Y \times I$ f.p.p., is valid in the categories $\mathscr{S}$ or $\mathscr{M}$ even though it is false for (simply connected) polyhedra in general.

The question

$$X \text{ f.p.p.}, Y \text{ f.p.p.} \Longrightarrow X \times Y \text{ f.p.p.?} \tag{1}$$

in the categories $\mathscr{S}$ or $\mathscr{M}$ remains open. In § 4, we prove two very special cases for the categories $\mathscr{S}_0$ and $\mathscr{M}_0$. In § 2 we provide the details of the examples announced in [1] which show that in $\mathscr{S}_0$ and $\mathscr{M}_0$ f.p.p. is not invariant under the suspension and join operations. In § 3 we use one of the examples of § 2 to construct a simply connected polyhedron X which has f.p.p. and with the curious property that with one choice of base point (a, a) the resulting smash product $X \wedge X = X \times X/a \times X \cup X \times a$ fails to have f.p.p., while constructing $X \wedge X$ with another choice of base point preserves f.p.p.

2. Two examples. If $F: X \to X$ is a self-map of a compact connected metric ANR, then for any field Λ

$$(1) \qquad L(f; \Lambda) = \sum_k (-1)^k \text{ Trace} f_k^*$$

is the Lefschetz number of f over Λ and $\bar{L}(f, \Lambda) = L(f, \Lambda) - 1$ is the *reduced* Lefschetz number of over Λ. When $\Lambda = Q$, the field of rational numbers, then $L(f) = L(f, Q)$ is the usual Lefschetz number of f. $\chi(X)$ and $\bar{\chi}(X) = \chi(X) - 1$ will denote the Euler characteristic and reduced Euler characteristic, respectively. All spaces in this paper will be connected compact metric ANR's.

We will make use of the following simple lemma.

LEMMA 2.1. *Suppose Λ is a field of characteristic $p \neq 2$ and X and Y are spaces with the property that for every self-map $f: X \to X$, $\bar{L}(f; \Lambda) = 0$ or 1 and every self-map $g: Y \to Y$, $\bar{L}(g, \Lambda) = 0$. Then any space $W \sim X \vee Y$ has* f.p.p.

Proof. Let

$$(2) \qquad X \xrightarrow{i_1} X \vee Y \xrightarrow{r_1} X$$

$$(3) \qquad Y \xrightarrow{i_2} X \vee Y \xrightarrow{r_2} Y$$

denote the natural inclusions and retractions. Then, if $\varphi: X \vee Y \to X \vee Y$ is any map, let $f = r_1\varphi i_1$ and $g = r_2\varphi i_2$. It is easy to verify that

$$(4) \qquad \bar{L}(\varphi, \Lambda) = \bar{L}(f, \Lambda) + \bar{L}(g, \Lambda) = 0 \text{ or } 1 .$$

Therefore, $L(\varphi, \Lambda) \neq 0$. Thus, $X \vee Y$ has the property that every self-map φ has nonzero Lefschetz number over Λ. Since this property is a homotopy type invariant, it follows that if $W \sim X \vee Y$, then W has f.p.p.

LEMMA 2.2. *If HP^4 is quaternionic projective 4-space, then for every self-map $f: HP^4 \to HP^4$, $\bar{L}(f, Z_3) = 0$ or 1.*

Proof. Let u denote a generator in $H^4(HP^4; Z_3)$. Then, if $f^*(u) = au$,

$$(5) \qquad \bar{L}(f; Z_3) = a + a^2 + a^3 + a^4 = 0 \text{ or } 1 .$$

LEMMA 2.3. *If SHP^3 is the suspension of quaternionic projective 3-space, then for every self-map $g\colon SHP^3 \to SHP^3$, $\bar{L}(g; Z_3) = 0$.*

Proof. Choose a generator $v \in H^5(SHP^3; Z^3)$ such that $P^1 v$ and $P^2 v$ generate the Z_3-cohomology in dimensions 9 and 13, respectively. P^i is the mod 3 Steenrod reduced power operator. Now, if $g\colon SHP^3 \to SHP^3$ and $g^*(v) = bv$,

$$\bar{L}(g; Z_3) = b + b + b = 0 . \tag{6}$$

PROPOSITION 2.4. *Any space $W \sim HP^4 \vee SHP^3$ has* f.p.p.

PROPOSITION 2.5. *Let*

$$K = HP^4 \cup_I SHP^3$$

denote the union of HP^4 and SHP^3 along an edge. Then, K is a simply connected polyhedron which has f.p.p. *and satisfies the Shi condition. Moreover, $\chi(K) = 2$.*

REMARK. $K' = (HP^4 \vee SHP^3) \times I$ has the same properties as K.

PROPOSITION 2.6. *The suspension SK and the join $K \circ K$ fail to have* f.p.p.

Proof. Since $\bar{\chi}(SK) = -\bar{\chi}(K)$ and $\bar{\chi}(K \circ K) = -\bar{\chi}(K)\bar{\chi}(K)$, both SK and $K \circ K$ have Euler characteristic 0. Since SK and $K \circ K$ satisfy the Shi condition, both admit maps homotopic to the identity map which are fixed point free [5].

THEOREM 2.7. *The* f.p.p. *is not invariant under suspension and join in the category $\mathscr{S}_0$.*

Our next example will verify the above theorem in the category $\mathscr{M}_0$.

Let $q\colon S^7 \to S^4$ denote the standard Hopf fibering and let $A = M_1(q)$, $B = M_2(q)$ denote two copies of the mapping cylinder of q. Then if $h\colon S^7 \to S^7$ is a reflection (degree -1), where S^7 is identified with one end of the mapping cylinder of q, we may represent the connected sum

$$M = HP^2 \# HP^2 \tag{7}$$

by

$$M = A \cup_h B . \tag{8}$$

There is a natural "flip" map $f: M \to M$ which takes A to B and B to A and which is the reflection on $S^7 = A \cap B$, where A and B are identified with the appropriate subsets of M. It is easy to see that f is a homeomorphism which preserves orientation. Furthermore, by identifying $S^7 = A \cap B$ we obtain an identification map

$$g: M \longrightarrow HP^2 \vee HP^2 \tag{9}$$

which allows us to compute the cohomology ring structure (Z-coefficients) as follows:

LEMMA 2.8. *The cohomology of $M = HP^2 \# HP^2$ is given by*

$$\begin{aligned} H^0(M) &= Z, \text{ generator } 1 \\ H^2(M) &= Z \oplus Z, \text{ generators } x, y \\ H^4(M) &= Z, \text{ generator } x^2 = y^2 \end{aligned} \tag{10}$$

with $H^q(M) = 0$ in the remaining dimensions and $xy = 0$.

THEOREM 2.9. *$M = HP^2 \# HP^2$ is a simply connected manifold with* f.p.p. *which admits a map f of Lefschetz number $L(f) = 2$.*

Proof. The natural "flip" map $f: M \to M$ defined above has $L(f) = 2$ so that the last part of the theorem is easy. Now, let

$$\varphi: M \longrightarrow M \tag{11}$$

denote an arbitrary map and suppose, using (10), that

$$\begin{aligned} \varphi^*(x) &= ax + by \\ \varphi^*(y) &= cx + dy \,. \end{aligned} \tag{12}$$

Then,

$$\varphi^*(x^2) = \varphi^*(y^2) = (a^2 + b^2)x^2 = (c^2 + d^2)y^2 \tag{13}$$

and

$$\varphi(xy) = 0 = (ac + bd)x^2 \tag{14}$$

which yields the conditions

$$a^2 + b^2 = c^2 + d^2\,, \qquad ac + bd = 0\,. \tag{15}$$

Furthermore,

$$L(\varphi) = 1 + a + d + a^2 + b^2\,. \tag{16}$$

We now consider individual cases.

Case 1. $a = 0, b = 0$. Here $L(\varphi) = 1$.

Case 2. $a^2 + b^2 \neq 0$, $(a, b) \neq (-1, 0)$. Using (15), we have

$$a^2(a^2 + b^2) = a^2(c^2 + d^2) = d^2(a^2 + b^2) \tag{17}$$

so that $a = \pm d$. If $a = -d$, $L(\varphi) = 1 + a^2 + b^2 > 0$. On the other hand if $a = d$, $L(\varphi) = (1 + a)^2 + b^2 > 0$.

Case 3. $a = -1, b = 0$. This case does not occur. To see this, choose $v \in H^4(HP^2; \mathbf{Z}_3)$ such that $P^1 v = v^2$. Then we may assume $g^*(v) = x$ (over $\mathbf{Z}_3$) and $P^1 x = x^2$ in $H^4(M; \mathbf{Z}_3)$. If $\varphi^*(x) = ax$ (over $\mathbf{Z}$), we must have

$$\varphi^*(P^1 x) = \varphi^*(x^2) = a^2 x^2 = a^2 P^1 x = a P^1 x = a x^2 \tag{18}$$

so that $a^2 \equiv a \pmod 3$. This precludes $a = -1$.

Thus, we see that for any map $\varphi\colon M \to M$, $L(\varphi) \neq 0$ and hence M has f.p.p.

THEOREM 2.10. *The* f.p.p. *is not invariant under suspension and join in the category* $\mathscr{M}_0$.

Proof. Let M denote the manifold in the previous theorem and $f\colon M \to M$ the map with $L(f) = 2$. Then,

$$Sf\colon SM \longrightarrow SM \quad \text{and} \quad f \circ f\colon M \circ M \longrightarrow M \circ M \tag{19}$$

yield

$$\bar{L}(Sf) = -\bar{L}(f) = -1 = -\bar{L}(f)\bar{L}(f) = \bar{L}(f \circ g) \tag{20}$$

so that

$$L(Sf) = 0 = L(f \circ f)\ . \tag{21}$$

Since we are in the simply connected case, the Nielson number of Sf (and $f \circ f$) is zero. Therefore again using [5], Sf and $f \circ f$ can be deformed to fixed point free maps so that SM and $M \circ M$ fail to have f.p.p.

3. The f.p.p. and smash product. Our objective in this section is to show that there is a simply connected polyhedron X with f.p.p. such that the smash product $X \wedge X = X \times X/X \vee X$ has f.p.p. with one choice of base point $x_0 \in X$ while it may fail to have f.p.p. if one employs another base point $x_1 \in X$.

We will make use of the polyhedron

$$K = HP^4 \cup_I SHP^3 \tag{1}$$

discussed in the previous section. If $N = SHP^2$ and

$$X = K \vee N = (HP^4 \cup_I SHP^3) \vee SHP^2 \tag{2}$$

we will show that $X \wedge X$ fails to have f.p.p. if the base point $x_0 \in X$ is chosen distinct from the *wedge point* $v \in X$. On the other hand, if the wedge point v is employed to form $X \wedge X$, then $X \wedge X$ retains f.p.p.

THEOREM 3.1. *If $x_0 \neq v$, then*

$$X \wedge X = X \times X/x_0 \times X \cup X \times x_0$$

fails to have f.p.p.

Proof. First we observe that since $\chi(X) = 0$, $\bar{L}(id) = -1$, where $\bar{L}$ is the reduced Lefschetz number. Since $\bar{\chi}(K) = 1$ (reduced Euler characteristic) we see that X admits a map g such that $\bar{L}(g) = 1$. Thus, $\bar{L}(id \wedge g) = \bar{L}(id)\bar{L}(g) = -1$, and we see that $f = id \wedge g$ is a self-map of $X \wedge X$ with $L(f) = 0$. $X \wedge X$ is simply connected and can be shown to satisfy the Shi condition (using the fact that $x_0 \times X \cup X \times x_0$ fails to separate $X \times X$). It follows that there is a map $g \sim f$ such that g has no fixed points. Thus, $X \wedge X$ fails to have f.p.p.

We now show that using the wedge point v

$$X \wedge X = X \times X/v \times X \cup X \times v \tag{3}$$

has f.p.p. Although the details are lengthy, the idea is quite simple. $X = K \cup N$ with $K \cap N = v$, the wedge point. Using v as base point in the formation of $X \wedge X$ yields

$$X \wedge X = (K \wedge K) \vee (K \wedge N) \vee (N \wedge K) \vee (N \wedge N) \tag{4}$$

where the four-fold wedge on the right is understood to have a single wedge point v' corresponding to $v \times X \cup X \times v$. Now, since f.p.p. is invariant under the wedge operation, it suffices to show that the four individual wedge factors $K \wedge K$, $K \wedge N$, $N \wedge K$, $N \wedge N$ have f.p.p.

LEMMA 3.2. *$HP^4 \wedge HP^4$ has* f.p.p. *Specifically, for any self map φ, $\bar{L}(\varphi, Z_3) = 0$ or* 1.

Proof. We will identify $H^*(A \wedge B)$ with $H^*(A \times B, A \vee B) \simeq H^*(A, a_0) \otimes H^*(B, b_0)$ using always field coefficients. Then, working over Z_3, $H^*(HP^4)$ has a basis of the form

$$1,\ \alpha,\ P^1\alpha,\ P^2\alpha,\ \alpha^4 \tag{5}$$

where P^i is the Steenrod reduced power operator. Then, we may arrange a basis for $H^*(HP^4 \wedge HP^4)$ in positive dimensions as follows:

$$\begin{array}{lll}
\alpha \times \alpha & \alpha \times P^1\alpha + P^1\alpha \times \alpha & \alpha \times P^2\alpha + P^1\alpha \times P^1\alpha + P^2\alpha \times \alpha \\
P^1\alpha \times \alpha & P^1\alpha \times P^1\alpha - P^2\alpha \times \alpha & -P^2\alpha \times P^1\alpha + P^1\alpha \times P^2\alpha \\
\alpha \times P^2\alpha & P^1\alpha \times P^2\alpha & P^2\alpha \times P^2\alpha \\
\alpha^4 \times \alpha & \alpha^4 \times P^1\alpha & \alpha^4 \times P^2\alpha \\
\alpha \times \alpha^4 & P^1\alpha \times \alpha^4 & P^2\alpha \times \alpha^4 . \\
\alpha^4 \times \alpha^4 & &
\end{array}$$

Notice that (for the first five rows) applying P^1 and P^2 to the first column yields the second and third columns. This means that for a self-map φ: of $HP^4 \wedge HP^4$, $\bar{L}(\varphi, Z_3) = \lambda^4$, where $\varphi^*(\alpha \times \alpha) = \lambda(\alpha \times \alpha)$. This concludes the proof.

LEMMA 3.3. *$HP^4 \wedge SHP^3$ has* f.p.p. *Specifically, for any self-map φ, $\bar{L}(\varphi, Z_3) = 0$.*

LEMMA 3.4. *$SHP^3 \wedge SHP^3$ has* f.p.p. *Specifically, for any self-map φ, $\bar{L}(\varphi, Z_3) = 0$.*

The proofs of these lemmas are modelled after the proof of Lemma 3.2 and consequently are left as exercises.

PROPOSITION 3.5. *$K \wedge K$ has* f.p.p.

Proof. Let $K' = HP^4 \vee SHP^3$, then using the above lemmas every self-map φ' of

$$\begin{aligned} &K' \wedge K' \\ &\quad = (HP^4 \wedge HP^4) \vee (HP^4 \wedge SHP^3) \vee (SHP^3 \wedge HP^4) \vee (SHP^3 \wedge SHP^3) \end{aligned} \tag{6}$$

has the property that $\bar{L}(\varphi, Z_3) = 0$ or 1(using the technique in the proof of Lemma 2.1). Since this property is a homotopy type invariant, every self-map φ of $K \wedge K$ has $L(\varphi, Z_3) \neq 0$. Thus, $K \wedge K$ has f.p.p.

LEMMA 3.6. *$HP^4 \wedge SHP^2$ has* f.p.p. *Specifically, for every self-map φ, $\bar{L}(\varphi, Z_2) = 0$.*

Proof. We may choose basis for the Z_2-cohomology of HP^4 and SHP^2, respectively, as follows

$$HP^4\colon 1,\ \alpha,\ Sq^4\alpha,\ \beta,\ Sq^4\beta \tag{7}$$

$$SHP^2\colon 1,\ u,\ Sq^4u\ . \tag{8}$$

Then, we may arrange a basis (in positive dimensions) for the Z_2-cohomology of $HP^4 \wedge SHP^2$ as follows

$$\begin{array}{rl} \alpha \times u & \alpha \times Sq^4u + Sq^4\alpha + Sq^4\alpha \times u \\ Sq^4\alpha \times u & Sq^4\alpha \times Sq^4u \\ \beta \times u & \beta \times Sq^4u + Sq^4\beta \times u \\ Sq^4\beta \times u & Sq^4\beta \times Sq^4u \end{array}$$

where S_q^4 applied to the first column yields the second column. This is enough to show that for every self-map φ, $\bar{L}(\varphi, Z_2) = 0$.

LEMMA 3.7. *$SHP^3 \wedge SHP^2$ has* f.p.p. *Specifically, for every self-map φ, $\bar{L}(\varphi, Z_2) = 0$.*

The proof of this lemma is similar to the proof of Lemma 3.6.

PROPOSITION 3.8. *$K \wedge N$ has* f.p.p.

Proof. $K \wedge N$ has the same homotopy type as

$$W = (HP^4 \vee SHP^3) \wedge SHP^2 = (HP^4 \wedge SHP^2) \vee (SHP^3 \wedge SHP^2)\ . \tag{9}$$

But by the previous lemmas, every self-map φ' of W has the property that $\bar{L}(\varphi', Z_2) = 0$ and hence every self-map of $K \wedge N$ has Lefschetz number 1 (over Z_2). Thus, $K \wedge N$ has f.p.p.

PROPOSITION 3.9. *$N \wedge N$ has* f.p.p.

Proof. Working with Z_2 coefficients, a basis for the cohomology of $N = SHP^2$ has the form $1, u, Sq^4u$. A basis for the cohomology (in positive dimensions) of $N \wedge N$ can be written

$$\begin{array}{rl} u \times u & Sq^4u \times u + u \times Sq^4u \\ Sq^4u \times u & Sq^4u \times Sq^4u \end{array}$$

where Sq^4 applied to column one yields column two. This, given any self-map φ of M, $\bar{L}(\varphi; Z_2) = 0$.

THEOREM 3.10. *Using the wedge point v of X*

$$X \wedge X = X \times X/v \times X \cup X \times v$$

has f.p.p.

4. Very special cases of the product theorem. Consider the following property:

Property F: X is said to have property F if, and only if, $L(f) \neq 0$ for every self-map $f: X \to X$.

In terms of this property we recall the following theorem [1]:

THEOREM 4.1. *If X belongs to $\mathscr{S}_0$ or $\mathscr{M}_0$, then X has* f.p.p. *if, and only if, X has property F.*

Thus for spaces in $\mathscr{S}_0$ (or $\mathscr{M}_0$), the question of the invariance of f.p.p. under Cartesian products (see (1) of § 1) is equivalent to the question

$$X \text{ and } Y \text{ have property } F \Rightarrow X \times Y \text{ has property } F? \tag{1}$$

Our next theorem answers (1) in the affirmative under quite special hypothesis. In the following we use rational singular cohomology.

THEOREM 4.2. *Suppose X and Y are spaces having property F. Suppose further that X has trivial cup products and X and Y have disjoint cohomology, i.e., $H^p(X) \neq 0$, $H^q(Y) \neq 0$, $p, q \geqq 1$, implies $p \neq q$. Then $X \times Y$ has property F.*

We will make use of the following lemma whose proof is left to the reader.

LEMMA 4.3. *Suppose $\psi: X \to Y$ is a map and $\psi_0: Y \to Y$ is defined by the diagram*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\psi} & X \times Y \\ \sigma \big\uparrow & & \big\downarrow \pi \\ Y & \xrightarrow{\psi_0} & Y \end{array}$$

where σ is a section given by $\sigma(y) = (x_0, y)$, $x_0 \in X$ and π is a projection on the second factor. Then, for $v \in H^n(Y)$

$$\psi^*(1 \times v) = 1 \times \psi_0^*(v) + E(v) \tag{2}$$

where $E(v)$ is a linear combination of terms of the form $a \times b$ where $\dim a \geqq 1$.

Proof of 4.2. Let $\varphi: X \times Y \to X \times Y$ denote an arbitrary map and let f and g be defined by the diagrams

$$\begin{array}{ccc} X \times Y & \xrightarrow{\varphi} & X \times Y \\ {\scriptstyle\sigma_1}\uparrow & & \downarrow{\scriptstyle\pi_1} \\ X & \xrightarrow{f} & X \end{array}$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{\varphi} & X \times Y \\ {\scriptstyle\sigma_2}\uparrow & & \downarrow{\scriptstyle\pi_2} \\ Y & \xrightarrow{g} & Y \end{array}$$

where σ_1 and σ_2 are sections and π_1 and π_2 are projections (see Lemma 4.3).

We choose bases $1 = u_1, \cdots, u_k$ and $1 = v_1, \cdots, v_l$ for the rational cohomology of X and Y, respectively. Then, elements of the form $u_i \times v_j$ form a basis for the cohomology of $X \times Y$. If u and v are typical basis elements, then using Lemma 4.3

$$\varphi^*(u \times 1) = f^*(u) \times 1 + E(u)$$
$$\varphi^*(1 \times v) = 1 \times g^*(v) + E(v)$$

where $E(u)$ is a linear combination of terms of the form $a \times b$ with $\dim b \geqq 1$ and $E(v)$ is a linear combination of terms of the form $a' \times b'$, $\dim a' \geqq 1$. Suppose $\dim u = m$ and $\dim v = n$. Then

$$(3)\quad \begin{aligned}&\varphi^*(u \times v)\\ &\quad = f^*(u) \times g^*(v) + E(u)(1 \times g^*(v)) + (f^*(u) \times 1)E(v) + E(u)E(v) .\end{aligned}$$

Now $E(u)$ is a linear combination of terms of the form $a \times b$ where $\dim a \leqq m-1$ so that $u \times v$ cannot appear in the term $E(u)(1 \times g^*(v))$. Similarly, $u \times v$ cannot appear in the term $(f^*(u) \times 1)E(v)$. In $E(u)E(v)$ a typical term has the form

$$(4)\qquad (a \times b)(a' \times b') = \pm aa' \times bb'$$

where $\dim a \leqq m-1, \dim b \geqq 1, \dim a' \geqq 1, \dim b' \leqq n-1$. If $\dim a \geqq 1$, $aa' = 0$ so that (4) is 0. On the other hand if $\dim a = 0$ then $\dim b = m$. Since $\dim u = m$ we see that $b = 0$ and hence (4) is 0 in this case. Thus $E(u)E(v) = 0$. Thus, we see that $\varphi^*(u \times v)$ and $(f \times g)^*(u \times v)$ have the same coefficient of $u \times v$. Thus,

$$(5)\qquad L(f \times g) = L(f)L(g) = L(\varphi) \neq 0 .$$

THEOREM 4.4. *Suppose X and Y belong to $\mathscr{S}_0$ (or $\mathscr{M}_0$) and have* f.p.p. *Then $X \times Y$ has* f.p.p. *if X or Y has trivial rational cup products and X and Y have disjoint rational cohomology.*

EXAMPLE. Using Theorem 4.4, we see that $CP^i \times SCP^j$ has f.p.p. for i and j even, $i, j \geqq 2$. To prove that CP^i has f.p.p., arrange a basis for the Z_2-cohomology of CP^i in the form (i even)

$$1, x_1, Sq^2x_1, x_2, Sq^2x_2, \cdots \tag{6}$$

so that for any self-map φ of CP^i we have $L(\varphi, Z_2) = 1$. Since S_q commutes with suspension the same argument works for SCP^i.

Theorem 4.4 raises the following question:

QUESTION 4.5. If $SX \times Y$ has f.p.p., does this imply that $X \times Y$ has f.p.p.?

An affirmative answer to this question would settle the following conjecture.

CONJECTURE 4.6. Suppose X and Y belong to $\mathscr{S}_0$ and X and all its suspension have f.p.p. Then if Y has f.p.p., so does $X \times Y$.

The technique used to prove Theorem 4.2 can also be used to prove the following.

THEOREM 4.7. *Suppose X and Y belong to $\mathscr{S}_0$ (or $\mathscr{M}_0$) and have* f.p.p. *Suppose further that $H^*(X)$ is a truncated polynomial ring on a single generator $u \in H^k(X)$. Then, if $H^k(Y) = 0$, $X \times Y$ has* f.p.p.

EXAMPLE. $CP^i \times HP^j$, where i is even ($i, j \geqq 2$) has f.p.p. The argument that HP^j has f.p.p. goes as follows. First of all, if φ is a self-map of HP^j, then working over the rational field

$$L(\varphi) = 1 + a + a^2 + \cdots + a^j \tag{7}$$

where $\varphi^*(u) = au$, u a generator in $H^4(HP^j)$. Of course, if j is even we're done, since $L(\varphi) \neq 0$ in this case. If j is odd, $j \geqq 3$ we need only preclude the case $a = -1$. Working over Z_3, we may assume that $P^1u = u^2$ in $H^8(HP^j; Z_3)$. This forces

$$a^2 \equiv a \pmod 3 \tag{8}$$

which precludes $a = -1$.

REMARK. G. Bredon was the first to observe that HP^3 has f.p.p. using the above argument.

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Received August 18, 1969. This paper was supported in part by the National Science Foundation under Grant NSF GR-8427 and the Wisconsin Alumni Research Foundation.

THE UNIVERSITY OF WISCONSIN

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

TANGENTIAL CAUCHY-RIEMANN EQUATIONS AND UNIFORM APPROXIMATION

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A smooth ($\mathscr{E}^\infty$) function on a smooth real submanifold M of complex Euclidean space $\mathbf{C}^n$ is a *CR function* if it satisfies the Cauchy-Riemann equations tangential to M. It is shown that each CR function admits an extension to an open neighborhood of M in $\mathbf{C}^n$ whose $\bar{z}$-derivatives all vanish on M to a prescribed high order, provided that the system of tangential Cauchy-Riemann equations has minimal rank throughout M. This result is applied to show that on a holomorphically convex compact set in M each CR fuction can be uniformly approximated by holomorphic functions.

1. Extension and approximation of CR functions.

Each point p of a smooth real submanifold M of $\mathbf{C}^n$ has a *complex tangent space* H_pM. It is the largest complex-linear subspace of the ordinary real tangent space T_pM; evidently $H_pM = T_pM \cap iT_pM$. Its complex dimension is the *complex rank* of M at p. The theorem of linear algebra relating the real dimensions of T_pM, iT_pM and their sum and intersection shows that if M has real codimension k its complex rank is not less than $n - k$.

DEFINITION 1.1. M is a *CR manifold* if its complex rank is constant. It is *generic* if in addition this rank is minimal; that is, equal to the larger of 0 and $n - k$. A smooth function f on M is a *CR function* if $\ker \bar{\partial}_p f \supset H_pM$ for each p in M.

Here f is assumed to be extended in a smooth manner to an open neighborhood of M and $\bar{\partial}_p f$ is regarded as the conjugate complex-linear part of the ordinary Fréchet differential d_pf. Since the condition on $\bar{\partial}_p f$ is independent of the extension chosen, the definition makes sense. Computational equivalents to it and some elaboration are given in § 2. A more comprehensive treatment of these ideas is found in the paper by S. Greenfield [1]. It should be mentioned that his definition [1, Definition II. A. 1] of CR manifolds also requires that the distribution $p \rightarrow H_pM$ be involutive. That assumption is not needed here.

If M is a complex submanifold of $\mathbf{C}^n$, then it is CR with complex rank equal to its complex dimension. It is not generic if it has positive codimension. Of course the CR functions on M are just its holomorphic functions.

At the other extreme, every real hypersurface is a generic CR

manifold of complex rank $n-1$. These frequently have no nontrivial complex submanifolds, which is true for example of the usual $2n-1$ sphere *in* $\mathbf{C}^n$.

M is a generic CR manifold if its complex rank is everywhere zero, which is the *totally real* [5] case.

An example of a proper generic CR submanifold which is neither totally real nor a hypersurface can of course only be found if $n \geqq 3$. There is one in $\mathbf{C}^3$, a 4-sphere S^4 given as the intersection of the usual 5-sphere and a real hyperplane transverse to it. Let

$$\rho_1 = |z_1|^2 + |z_2|^2 + |z_3|^2 - 1$$

and $\rho_2 = z_3 + \bar{z}_3$, where z_1, z_2, z_3 are the usual coordinates for $\mathbf{C}^3$, and let $S^4 = \{\rho_1 = \rho_2 = 0\}$. It follows from (2.2) below that S^4 has the requisite properties. Furthermore, S^4 has no nontrivial complex submanifolds (since the 5-sphere does not).

THEOREM 1.2. *If f is a CR function on a generic CR manifold M in $\mathbf{C}^n$ and m is a nonnegative integer, then there is an extension of f to a smooth function f_m on an open set $U \supset M$ such that $\bar{\partial} f_m$ vanishes on M to order m in all directions.*

This result is known [3, Lemma 4.3] and [5, Lemma 3.1] when M is totally real. It is also proved in [2, Th. 2.3.2′] when M is a real hypersurface. A local version which does not require that M be generic is proved in [5, Lemma 3.3].

Theorem 1.2 plays a key role in a program outlined by L. Hörmander for showing that CR functions can be uniformly approximated by holomorphic functions. The basic idea is to take a compact set K in M and a given CR function f on M and find a solution g of $\bar{\partial} g = \bar{\partial} f$ with $\sup_K |g|$ small. Then $u = f - g$ is holomorphic and approximates f uniformly on K with error no larger than $\sup_K |g|$.

In Hörmander's implementation of this idea, Theorem 1.2 implies that a certain bound on an L^2 norm of the Sobolev type is imposed on $\bar{\partial} g$. The existence of solutions to $\bar{\partial} g = \bar{\partial} f$ subject to the same a priori bound [2] and a Sobolev inequality are used to estimate $\sup_K |g|$. This proof appears in [3] and [5] for the cases cited above. Since the only step of it which depends on the complex rank of M is the conclusion of Theorem 1.2, this proof will, without further modification, yield a result on uniform approximation.

THEOREM 1.3. *If M is a closed generic CR submanifold of a domain of holomorphy U in $\mathbf{C}^n$ and K is a compact subset of M holomorphically convex with respect to U, then each smooth CR func-*

tion on M is a uniform limit on K of functions holomorphic on U.

In fact, the same method in conjunction with Theorem 1.2 will prove the stronger statement that approximation holds in the $\mathscr{C}^\infty$ topology; c.f. [5, Th. 6.1]. One merely replaces $\sup_K |g|$ by a $\mathscr{C}^k$ norm of g on K.

In the totally real case, it is known that the holomorphic convexity of any given compact subset K with respect to *some* domain of holomorphy is a consequence of the absence of complex tangent vectors. This follows from the fact [3, Th. 3.1] and [5, Corollary 4.2] that each K has arbitrarily small tubular neighborhoods which are domains of holomorphy. However, the case of the $2n-1$ sphere in $\mathbf{C}^n$ shows that in the presence of complex tangent vectors holomorphic convexity must be assumed. When there is complex tangency, the problem of determining holomorphic convexity of a given compact subset of M is very difficult, even for the examples mentioned above.

It should be remarked that in Definition 1.1 and Theorem 1.2 $\mathbf{C}^n$ may be replaced by any complex manifold, and if this manifold is Stein [2], it may replace U in Theorem 1.3. No significant modification of the exposition is required.

2. *CR* manifolds and functions. Each real-linear map $L: \mathbf{C}^n \to \mathbf{C}^k$ is uniquely expressible as a sum $L = S + T$ where $S, T: \mathbf{C}^n \to \mathbf{C}^k$, S is complex linear, and T is conjugate complex linear. If $J: v \to iv$, a direct computation shows that $S = \frac{1}{2}(L - JLJ)$ and $T = \frac{1}{2}(L + JLJ)$. Applying this result to the Fréchet differential $d_p\rho$ of a smooth map $\rho: \mathbf{C}^n \to \mathbf{C}^k$ at p there results

$$d_p\rho = \partial_p\rho + \bar{\partial}_p\rho$$

in which $\partial_p\rho$ is the complex linear part of $d_p\rho$ and $\bar{\partial}_p\rho$ the conjugate complex linear part.

Each point of M has an open neighborhood U in $\mathbf{C}^n$ on which there exists a smooth map $\rho = (\rho_1, \cdots, \rho_k): U \to \mathbf{R}^k$ with maximal rank k on U and satisfying

$$M \cap U = \{z \in U : \rho(z) = 0\} . \tag{2.1}$$

Regarding $\mathbf{R}^k$ as contained in $\mathbf{C}^k$ in the usual way, and applying the remarks above to Definition 1.1, it follows that M is *CR* if and only if $\bar{\partial}\rho$ has constant complex rank on $M \cap U$, and is generic exactly when this rank is maximal. When $k \geqq n$ this means that $H_pM = 0$, which is the totally real case. The case of interest here is $k \leqq n$, when M is generic if and only if $\bar{\partial}\rho$ has complex rank k on $M \cap U$. Henceforth, it is assumed that $k \leqq n$. Since it is clear that $\bar{\partial}\rho = (\bar{\partial}\rho_1, \cdots, \bar{\partial}\rho_k)$ it follows that the condition

$$\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \quad \text{has no zeros on } M \cap U \tag{2.2}$$

is necessary and sufficient that M be a generic CR manifold.

From Definition 1.1 and (2.2) it follows that a smooth function f on M is CR if and only if

$$\bar{\partial}f \wedge \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k = 0 \quad \text{on } M. \tag{2.3}$$

Equivalently, since $\{\bar{\partial}\rho_1, \cdots, \bar{\partial}\rho_k\}$ is, at points of M, by virtue of (2.2) part of a basis for the space of conjugate-linear functionals on $\mathbf{C}^n$, there exist smooth functions $h_1, \cdots, h_k$ on U such that

$$\bar{\partial}f = \sum_{j=1}^{k} h_j \bar{\partial}\rho_j + O(\rho) . \tag{2.4}$$

Here $O(\rho)$ denotes a form which vanishes on $M \cap U$. It is a standard result [4, Lemma 2.1] that if g is a smooth $O(\rho)$-form there exist smooth forms $g_1, \cdots, g_k$ such that

$$g = \sum_{j=1}^{k} \rho_j g_j . \tag{2.5}$$

More generally, $O(\rho^m)$ will denote a smooth form on U which vanishes on $M \cap U$ to order m. Induction on m using (2.5) shows that if g is such a form there are smooth forms g_α on U satisfying

$$g = \sum_{|\alpha|=m} \rho^\alpha g_\alpha , \tag{2.6}$$

in which the standard multi-index notation has been used. Thus $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a k-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_k$, and $\rho^\alpha = \rho_1^{\alpha_1} \cdots \rho_k^{\alpha_k}$. The coefficients g_α are not unique on U, but the fact that they are determined on $M \cap U$ will be essential.

LEMMA 2.1. *If smooth forms g, g_α are related on U by*

$$g = \sum_{|\alpha|=m} \rho^\alpha g_\alpha + O(\rho^{m+1})$$

then for each α, $D^\alpha g \mid M \cap U = \alpha! g_\alpha \mid M \cap U$. In particular, if $g = 0$ on U then each $g_\alpha \mid M \cap U = 0$.

Here $D^\alpha = D_1^{\alpha_1} \cdots D_k^{\alpha_k}$, where D_j denotes differentiation with respect to ρ_j and $\alpha! = \alpha_1! \cdots \alpha_k!$.

Proof. The statement is local and since ρ has rank k, the proof can be reduced to the case where each $\rho_j = x_j$, the jth ordinary Euclidean coordinate function. Then the lemma follows from the general Leibniz formula

$$D^\alpha(fg) = \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} D^\gamma f \cdot D^{\alpha-\gamma} g$$

with $f = x^\alpha$, noting that $D^\gamma x^\alpha = 0$ on $M \cap U$ if $\gamma < \alpha$ and $D^\alpha x^\alpha = \alpha!$. Here $\binom{\alpha}{\gamma} = \alpha!/\gamma!(\alpha - \gamma)!$ and $\gamma < \alpha$ means that $\gamma_j < \alpha_j$ for some j.

3. **Proof of Theorem 1.2.** The proof is an induction on m in which f_{m+1} is obtained by subtraction of an $O(\rho^{m+1})$ function from f_m. Similar procedures have been used in [2, Th. 2.3.2′], [3, Lemma 4.3], and [5, Lemmas 3.1 and 3.3]. The one used here borrows ideas from all of these. Since the totally real generic cases where $k \geqq n$ are treated in [3] and [5], it will be assumed that $k \leqq n$. However, the proof below can be read with $k \geqq n$, with some slight modifications.

In the presence of complex tangent vectors, the only known result is local in nature [5, Lemma 3.3]. Its proof refers to a particular local coordinate system for $\mathbf{C}^n$ and uses an initial extension f_0 which is independent of the coordinates normal to M. This feature is clearly not preserved by the patching construction intended here, so an arbitrary extension of f must be admitted at each step. This introduces remainder terms of the form $O(\rho^m)$, and it is necessary to keep an accurate account of their effects.

To begin the induction, extend a given CR function f from M to a smooth function f_0 on an open set $U \supset M$.

First assume that the representation (2.1) holds on U. Then $\bar\partial f_0$ is of the form (2.4) and if $u = \sum_{j=1}^k \rho_j h_j$ it is clear that $\bar\partial(f_0 - u) = O(\rho)$.

In general U has a locally finite cover by open sets U_ι on each of which there exists a defining function ρ_ι presenting $M \cap U_\iota$ as in (2.1) and a $O(\rho_\iota)$ function u_ι satisfying $\bar\partial(f_0 - u_\iota) = O(\rho_\iota)$ on U_ι. If $\{\varphi_\iota\}$ is a partition of unity subordinate to $\{U_\iota\}$ and

$$u = \sum_\iota \varphi_\iota u_\iota \tag{3.1}$$

then

$$\bar\partial(f_0 - u) = \sum_\iota \varphi_\iota \bar\partial(f_0 - u_\iota) - \sum_\iota u_\iota \bar\partial\varphi_\iota \,. \tag{3.2}$$

By construction each term of either sum in (3.2) vanishes on M. Therefore so does $\bar\partial f_1$ if $f_1 = f_0 - u$.

For the inductive step assume that $m > 0$ and f has an extension f_m to U such that $\bar\partial f_m$ vanishes on M to order m. A global modification of f_m will again be obtained by patching local ones, so the construction is again begun by assuming that M is globally presented by (2.1).

Then by (2.6) there are smooth (0, 1) forms g_α such that

$$\bar\partial f_m = \sum_{|\alpha|=m} \rho^\alpha g_\alpha . \tag{3.3}$$

Hence

$$0 = \bar\partial^2 f_m = \sum_{|\alpha|=m} \sum_{j=1}^{k} \alpha_j \rho^{\alpha-j} \bar\partial\rho_j \wedge g_\alpha + O(\rho^m) , \tag{3.4}$$

in which $\alpha - j$ denotes $(\alpha_1, \cdots, \alpha_j - 1, \cdots, \alpha_k)$ if $\alpha_j > 0$. Wedge this equation with $\bar\partial\rho_1 \wedge \cdots \wedge \widehat{\bar\partial\rho_j} \wedge \cdots \wedge \bar\partial\rho_k$ ($\bar\partial\rho_j$ is missing) to show that for each j

$$0 = \sum_{|\alpha|=m} \alpha_j \rho^{\alpha-j} \bar\partial\rho_1 \wedge \cdots \wedge \bar\partial\rho_k \wedge g_\alpha + O(\rho^m) . \tag{3.5}$$

Now for fixed j, the map $\alpha \to \alpha - j$ is a one-to-one correspondence of $\{\alpha: |\alpha| = m \text{ and } \alpha_j > 0\}$ with $\{\beta: |\beta| = m - 1\}$. Therefore (3.5) may be rewritten as

$$0 = \sum_{|\beta|=m-1} (\beta_j + 1)\rho^\beta \bar\partial\rho_1 \wedge \cdots \wedge \bar\partial\rho_k \wedge g_{\beta+j} + O(\rho^m)$$

and Lemma 2.1 applied to deduce that $g_{\beta+j} \wedge \bar\partial\rho_1 \wedge \cdots \wedge \bar\partial\rho_k = 0$ on M. Since this holds for every j and β, it follows from the linear independence of $\bar\partial\rho_1, \cdots, \bar\partial\rho_k$ on M that for each α, $|\alpha| = m$, and each j, $1 \leqq j \leqq k$, there is a function $h_{\alpha j}$ such that

$$g_\alpha = \sum_{j=1}^{k} h_{\alpha j} \bar\partial\rho_j + O(\rho) . \tag{3.6}$$

When substituted for g_α in (3.3) and (3.4) this relation yields

$$\bar\partial f_m = \sum_{|\alpha|=m} \sum_{j=1}^{k} \rho^\alpha h_{\alpha j} \bar\partial\rho_j + O(\rho^{m+1}) \tag{3.7}$$

and

$$0 = \sum_{|\alpha|=m} \sum_{i,j=1}^{k} \alpha_j \rho^{\alpha-j} h_{\alpha i} \bar\partial\rho_j \wedge \bar\partial\rho_i + O(\rho^m) . \tag{3.8}$$

The expression (3.7) suggests modifying f_m by

$$u = \frac{1}{n+1} \sum_{|\alpha|=m} \sum_{j=1}^{k} \rho^\alpha \rho_j h_{\alpha j}$$

(the need for the constant $1/(n + 1)$ will appear as a consequence of (3.11)). Now

$$(n + 1)\bar\partial u = \sum_{\alpha,j} \rho^\alpha h_{\alpha j} \bar\partial\rho_j + \sum_{\alpha,j} \sum_{i=1}^{k} \rho_j \alpha_i \rho^{\alpha-i} h_{\alpha j} \bar\partial\rho_i + \sum_{\alpha,j} \rho^\alpha \rho_j \bar\partial h_{\alpha j} . \tag{3.9}$$

The first term of this is $\bar{\partial} f_m$. The second is

$$\sum_{i,j=1}^{k} \rho_j \Bigl(\sum_{|\alpha|=m} \alpha_i \rho^{\alpha-i} h_{\alpha j} \Bigr) \bar{\partial} \rho_i \,, \tag{3.10}$$

which will be shown to equal $n\bar{\partial} f_m + O(\rho^{m+1})$.

To that end, for each $i < j$, wedging (3.8) with

$$\bar{\partial}\rho_1 \wedge \cdots \wedge \widehat{\bar{\partial}\rho_i} \wedge \cdots \wedge \widehat{\bar{\partial}\rho_j} \wedge \cdots \wedge \bar{\partial}\rho_k$$

($\bar{\partial}\rho_i$ and $\bar{\partial}\rho_j$ are missing) gives the symmetry relation

$$0 = \sum_{|\alpha|=m} (\alpha_j \rho^{\alpha-j} h_{\alpha i} - \alpha_i \rho^{\alpha-i} h_{\alpha j}) + O(\rho^m) \,. \tag{3.11}$$

Using this in (3.10) it becomes

$$\sum_{i,j=1}^{k} \rho_j \Bigl(\sum_{|\alpha|=m} \alpha_j \rho^{\alpha-j} h_{\alpha i} \Bigr) \bar{\partial}\rho_i + O(\rho^{m+1})$$

which when the summation over j is performed first is

$$\sum_{|\alpha|=m} \sum_{i=1}^{k} \Bigl(\sum_{j=1}^{k} \alpha_j \Bigr) \rho^{\alpha} h_{\alpha i} \bar{\partial}\rho_i + O(\rho^{m+1}) \,.$$

Noting that $\sum_{j=1}^{k} \alpha_j = n$ completes the argument that the second term of (3.9) is $n\bar{\partial} f_m + O(\rho^{m+1})$. Therefore $\bar{\partial} u = \bar{\partial} f_m + O(\rho^{m+1})$.

Thus on each U_ι there is a function $u_\iota = O(\rho_\iota^{m+1})$ such that $\bar{\partial}(f_m - u_\iota) \,|\, U_\iota = O(\rho_\iota^{m+1})$. With u defined again by (3.1) and $f_{m+1} = f_m - u$ it follows as before from (3.2) that $\bar{\partial} f_{m+1}$ vanishes on M to order $m+1$. This completes the proof.

4. **Remarks.** We know of no nongeneric examples where Theorem 1.2 fails. However, when M is not generic, the above proof breaks down at the inductive step from $m = 1$ to $m = 2$: Since $\bar{\partial}\rho$ does not have maximal rank it may be assumed that there is an integer $l < k$ such that $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_l$ has no zeros on M but $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_j = 0$ on M if $j > l$. Thus there are more unknowns g_α than equations available from (3.4). There are very simple cases where this occurs:

EXAMPLE 4.1. If the usual coordinates of $\mathbf{C}^2$ are denoted z_1, z_2 and $M = \{z\colon z_2 = 0\}$ then the function $f = z_2 \bar{z}_1$ is CR, for $\bar{\partial} f = z_2 d\bar{z}_1$. The most general function u vanishing to second order on M is by (the complex analogue of (2.5)) of the form

$$u = z_2^2 g_1 + z_2 \bar{z}_2 g_2 + \bar{z}_2^2 g_3$$

for suitable smooth functions g_1, g_2, and g_3. Therefore

$$\bar{\partial} u = z_2^2 \bar{\partial} g_1 + z_2 g_2 d\bar{z}_2 + z_2 \bar{z}_2 \bar{\partial} g_2 + 2\bar{z}_2 g_3 d\bar{z}_2 + \bar{z}_2^2 \bar{\partial} g_3 \,.$$

Each of these terms either vanishes to second order on M or is linearly independent of $\bar{\partial} f$. Therefore no such u will satisfy $\bar{\partial}(f - u) = O(\rho^2)$.

However since f is zero on M, it obviously satisfies the conclusion of Theorem 1.2. In fact, if M is a complex manifold, each CR function f is holomorphic, so if U is a domain of holomorphy Theorem 1.2 for U and $M \cap U$ follows from Cartan's Theorem B [2], which implies that f has a holomorphic extension to U. Moreover, standard results in several complex variables show that Theorem 1.3 is true for any complex manifold M. Thus Theorem 1.2 and a consequent Theorem 1.3 may still hold in the nongeneric case, but some new ideas for proof are necessary.

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Received November 21, 1969. This work was supported by NSF Grant GP-8997.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

TORSION CLASSES AND PURE SUBGROUPS

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In this note we obtain a classification of the classes $\mathscr{T}$ of abelian groups satisfying the following closure conditions:
(i) If $\{A_\mu \mid \mu \in M\} \subseteqq \mathscr{T}$, then $\mathscr{T}$ contains the direct sum $\sum A_\mu$.

For a short exact sequence

$$(*) \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

(ii) $C \in \mathscr{T}$ if $B \in \mathscr{T}$
(iii) $B \in \mathscr{T}$ if $A, C \in \mathscr{T}$
(iv) $A \in \mathscr{T}$ if $B \in \mathscr{T}$ and (*) is pure.

Classes satisfying (i), (ii) and (iii) are called *torsion classes* (of abelian groups) and were first studied by Dickson [2], who classified those which contain only torsion groups and showed that the general classification problem reduces, essentially, to that for torsion classes determined (in the sense of § 2 below) by torsion-free groups. The torsion classes which are closed under taking subgroups (called *strongly-complete Serre classes*) can be described quite simply ([1], [2], [10]). A possible approach to the general problem is to investigate torsion classes closed under taking the subgroups corresponding to *proper classes* of monomorphisms as used in relative homological algebra (see for example [8], pp. 367 *et seqq.*), and herein lies the motivation for the present paper.

1. Notation. "Group" means "abelian group" throughout. $h(x)$ denotes the height of an element of a torsion-free group $\tau(x)$ its type and $\tau(X)$ the type of a rational group X. An *S-group*, where S is a set of primes, is a group whose elements have orders belonging to the multiplicative semigroup S^* generated by S. A group A is *p-divisible* for a prime p if $pA = A$ and *S-divisible* if p-divisible for each $p \in S$. $\mathscr{T}_0$, $\mathscr{F}_0$ are the classes of all torsion and torsion-free groups respectively. For a group A, A_t is the torsion subgroup, A_p its p-primary component. The direct sum (or discrete direct sum) of a set of groups $\{A_\mu \mid \mu \in M\}$ is denoted by $\sum A_\mu$, the direct product (or complete direct sum) by $\sum^* A_\mu$ and an element of either by (a_μ). $[A, B]$ is the group of homomorphisms from a group A to a group B. If a is an element of a torsion-free group A, $[a]$ denotes the cyclic subgroup it generates, $[a]_*$ the smallest pure subgroup containing it. Z is the group of integers, Q the (additive) group of rational numbers, $Z(p)$ the cyclic group of order p, $Z(p^\infty)$ the quasicyclic p-group. For a set S of primes, $Q(S)$ is the subgroup $\{m/n \mid m \in Z, n \in S^*\}$ of Q and for a prime p, $Q(p) = \{m/p^n \mid m, n \in Z, n \geqq 0\}$. I_p is the group or ring of p-adic integers.

For unexplained terms see [4].

2. Torsion classes.

We begin by listing some properties of torsion classes for later use.

For a class $\mathscr{C}$ of groups we write $T(\mathscr{C})$ for the torsion class determined by $\mathscr{C}$, i.e. the smallest torsion class $\mathscr{T}$ with $\mathscr{C} \subseteqq \mathscr{T}$ but if $\mathscr{C}$ has a single member C, $T(C)$ rather than $T(\{C\})$ will be used.

T1. *$A \in T(\mathscr{C})$ if and only if $[A, B] = 0$ whenever $[C, B] = 0$ for all $C \in \mathscr{C}$.* [3].

$T(\mathscr{C})$ is also the lower radical class determined by $\mathscr{C}$, in the sense of Kurosh [7]-Shul'geifer [9], so by the simplified version of the Kurosh construction which applies in an abelian category, we obtain

T2. *$A \in T(\mathscr{C})$ if and only if every nonzero homomorphic image B of A has a nonzero subgroup which is a homomorphic image of some $C \in \mathscr{C}$, i.e., $[C, B] \neq 0$.*

A torsion class $\mathscr{T}$ will be called a *t-torsion class* if it contains only torsion groups.

T3. *Let S_1, S_2 be disjoint sets of primes and let $\mathscr{T}$ be the class of all groups of the form $A_1 \oplus A_2$, where A_1 is an S_1-group and A_2 a divisible S_2-group. Then $\mathscr{T}$ is the t-torsion class*

$$T(\{Z(p) \mid p \in S_1\} \cup \{Z(p^\infty) \mid p \in S_2\}) .$$

Any t-torsion class is uniquely represented in this way. [2].

T4. *Let $\mathscr{T}$ be a torsion class and p a prime. Then either $Z(p) \in \mathscr{T}$ or every group in $\mathscr{T}$ is p-divisible* [2].

PROPOSITION 2.1. *If $\mathscr{T}$ is a torsion class containing a torsion-free group A, then $Z(p^\infty) \in \mathscr{T}$ for every prime p.*

Proof. If $Z(p) \in \mathscr{T}$, then $\mathscr{T}$ contains all p-groups (T3); if not, then A is p-divisible, so $\tau([a]_*) \geqq \tau(Q(p))$ for any nonzero $a \in A$. Thus $A/[a]$ has a subgroup and therefore a direct summand isomorphic to $Z(p^\infty)$, i.e. $Z(p^\infty)$ is a homomorphic image of A.

T5. *A torsion class $\mathscr{T}$ contains a group A if and only if A_t*

and $A/A_t \in \mathscr{T}$ [2].

T6. *Any torsion class* $\mathscr{T}$ *satisfies the equality*

$$\mathscr{T} = T([\mathscr{T} \cap \mathscr{T}_0] \cup [\mathscr{T} \cap \mathscr{F}_0]) .$$

[2].

T7. *$T(Q(S))$ is the class of S-divisible groups, for any set S of primes.* (Cf. [2], Proposition 4.1.)

3. A simplification of the problem. As a first step, we show that every torsion class closed under taking pure subgroups is either a t-torsion class or is determined by rational and torsion groups. A class of the latter kind will be called an *r.t.-torsion class.*

PROPOSITION 3.1. *All t-torsion classes are closed under taking pure subgroups.*

Proof. Let S_1, S_2 be disjoint sets of primes. If A_1 is an S_1-group and A_2 a divisible S_2-group, then clearly any pure subgroup of $A_1 \oplus A_2$ is the direct sum of an S_1-group and a divisible S_2-group.

THEOREM 3.2. *A torsion class* $\mathscr{T}$ *is closed under taking pure subgroups if and only if* $\mathscr{T} \cap \mathscr{F}_0$ *is.*

Proof. Let A' be a pure subgroup of $A \in \mathscr{T}$, and consider the induced diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A'_t \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A'/A'_t \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A_t \arrow[r] & A \arrow[r] & A/A_t \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns, where g is defined by $g(a' + A'_t) = a' + A_t$. A'_t is pure in A' and hence in A. Therefore A'_t is pure in A_t so by Proposition 3.1, $A'_t \in \mathscr{T} \cap \mathscr{T}_0$. The kernel of g is $A' \cap A_t/A'_t = 0$. If, for some nonzero $n \in Z$, $a' \in A'$ and $a \in A$ we have $g(a' + A'_t) = n(a + A_t)$, then $m(a' - na) = 0$ for some nonzero $m \in Z$, i.e. $ma' = mna$. Since A' is pure in A, there exists $a'' \in A'$ with $ma' = mna''$. But then $g(a' + A'_t) = ng(a'' + A'_t)$, so that g is a pure monomorphism. Thus if $\mathscr{T} \cap \mathscr{F}_0$ is closed under taking pure subgroups, $A'/A'_t \in \mathscr{T} \cap \mathscr{F}_0$

so $A' \in \mathscr{T}$ and $\mathscr{T}$ is therefore closed under taking pure subgroups. The converse is obvious.

THEOREM 3.3. *If a torsion class $\mathscr{T}$ is closed under taking pure subgroups, then*

$$\mathscr{T} = T([\mathscr{T} \cap \mathscr{T}_0] \cup \overline{\mathscr{T}})$$

where $\overline{\mathscr{T}}$ is the class of rational groups in $\mathscr{T}$.

The proof uses the following lemmas:

LEMMA 3.4. *For $\mathscr{T}$ and $\overline{\mathscr{T}}$ as in Theorem 3.3, $\mathscr{T} \cap \mathscr{F}_0 = T(\overline{\mathscr{T}}) \cap \mathscr{F}_0$.*

Proof. Clearly $\mathscr{T} \cap \mathscr{F}_0 \supseteqq T(\overline{\mathscr{T}}) \cap \mathscr{F}_0$. Let A be any group in $\mathscr{T} \cap \mathscr{F}_0$. Then A is a homomorphic image of $\sum [a]_*$ where the sum extends over all $a \in A$ and each $[a]_* \in \overline{\mathscr{T}}$, so $A \in T(\overline{\mathscr{T}})$.

LEMMA 3.5. *For any two classes $\mathscr{C}_1$, $\mathscr{C}_2$ of groups, $T(\mathscr{C}_1 \cup \mathscr{C}_2) = T(T[\mathscr{C}_1] \cup T[\mathscr{C}_2])$.*

To complete the proof of Theorem 3.3, we observe that

$$\begin{aligned}\mathscr{T} &= T([\mathscr{T} \cap \mathscr{T}_0] \cup [\mathscr{T} \cap \mathscr{F}_0]) = T([\mathscr{T} \cap \mathscr{T}_0] \cup [T(\overline{\mathscr{T}}) \cap \mathscr{F}_0]) \\ &\subseteqq T([\mathscr{T} \cap \mathscr{T}_0] \cup T(\overline{\mathscr{T}})) \qquad = T([\mathscr{T} \cap \mathscr{T}_0] \cup \overline{\mathscr{T}}) \subseteqq \mathscr{T} .\end{aligned}$$

We conclude this section by showing that not every r.t. torsion class is closed under taking pure subgroups.

PROPOSITION 3.6. *Let $\mathscr{T}$ be a torsion class closed under taking pure subgroups and Γ the set of types of rational groups in $\mathscr{T}$. If $\gamma, \delta \in \Gamma$, then $\gamma \cap \delta \in \Gamma$.*

Proof. Let X and Y be rational groups with $\tau(X) = \gamma$ and $\tau(Y) = \delta$. Then $X \oplus Y$ has elements and therefore pure rational subgroups of type $\gamma \cap \delta$.

Thus for example if p and q are distinct primes, $T(\{Q(p), Q(q)\})$ is not closed under taking pure subgroups since $\tau(Q(p)) \cap \tau(Q(q)) = \tau(Z)$ and $[Q(p), Z] = 0 = [Q(q), Z]$.

4. The main results. In this section we obtain an explicit characterization of the torsion classes closed under taking pure subgroups.

LEMMA 4.1. *Let X be a rational group and $S = \{p \text{ prime} \mid X$ is*

p-divisible}. Then $I_p \in T(X)$ whenever $p \notin S$.

Proof. Let P be the set of primes distinct from p. Then $I_p \in T(Q(P))$(T7). Also, there is a short exact sequence

$$0 \longrightarrow X \longrightarrow Q(P) \longrightarrow \sum Z(q^\infty) \longrightarrow 0$$

where q ranges over $P - S$. Since $\sum Z(q^\infty) \in T(X)$(Proposition 2.1), it follows that $T(X)$ contains $Q(P)$ and hence I_p.

The main result can now be stated.

THEOREM 4.2. *A torsion class $\mathscr{T}$ is closed under taking pure subgroups if and only if either*
(i) $\mathscr{T}$ is a t-torsion class
or (ii) $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$, where P and S are sets of primes with $P \subseteqq S$.

The proof of Theorem 4.2 will be accomplished in several stages. We first prove

LEMMA 4.3. *Let $\{X_\mu \mid \mu \in M\}$ be a set of rational groups. Let $A = \sum X_\mu$ and $S = \{p$ prime $\mid A$ is p-divisible$\}$. Then $T(\{X_\mu \mid \mu \in M\})$ contains $\sum {}^*A_i$, $i = 1, 2, 3, \cdots$, where each $A_i = A$.*

Proof. Let $f\colon \sum {}^*A_i \rightarrow Y$ be a nonzero epimorphism. We show that $[X_\mu, Y] \neq 0$ for at least one value of μ.

If $Y_p \neq 0$ for some p, then since Y is S-divisible, so is Y_p. If $p \in S$, Y_p is therefore a direct sum of copies of $Z(p^\infty)$ so by Proposition 2.1, $Y_p \in T(X_\mu)$ for each μ and *a fortiori* $[X_\mu, Y] \neq 0$ for all μ. If $p \notin S$, then at least one X_μ is p-reduced, whence $[X_\mu, Y_p] \neq 0$.

If Y is torsion-free, there are two possibilities. If $f((a_i)) \neq 0$ for some (a_i) with almost all $a_i = 0$, then f induces a nonzero map from some A_i, and hence from some X_μ, into Y, while if $f((a_i)) = 0$ whenever $a_i = 0$ for almost all values of i, then f factorizes as

$$\begin{tikzcd} \sum^* A_i \arrow[r, "f"] \arrow[d] & Y \\ \sum^* A_i/\sum A_i \arrow[ur, dashed] & \end{tikzcd}$$

where the other maps are epimorphisms. $\sum^* A_i/\sum A_i$ is algebraically compact (see [6]), and also torsion-free, since $\sum A_i$ is a pure subgroup of $\sum^* A_i$. Thus $\sum^* A_i/\sum A_i$ is the direct sum of a divisible group and a (reduced) cotorsion group [5]; so, therefore, is Y, which being torsion-free is algebraically compact [5]. Since Y is S-divisible, it has the form $D \oplus \sum^* R_p$, $p \notin S$ where each R_p is *inter alia* a reduced I_p-module and D is divisible. If $D \neq 0$ then for each $\mu \in M$ there are monomorphisms $X_\mu \to Q \to D$. If $D = 0$, let $R_p \neq 0$. Then at least one X_μ is p-reduced, so by Lemma 4.1, $I_p \in T(X_\mu)$. Since there is an epimorphism (an I_p-epimorphism) from a direct sum of copies of I_p to R_p, we have $R_p \in T(I_p) \subseteqq T(X_\mu)$, so $[X_\mu, R_p] \neq 0$ and the proof is complete.

The next step is to show when $T(\{X_\mu \mid \mu \in M\})$ is closed under taking pure subgroups.

LEMMA 4.4. *With the notation of Lemma* 4.3, *if* $T(\{X_\mu \mid \mu \in M\})$ *is closed under taking pure subgroups, it contains* $Q(S)$.

Proof. Let $p_1, p_2, p_3, \cdots$ be the natural enumeration of the primes, and let $J = \{i \mid p_i \notin S\}$. For each $j \in J$, choose $a_j \in A$ with $h_j(a_j) = 0$, where h_j denotes height at p_j. For example, let $a_j = (x_{j\mu})$ with $x_{j\mu} \in X_\mu$ satisfying the following conditions: (i) $x_{j\lambda} \neq 0$ for some $\lambda \in M$ for which X_λ is p_j-reduced; (ii) $h_j(x_{j\lambda}) = 0$; (iii) $x_{j\mu} = 0$ for $\mu \neq \lambda$. For a natural number $i \notin J$, let a_i be an arbitrary element of A, and regard the resulting (a_i) as an element of a group $\sum^* A_i$, $i = 1, 2, 3, \cdots$. Then $h((a_i)) = \bigcap_{i=1}^{\infty} h(a_i)$. In particular, $h_j((a_i)) = 0$. Therefore, since $\sum^* A_i$ is S-divisible, the height of (a_i) at a prime p is infinite if $p \in S$ and zero otherwise, i.e., $\tau((a_i)) = \tau(Q(S))$ and $\sum^* A_i$ has a pure subgroup isomorphic to $Q(S)$. By Lemma 4.3 and assumption, therefore, $Q(S) \in T(\{X_\mu \mid \mu \in M\})$.

Since each X_μ is S-divisible and $T(Q(S))$ is the class of all S-divisible groups (T7) we have

COROLLARY 4.5. *With the notation of Lemma* 4.3, *if* $T(\{X_\mu \mid \mu \in M\})$ *is closed under taking pure subgroups, it is the class of all S-divisible groups.*

Proof of Theorem 4.2. Let $\mathscr{T}$ be a torsion class closed under taking pure subgroups. If $\mathscr{T}$ is not a t-torsion class, let Γ be the set of types of rational groups in $\mathscr{T}$ and for each $\gamma \in \Gamma$ let X_γ be a rational group of type γ. Then

$$\mathscr{T} = T([\mathscr{T} \cap \mathscr{T}_0] \cup \{X_\gamma \mid \gamma \in \Gamma\}) \qquad \text{(Theorem 3.3)}$$
$$\text{and } \mathscr{T} \cap \mathscr{F}_0 = T(\{X_\gamma \mid \gamma \in \Gamma\}) \cap \mathscr{F}_0 \qquad \text{(Lemma 3.4).}$$

By Theorem 3.2, $T(\{X_\gamma \mid \gamma \in \Gamma\})$ is closed under taking pure subgroups and therefore, by Corollary 4.5, is the class of all S-divisible groups, where S is the set of all primes dividing $\sum X_\gamma$. Thus $\mathscr{T} =$

$T([\mathscr{T} \cap \mathscr{T}_0] \cup \{Q(S)\})$. Let $P = \{p \in S \mid Z(p) \in \mathscr{T}\}$. Since $T(Q(S)) \subseteqq \mathscr{T}$, $\mathscr{T}$ contains the groups $Z(p^\infty)$ for all primes p as well as $Z(p)$ for primes $p \notin S$. Thus by T3 and Lemma 3.5

$$\begin{aligned}\mathscr{T} &= T(\{Z(p) \mid p \notin S\} \cup \{Z(p) \mid p \in P\} \cup \{Z(p^\infty) \mid \text{all } p\} \cup \{Q(S)\}) \\ &= T(\{Z(p) \mid p \in P\} \cup \{Q(S)\}) .\end{aligned}$$

Conversely, that any class $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ with $P \subseteqq S$ is closed under taking pure subgroups follows from Theorem 3.2, Lemma 3.4 and the observation that $T(Q(S))$ is closed under taking pure subgroups. By Proposition 3.1, the proof is now complete.

Note that by T1, for a torsion class $\mathscr{T}$ which is not a t-torsion class, the representation $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ is unique. We conclude with a characterization of the groups in such a class:

PROPOSITION 4.6. *A group A belongs to $\mathscr{T} = T(\{Z(p) \mid p \in P\} \cup \{Q(S)\})$ where P and S are sets of primes with $P \subseteqq S$, if and only if there is a short exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

where A' is a P-group and A'' is S-divisible.

Proof. Let $A \in \mathscr{T}$ and $A' = \sum A_p$ where the sum extends over all $p \in P$, $A'' = A/A'$. Then A''_t has no P-component and belongs to $\mathscr{T}$(T5) so therefore has divisible S-component. Thus A''_t is S-divisible. A''/A''_t is torsion-free and belongs to $\mathscr{T}$. If not S-divisible, it has a non-zero S-reduced torsion free homomorphic image B. But then $B \in \mathscr{T}$ and $[Q(S), B] = 0 = [Z(p), B]$ for each $p \in P$ and this contradicts T1, so A''/A''_t is S-divisible, whence A'' is also. The converse is obvious.

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Received July 2, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

BOUNDS FOR THE SOLUTIONS OF A CERTAIN CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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This paper is a study of boundedness and other properties of the solutions of nonlinear partial differential equations of the form

$$\Delta u = P(x_1, x_2, \cdots, x_n)f(u) \tag{1.1}$$

where $P(x_1, x_2, \cdots, x_n)$ is positive, and $u(x_1, x_2, \cdots x_n)$ is to be defined in some region of Euclidean n-space, and $\Delta u = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2$ is the Laplacian of u. In particular, we consider the case $f(u) = e^u$.

Our principal result is concerned with the nonexistence of entire solutions. An entire solution $u = u(x_1, x_2, \cdots, x_n)$ will be defined as a solution which though continuous for $0 \leqq r < \infty$ is twice continuously differentiable for $0 < r < \infty$. Other results are concerned with the general form of and explicit bounds for solutions.

In the literature on the subject [3, 4, 5, 8, 9, 11, 12] conditions have been given on $f(u)$ in order that the equation

$$\Delta u = f(u) \tag{1.2}$$

or, more generally, the differential inequality

$$\Delta u \geqq f(u) \tag{1.3}$$

will have no solutions $u = u(x_1, x_2, \cdots, x_n)$ having two continuous derivatives for all finite values of $x_1, x_2, \cdots, x_n$. The most general conditions which exclude such solutions, obtained by Keller [5], are: $f(u) > 0$, $f'(u) \geqq 0$ for $-\infty < u < \infty$ and

$$\int_0^\infty \left[\int_0^u f(t)dt\right]^{-1/2} du < \infty .$$

For $n = 2$ Redheffer [10] showed that the monotonicity of $f(u)$ may be dispensed with.

In § 2 we shall consider a more general question for the equation

$$\Delta u = P(x, y)e^u, \ P(x, y) > 0, \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \tag{1.4}$$

While the coefficient $P(x, y)$ will be assumed to be positive and twice continuously differentiable for $0 < r < \infty$, $P(x, y)$ will be permitted to vanish or to become singular in a manner specified in the statement of the Theorem 2.1. If $P(x, y)$ has such a singularity it will, of course, be reflected in the singular behaviour of the solutions of (1.4). We shall thus give conditions on $P(x, y)$ which exclude entire solutions of (1.4). An example of such a solution is $u = r$ which solves equation (1.4) with $P(x, y) = e^{-r}/r$.

For $n = 2$ it is well known that the function

$$u(z, \bar{z}) = \log\frac{|f'(z)|}{1 - |f(z)|^2} \tag{1.5}$$

is a solution of

$$\Delta u = 4e^{2u} \tag{1.6}$$

if $f(z)$ is an analytic function satisfying $|f(z)| < 1$ and $|f(z)| \neq 0$ in the domain considered. In § 3 we show, conversely, that every solution of (1.6) is essentially of this form. This converse result is necessary if it desired to use (1.5) and the theory of bounded analytic functions to obtain general properties of the regular solutions of (1.6). If the solution $u(z, \bar{z})$ of (1.6) is regular in a disk $|z| < R$, Theorem 3.1 leads to a bound for u in this disk. If $|f(z)| < 1$ in $|z| < R$ then, by Schwarz' lemma $|f'(z)|/1 - |f(z)|^2 \leqq R/R^2 - |z|^2$. Hence, a solution of (1.6) which is regular for $|z| < R$ is subject to the inequality.

$$u(z, \bar{z}) \leqq \log \frac{R}{R^2 - |z|^2} .$$

For $z = 0$, this leads, in particular, to the well known fact that the equation (1.6) can not have twice continuously differentiable solutions.

In § 4 comparison theorems are proved and explicit bounds are obtained for the solutions of

$$\Delta u = P(r)f(u) \tag{1.7}$$

or, more generally

$$\Delta u \geqq P(r)f(u) . \tag{1.8}$$

The behaviour of these solutions at an isolated singularity is investigated.

2. Entire solutions.

The main result is:

THEOREM 2.1. *Let*

$$\iint_{r<r_0} P(x, y)dxdy = O(r_0) \qquad \text{(for small } r_0\text{)} \tag{2.1}$$

and

(2.2) $$\int_0^r t\sigma(t)dt = O(r^{\varepsilon}) , \qquad \varepsilon > 0$$

where

(2.3) $$\sigma(r) = \frac{1}{2\pi}\int_0^{2\pi} \Delta(\log P)d\theta .$$

If either

(2.4) $$\int^{\infty} e^{(1-\beta)g(r)} r^{c-1} (\log r)^{1-3\beta} dr = \infty$$

or

(2.4)′ $$\int^{\infty} e^{(1-\beta)g(r)} r^{(1-2\beta)+\varepsilon^2-\varepsilon/2} (\log r)^{-\beta-\varepsilon} dr = \infty$$

where

(i) *c is a constant such that $c = (2-\varepsilon)(1-\beta)$ where $1/2 < \beta < 1$ and $\varepsilon > 0$ but small. And*

(ii) *the function $g(r)$ is a solution of*

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} \Delta (\log P)d\theta$$

such that $rg'(r) \to 0$ as $r \to 0$.

Then (1.4) *cannot have a solution which is twice continuously differentiable for $0 < r < \infty$ and continuous for $0 \leqq r < \infty$.*

That such solutions of (1.4) may exist for certain $P(x, y)$ is shown by the example $u = r^n$, $n \geqq 2$ where $P(x, y) = n^2 r^{n-2} e^{-r^n}$.

Proof. If we set

(2.5) $$u = v - \log P$$

equation (1.4) becomes

(2.6) $$\Delta v = e^v + \Delta(\log P) .$$

We introduce the notation

(2.7) $$\omega(r) = \frac{1}{2\pi}\int_0^{2\pi} v(r, \theta)d\theta .$$

By Green's formula for the circle $|z| \leqq r < R$

$$\iint_{|z|\leqq r} \Delta v \, dxdy = \int_{|z|=r} \frac{\partial v}{\partial n} ds$$

where n is the exterior normal. On account of $\partial/\partial n = \partial/\partial r$ it follows that

$$\int_0^r\int_0^{2\pi} \Delta v r d\theta dr = \int_0^{2\pi} \frac{\partial v}{\partial r} r d\theta = r\frac{\partial}{\partial r}\int_0^{2\pi} v(r, \theta)d\theta .$$

With the help of (2.6) and (2.7), this yields

$$r\frac{d}{dr}\omega(r) = \frac{1}{2\pi}\int_0^r\int_0^{2\pi}(e^v + \Delta(\log P))rd\theta dr . \tag{2.8}$$

$\omega(r)$ is single valued and twice continuously differentiable for $r < R$. Because of (2.3) and (2.5), (2.8) is equivalent to

$$\frac{rd\omega(r)}{dr} = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} P(x, y)e^u r d\theta dr + \int_0^r t\sigma(t)dt . \tag{2.9}$$

Since u is continuous, it follows from assumption (2.1) and (2.2) that

$$r\omega'(r) \longrightarrow 0 \tag{2.10}$$

as $r \to 0$.

Differentiating (2.8) with respect to r and using (2.3), we obtain

$$\frac{1}{r}\frac{d}{dr}\left(d\frac{d\omega}{dr}\right) = \sigma(r) + \frac{1}{2\pi}\int_0^{2\pi} e^v d\theta . \tag{2.11}$$

Since e^ξ is convex for all ξ, the right hand side of (2.11) can be estimated by

$$\frac{1}{2\pi}\int_0^{2\pi} e^{v(r,\theta)}d\theta \geqq e^{1/2\pi}\int_0^{2\pi\int_0^{2\pi} v(r,\theta)d\theta} = e^{\omega(r)} .$$

Hence (2.11) yields

$$\frac{d}{dr}\left(r\frac{d\omega}{dr}\right) \geqq r\sigma(r) + re^{\omega(r)} . \tag{2.12}$$

We now set

$$\omega(r) = g(r) + f(r) \tag{2.13}$$

where $g(r)$ is a solution of

$$\frac{d}{dr}\left(r\frac{dg}{dr}\right) = r\sigma(r)$$

which is continuous at the origin; that is, we compute $g(r)$ from

$$r\frac{d}{dr}(g(r)) = \int_0^r t\sigma(t)dt . \tag{2.14}$$

Because of our assumption on the behaviour of $\sigma(r)$ at $r = 0$, $g(r)$ will be continuous at $r = 0$. Inequality (2.12) then takes the from

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) \geqq r\tau(r)e^{f} \tag{2.15}$$

where $\tau(r) = e^{g(r)}$. Introducing the new independent variable by $\rho = \log r$ and setting

$$F = f + 2\rho \tag{2.16}$$

inequality (2.15) yields

$$\ddot{F} \geqq \tau(\rho)e^{F} \tag{2.17}$$

where dot denotes the differentiation with respect to ρ. Since the right hand side of (2.17) is always positive $F(\rho)$ is convex in ρ therefore, $\omega(r)$ is convex in $\log r$.

Now suppose (1.4) and, therefore, also (2.17) has entire solutions.

We observe that $\dot{F}(\rho)$ must be positive for all ρ in $(-\infty, \infty)$. Indeed, from (2.16), we get, $\dot{F}(\rho) = 2 + e^{\rho}(df(e^{\rho}))/dr$. Since by (2.14) and the assumption (2.2), $g'(r) = O(r^{\varepsilon-1})$ we have, $\lim_{r\to 0} rg'(r) = 0$. Hence, by (2.10) and (2.13) $\lim_{r\to 0} r\omega'(r) = \lim_{r\to 0} rf'(r) = 0$. It follows, therefore, that $\lim_{\rho\to -\infty} \dot{F}(\rho) = 2$. But, by (2.17) $F(\rho)$ is convex in ρ and we have, consequently,

$$\dot{F}(\rho) \geqq 2 \tag{2.18}$$

throughout $(-\infty, \infty)$. It, therefore, follows that $F(\rho)$ is ultimately positive. We choose ρ_0 large enough so that $F(\rho) > 0$ for $\rho > \rho_0$ and set

$$\phi = F\dot{F}\,. \tag{2.19}$$

Differentiating with respect to ρ and using (2.17) we have

$$\dot{\phi}\phi^{-\gamma} \geqq \tau F^{1-\gamma}e^{F}\dot{F}^{-\gamma} + F^{-\gamma}\dot{F}^{2-\gamma} \tag{2.20}$$

where γ is a constant to be chosen later.

Using the inequality [Hardy-Littlewood-Polya] $A + B \geqq (A/\alpha)^{\alpha}(B/\beta)^{\beta}$ where $\alpha + \beta = 1$, $0 \leqq \alpha$, $\beta \leqq 1$. the inequality (2.20) yields

$$\dot{\phi}\phi^{-\gamma} \geqq \tau^{1-\beta}(1-\beta)^{\beta-1}\beta^{-\beta}e^{(1-\beta)F}F^{1-\beta-\gamma}\dot{F}^{2\beta-\gamma}\,. \tag{2.21}$$

Now we consider two cases:

Case I. Let $2\beta - \gamma = 0$, $1/2 < \beta < 1$. Then the inequality (2.21) becomes

$$\dot{\phi}\phi^{-2\beta} \geqq C_1\tau^{1-\beta}e^{(1-\beta)F}F^{1-3\beta} \tag{2.22}$$

where $c_1 = (1-\beta)^{\beta-1}\beta^{-\beta}$. Since $\dot{F} \geqq 2$ we have $F \geqq (2-\varepsilon)\rho$ if ρ is sufficiently large. Moreover, since $e^{(1-\beta)F}F^{(1-3\beta)}$ is increasing for $F > 3\beta - 1/1 - \beta$, inequality (2.22) yields

$$\dot{\phi}\phi^{-2\beta} \geqq c_2\tau^{1-\beta}\rho^{1-3\beta}e^{c\rho}$$

provided $(2-\varepsilon)\rho > 3\beta - 1/1 - \beta$, $c_2 = c_1(2-\varepsilon)^{1-3\beta}$ and $c = (2-\varepsilon)(1-\beta)$. Integration of (2.22) gives

$$\text{(2.23)} \quad \frac{1}{2\beta - 1}\left[\frac{1}{\phi^{2\beta-1}(\rho_0)} - \frac{1}{\phi^{2\beta-1}(\rho)}\right] \geqq c_2\int_{c\rho_0}^{e\rho} e^{(1-\beta)g(r)}r^{c-1}(\log r)^{1-3\beta}dr\ .$$

Since F is convex and increasing in ρ, $\phi^{1-2\beta}(\rho)$ tends to zero as $\rho \to \infty$. Hence, the left hand side of (2.23) is bounded as $\rho \to \infty$. This contradicts the assumption (2.4).

Hence the inequality (2.17) and also (1.4) does not have entire solutions.

Case II. Let $2\beta - \gamma > 0$, $1/2 < \beta < 1$. The inequality (2.21) becomes in this case

$$\dot{\phi}\phi^{-\gamma} \geqq c_1\tau^{1-\beta}F^{1-\beta-\gamma}e^{(1-\beta)F}2^{2\beta-\gamma}$$

where we have used (2.18). But since

$$F^{1-\beta-\gamma}e^{(1-\beta)F} > e^{(1-\beta)(2-\varepsilon)\rho}\{(2-\varepsilon)\rho\}^{1-\beta-\gamma}$$

provided $(2-\varepsilon)\rho > (\gamma + \beta - 1)(1-\beta)^{-1}$, we have

$$\dot{\phi}\phi^{-\gamma} \geqq c_1 2^{(2\beta-\gamma)}\tau^{1-\beta}e^{(1-\beta)(2-\varepsilon)\rho}[\rho(2-\varepsilon)]^{1-\beta-\gamma}\ .$$

Choose $\gamma = 1 + \varepsilon$, $\varepsilon > 0$. Then $\beta > (1+\varepsilon)/2$. Therefore, integration with respect to ρ gives

$$\text{(2.24)} \quad \frac{1}{\varepsilon}\left[\frac{1}{\phi^{\varepsilon}(\rho_0)} - \frac{1}{\phi^{\varepsilon}(\rho)}\right] \geqq c_3\int^{\rho} e^{(1-\beta)g(r)}r^{(1-2\beta)+(\varepsilon^2-\varepsilon/2)}(\log r)^{-\beta-\varepsilon}dr$$

where $c_3 = c_1(2-\varepsilon)^{-\beta-\varepsilon}$.

If it were true that $u = u(x, y)$ is entire, the left-hand side of (2.24) would remain bounded as $\rho \to \infty$. Since by (2.4)′ the right hand side of (2.24) is unbounded, this leads to a contradiction.

This completes the proof of Theorem 2.1.

3. **General solution.** Let $u(x, y)$ be of class C^2 in the region D of x, y-plane and satisfy (1.6). Introducing the new independent variables $z = x + iy$ and $\bar{z} = x - iy$ equation (1.6) becomes

$$\text{(3.1)} \qquad u_{z\bar{z}} = e^{2u}$$

where $\partial/\partial z = 1/2(\partial/\partial x - i(\partial/\partial y))$ and $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i(\partial/\partial y))$. How we prove

THEOREM 3.1. *Every solution of* (1.6) *which is twice continuously differentiable in a given region D can be written in the form*

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 - |f(z)|^2}$$

where $f(z)$ is analytic in D such that $|f'(z)| \neq 0$ and $|f(z)| < 1$.

Proof. According to an observation which goes back to Bieberbach [1] a regular solution of (1.6) can be associated with an analytic function of z in the following manner: We set

$$Q = u_{zz} - u_z^2$$

where u is a solution of (1.6) or, equivalently, of (3.1) and we compute $Q_{\bar{z}}$. We have, with the help of (3.1), $Q_{\bar{z}} = 0$. Thus, Q is found to satisfy the Cauchy-Riemann equations. Since Q is continuous, it must therefore be regular analytic function $\omega(z)$.

If we set

$$\psi = \bar{e}^{u} \tag{3.2}$$

and observe that

$$\psi_{zz} = \bar{e}^{u}(u_z^2 - u_{zz})$$

we find that ψ is a solution of the linear differential equation

$$\psi_{zz} + \omega(z)\psi = 0\,. \tag{3.3}$$

Since $\omega(z)$ is analytic in z the general solution of (3.3) contains the analytic solutions of the equation

$$F''(z) + \omega(z)F(z) = 0 \tag{3.3$'$}$$

because, for an analytic F, we have $F'(z) = \partial F/\partial z$. The general solution of (3.3) can, therefore, be written in the form

$$\psi = A^*\psi_1(z) + B^*\psi_2(z)$$

where ψ_1 and ψ_2 are two linearly independent (analytic) solutions of (3.3)$'$ which may be assumed to be normalized by

$$\psi_1\psi_2' - \psi_2\psi_1' = 1 \tag{3.4}$$

and A^* and B^* are constants with respect to $\partial/\partial z$ — differentiation used in (3.3) i.e., $\partial A^*/\partial z = \partial B^*/\partial z = 0$. Since these are Cauchy-

Riemann equations for functions in $\bar{z}$ we have $A^* = \overline{A(z)}$, $B^* = \overline{B(z)}$ where A and B are analytic. The general solution of (3.3) is, therefore, found to be of the form

$$\psi = \overline{A(z)}\psi_1(z) + \overline{B(z)}\psi_2(z) \tag{3.5}$$

where A, B, ψ_1 and ψ_2 are analytic functions in D. In view of (3.2), equation (3.5) can be written

$$\bar{e}^{u} = \bar{A}(z)\psi_1(z) + \bar{B}(z)\psi_2(z) . \tag{3.6}$$

Now the proof of the theorem will follow from the following lemma:

LEMMA 3.1. *Let ψ_1 and ψ_2 be linearly independent solutions of the differential equation* (3.3)′ *where $\omega(z)$ is analytic in D. If $A(z)$ and $B(z)$ are analytic in D and if the expression*

$$K(z, \bar{z}) = \bar{A}(z)\psi_1(z) + \bar{B}(z)\psi_2(z) \tag{3.7}$$

is real throughout D but does not vanish identically then $K(z, \bar{z})$ can be written in the form

$$K(z, \bar{z}) = \pm|\sigma(z)|^2 \mp |\tau(z)|^2$$

where $\sigma(z)$ and $\tau(z)$ are two linearly independent solutions of (3.3)′ *for which*

$$\tau(z)\sigma'(z) - \sigma(z)\tau'(z) = 1 . \tag{3.8}$$

Proof. Since $K(z, \bar{z})$ is real, we have

$$\bar{A}(z)\psi_1(z) + \bar{B}(z)\psi_2(z) = A(z)\overline{\psi_1(z)} + B(z)\overline{\psi_2(z)} . \tag{3.9}$$

Differentiation with respect to z and (3.4) give

$$\bar{\psi}_1(z)[\psi_1'(z)A(z) - \psi_1(z)A'(z)] + \bar{\psi}_2(z)[\psi_1'(z)B(z) - B'(z)\psi_1(z)] = -\bar{B}(z) .$$

Setting

$$g(z) = \psi_1'(z)A(z) - \psi_1(z)A'(z) \tag{3.10}$$

and

$$h(z) = \psi_1'(z)B(z) - \psi_1(z)B'(z) \tag{3.11}$$

we have

$$\psi_1(z)\bar{g}(z) + \psi_2(z)\bar{h}(z) = -B(z) . \tag{3.12}$$

But the left-hand side of (3.12) is a solution of (3.3)′; hence $(-B(z))$ satisfies

$$B_{zz} + \omega(z)B = 0$$

where $\omega(z)$ is an analytic function. But since $B(z)$ is analytic in z,

$$B''(z) + \omega(z)B(z) = 0 \ ,$$

consequently, B is of the form

$$B(z) = \alpha\psi_1(z) + \beta\psi_2(z) \tag{3.13}$$

where α and β are constants. Arguing in the same manner (3.4) and (3.9) give

$$A(z) = \gamma\psi_1(z) + \delta\psi_2(z) \tag{3.14}$$

where γ and δ are constants.

Also from (3.12) and (3.13), $\psi_1(z)/\psi_2(z) = -\overline{((h(z) + \beta)/(g(z) + \alpha))}$. But since $\psi_1(z)/\psi_2(z)$ is analytic in z and, moreover, since ψ_1 and ψ_2 are linearly independent, we must have $\bar{g}(z) + \bar{\alpha} \equiv 0$ and $\bar{h}(z) + \bar{\beta} \equiv 0$, or equivalently

$$(\gamma\psi_1 + \delta\psi_2)\psi_1' - (\gamma\psi_1' + \delta\psi_2')\psi_1 = -\bar{\alpha} \tag{3.15}$$

and

$$(\alpha\psi_1 + \beta\psi_2)\psi_1' - (\alpha\psi_1' + \beta\psi_2')\psi_1 = -\bar{\beta} \tag{3.16}$$

respectively. With the help of (3.12), (3.14), (3.15) and (3.16) the equation (3.7) becomes

$$K(z, \bar{z}) = \gamma \,|\, \psi_1 \,|^2 + \beta \,|\, \psi_2 \,|^2 + \bar{\alpha}\bar{\psi}_1\psi_2 + \alpha\bar{\psi}_2\psi_1 \ . \tag{3.17}$$

Now let $\sigma(z)$ and $\tau(z)$ be any other solutions of (3.3)' such that $\psi_1(z) = a\sigma(z) + b\tau(z)$ and $\psi_2(z) = c\sigma(z) + d\tau(z)$ where a, b, c and d are constants satisfying

$$ad - bc = 1 \tag{3.18}$$

and

$$b(\gamma\bar{a} + \alpha\bar{c}) + d(\bar{c}\beta + \bar{a}\bar{\alpha}) = 0 \ . \tag{3.19}$$

This is possible if the determinant

$$D = \gamma \,|\, a \,|^2 + \beta \,|\, c \,|^2 + 2Re(a\alpha\bar{c})$$

does not vanish. Evidently this can always be achieved as long as not all numbers α, β and γ are zero. However α, β and γ cannot all be zero since, in view of (3.17) $K(z, \bar{z})$ would then be identically zero, and this case is excluded.

Substituting ψ_1 and ψ_2 in (3.17) and using (3.19) we obtain

$$
\begin{aligned}
K(z, \bar{z}) = |\,\sigma(z)\,|^2\{\gamma\,|\,a\,|^2 + \beta\,|\,c\,|^2 + \bar{a}c\bar{\alpha} + a\bar{c}\alpha\} \\
+ |\,\tau(z)\,|^2\{\gamma\,|\,b\,|^2 + \beta\,|\,d\,|^2 + \bar{b}d\bar{\alpha} + b\bar{d}\alpha\}\,.
\end{aligned} \tag{3.20}
$$

Now we consider the following two cases:

Case I. Let $\beta \neq 0$, $\gamma \neq 0$. We set $a \neq 0$ and $c = 0$ then, with the help of (3.18) and (3.19), (3.20) becomes

$$K(z, \bar{z}) = |\,\sigma(z)\,|^2\,\gamma\,|\,a\,|^2 + |\,\tau(z)\,|^2\,|\,d\,|^2\gamma^{-1}(\beta\gamma - |\,\alpha\,|^2)\,.$$

(i) Let $\gamma > 0$, $\beta\gamma - |\,\alpha\,|^2 = m$ (m is a positive integer). Hence,

$$K(z, \bar{z}) = |\,\sigma^*\,|^2 + |\,\tau^*\,|^2$$

where $\sigma^* = \sigma(\gamma\,|\,a\,|^2)^{1/2}$ and $\tau^* = \tau m^{1/2}(\gamma\,|\,a\,|^2)^{-1/2}$ are solutions of (3.3)$'$.

(ii) $\gamma > 0$, $\beta\gamma - |\,\alpha\,|^2 = -m$. In this case

$$K(z, \bar{z}) = |\,\sigma^*\,|^2 - |\,\tau^*\,|^2\,.$$

(iii) Let $\gamma < 0$, $\beta\gamma - |\,\alpha\,|^2 = m$. Then

$$K(z, \bar{z}) = -|\,\sigma^*\,|^2 - |\,\tau^*\,|^2$$

(iv) $\gamma < 0$, $\beta\gamma - |\,\alpha\,|^2 = -m$. This gives

$$K(z, \bar{z}) = -|\,\sigma^*\,|^2 + |\,\tau^*\,|^2\,.$$

Case II. Let $\beta = 0$, $\gamma = 0$. We set $a, b \neq 0$. With this choice (3.18) and (3.19) reduce (3.20) to

$$K(z, \bar{z}) = -|\,\sigma_1\,|^2 + |\,\tau_1\,|^2$$

where $|\,\sigma_1\,| = |\,\sigma\,|\,(\bar{a}\bar{\alpha})^{1/2}b^{-1/2}$ and $|\,\tau_1\,| = a^{-1/2}\,|\tau\,|(\bar{\alpha}\bar{\beta})^{1/2}$ and are solutions of (3.3)$'$.

Summing up, we have thus proved that, if the function $K(z, \bar{z})$ is real, it must have either of the three following forms

$$
\left.
\begin{aligned}
&(1) \qquad K(z, \bar{z}) = |\,\tau\,|^2 - |\,\sigma\,|^2 \\
&(2) \qquad K(z, \bar{z}) = |\,\tau\,|^2 + |\,\sigma\,|^2 \\
&(3) \qquad K(z, \bar{z}) = -|\,\tau\,|^2 - |\,\sigma\,|^2
\end{aligned}
\right\}(S)
$$

where σ and τ are solutions of the differential equation (3.3)$'$ normalized by (3.8). The case $K(z, \bar{z}) = |\,\sigma\,|^2 - |\,\tau\,|^2$ is evidently not essentially different from case (1). Case (3) can be excluded immediately, since beacuse of (3.6) and (3.7) $K(z, \bar{z})$ must be positive. This also shows that, in case (1), we necessarily must have

$$|\,\tau(z)\,| > |\,\sigma(z)\,|. \tag{3.21}$$

We now define

$$f(z) = \frac{\sigma(z)}{\tau(z)} . \tag{3.22}$$

In view of (3.8) we have

$$f'(z) = \frac{1}{\tau^2(z)} \tag{3.23}$$

and thus $|\sigma|^2 + |\tau|^2 = (1 + |f(z)|^2)/|f'(z)|$ in case (2) and $|\tau|^2 - |\sigma|^2 = (1 - |f(z)|^2)/|f'(z)|$ in case (1). Comparing this with (3.6), (3.7) and (S) we find that $u(z, \bar{z})$ must be either of the forms

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 - |f(z)|^2}$$

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 + |f(z)|^2}$$

$$u(z, \bar{z}) = \log \frac{1 + |f(z)|^2}{|f'(z)|} .$$

Since the last two functions are not solutions of (1.6), these cases are excluded. Hence any real solution of (1.6) must be of the from

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 - |f(z)|^2}$$

where because of (3.21) and (3.22) $|f(z)| < 1$ and in view of (3.23) $|f'(z)| \neq 0$.

This completes the proof of Theorem 3.1.

4. Bounds for the solutions of $\Delta_u \geqq P(r)f(u)$.

Let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

denote the n-dimensional Laplace operator and let D_r and S_r stand for the open sphere $x_1^2 + x_2^2 + \cdots + x_n^2 < r^2$ and its boundary

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$$

respectively. We are concerned here with functions

$$\omega = \omega(Q)(Q \in D_r, 0 < r < R)$$

which are of class C^2 in D_r and satisfy the differential equation

$$\Delta\omega = P(r)F(\omega)$$

or, more generally, the differential inequality

$$\Delta\omega \geqq P(r)F(\omega) . \tag{4.1}$$

Nehari [6] found explicit bounds for the solutions of the differential equation $\Delta u = F(u)$ or, more generally the differential inequality $\Delta u \geqq F(u)$ which are regular in a disk. We shall obtain here a more general result, which also applies to certain equations of the form (4.1).

LEMMA 4.1. *Let $F(t)$ and $G(t)$ be positive and differentiable functions for $-\infty < t < \infty$ and such that the integrals*

$$\int_\omega^\infty \frac{dt}{F(t)}, \quad \int_v^\infty \frac{dt}{G(t)}$$

exist, and let $\omega = \omega(x_1, x_2, \cdots, x_n)$ and $v = v(x_1, x_2, \cdots x_n)$ be two functions related by the identity

$$\int_\omega^\infty \frac{dt}{F(t)} = \int_v^\infty \frac{dt}{G(t)} . \tag{4.2}$$

Then

$$\frac{\Delta\omega}{F(\omega)} \geqq \frac{\Delta v}{G(v)} \tag{4.3}$$

provided $F'(\omega) \geqq G'(v)$.

Proof. We write x for one of the variables $x_1, x_2, \cdots, x_n$ and differentiate (4.2) twice with respect to x. This yields

$$-\frac{v_x}{G(v)} = -\frac{\omega_x}{F(\omega)}$$

$$-\frac{v_{xx}}{G(v)} + \frac{v_x^2 G'(v)}{G^2(v)} = -\frac{\omega_{xx}}{F(\omega)} + \frac{v_x^2 F'(\omega)}{G^2(v)} .$$

Summing over all x_n and using the fact that $F'(\omega) \geqq G'(v)$, we get (4.3).

We derive the following corollary.

COROLLARY 5.1. *If $v = v(x_1, x_2, \cdots, x_n)$ is a function satisfying the differential inequality*

$$\Delta v \leqq P v^k , \qquad k > 1 \tag{4.4}$$

where $P = P(x_1, x_2, \cdots, x_n)$ is positive, and if $F(u)$ is such that

$$F'(u) \int_u^\infty \frac{dt}{F(t)} \leqq \frac{k}{k-1} \tag{4.5}$$

then, the function u defined by

$$\frac{1}{(k-1)v^{k-1}} = \int_u^\infty \frac{dt}{F(t)} \tag{4.6}$$

is subject to the inequality

$$\Delta u \leqq PF(u) . \tag{4.7}$$

Setting $G(v) = v^k$ in Lemma 4.1, the proof of the Corollary 4.1 is immediate.

As an application of Corollary 4.1, we prove the following result.

THEOREM 4.1. *If the function* $\omega = \omega(x_1, x_2, \cdots, x_n)$ *satisfies the inequality*

$$\Delta\omega \geqq r^2 F(\omega) \tag{4.8}$$

where $F(\omega)$ *is such that* $F'(\omega)\int^\infty dt/F(t) \leqq 9/8$ *and* $F'(\omega) \geqq 0$ *then the function* u *defined by*

$$\frac{(r^2-\rho^2)^2(R^2-r^2)^2}{20R^4} = \int_u^\infty \frac{dt}{F(t)} \qquad 0 < \rho < r < R$$

is such that

$$\omega \leqq u .$$

Proof. Consider the function v defined by

$$v = \frac{1}{(r^2-\rho^2)^\alpha (R^2-r^2)^\alpha} \tag{4.9}$$

where α is a constant to be determined later. Differentiating (4.9) twice with respect to one of the variables $x = x_k$, we obtain

$$\begin{aligned} v_x &= -\frac{2x\alpha}{(r^2-\rho^2)^{\alpha+1}(R^2-r^2)^\alpha} + \frac{2x\alpha}{(r^2-\rho^2)^\alpha (R^2-r^2)^{\alpha+1}} \\ v_{xx} &= -\frac{2\alpha}{(R^2-r^2)^\alpha (r^2-\rho^2)^{\alpha+1}} + \frac{4x^2\alpha(\alpha+1)}{(R^2-r^2)^\alpha (r^2-\rho^2)^{\alpha+2}} \\ &\quad + \frac{2\alpha}{(r^2-\rho^2)^\alpha (R^2-r^2)^{\alpha+1}} - \frac{8x^2\alpha^2}{(r^2-\rho^2)^{\alpha+1}(R^2-r^2)^{\alpha+1}} \\ &\quad + \frac{4x^2\alpha(\alpha+1)}{(r^2-\rho^2)^\alpha (R^2-r^2)^{\alpha+2}} . \end{aligned}$$

Summing over all $x = x_k$ and choosing $\alpha \geqq 1/4$ we get,

$$\Delta v \leqq \frac{5}{2} r^2 R^4 v^9 .$$

Now let $v = (2^{1/8}y)/(5^{1/2}R^2)^{1/4}$ then we have

$$\Delta y \leqq r^2 y^9 \tag{4.10}$$

where y is given by

$$y = \left(\frac{R^2 5^{1/2} 2^{-1/2}}{(R^2 - r^2)(r^2 - \rho^2)}\right)^{1/4}.$$

Now applying Corollary 4.1 to (4.10), we obtain,

$$\Delta u \leqq r^2 F(u)$$

when u is defined by

$$\frac{(r^2 - \rho^2)^2 (R^2 - r^2)^2}{20R^4} = \int_u^\infty \frac{dt}{F(t)}.$$

Clearly, $u'(0) = 0$ and $u \to \infty$ as $r \to R$ or $\rho \to r$. The fact that $\omega \leqq u$ now follows from Osserman's lemma [8]. This proves our assertion.

THEOREM 4.2. *Let $f(\omega)$ be positive, nondecreasing, differentiable function in $(-\infty, \infty)$ for which*

$$\int_\omega^\infty \frac{dt}{f(t)} \qquad (\omega > -\infty)$$

exists and

$$f'(\omega)\int_\omega^\infty \frac{dt}{f(t)} \leqq 1 + \lambda \qquad (\lambda > 0)\,. \tag{4.11}$$

If

$$u(r) = \sup_{Q \in S_r} \omega(Q) \tag{G}$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r which satisfy (4.1). Then

$$\frac{C(\lambda)\alpha(R^2 - r^2)^2}{R^2} \leqq \int_{u(r)}^\infty \frac{dt}{f(t)} \tag{4.12}$$

in case $P(r) = \alpha$ $(\alpha > 0)$.

$$\frac{C(\lambda)\beta r^{n/1+\lambda}(R^2 - r^2)^2}{R^2} \leqq \int_{u(r)}^\infty \frac{dt}{f(t)} \tag{4.13}$$

if $P(r) = \beta r^{n/1+\lambda}$ $(\beta > 0)$ and

$$\frac{C(\lambda)\gamma r^{n-2/\lambda}(R^2 - r^2)^2}{R^2} \leqq \int_{u(r)}^\infty \frac{dt}{f(t)} \tag{4.14}$$

if $P(r) = \gamma r^{n-2/\lambda}$ $(\gamma > 0)$

where

$$C(\lambda) = \frac{1}{4n} \qquad (4\lambda \leqq n - 2) \tag{4.15}$$

and

$$C(\lambda) = \frac{1}{8(2\lambda + 1)} \qquad (4\lambda > n - 2)\,. \tag{4.16}$$

The inequalities (4.12), (4.13) *and* (4.14) *are sharp.*

The case $\lambda = 0$ had been considered by the author in [2].

Proof. Consider the function $g = g(r)$ defined by

$$\frac{C(\lambda)(R^2 - r^2)^2}{R^2} = \frac{1}{p(r)}\int_g^\infty \frac{dt}{f(t)} \tag{4.17}$$

where $p(r)$ is positive, monotonically increasing and twice continuously differentiable and C is a positive constant to be chosen later. Denoting by x one of the variables x_k and differentiating twice with respect to x we have

$$-\frac{4cx(R^2 - r^2)}{R^2} = -\frac{g_x}{p(r)f(g)} - \frac{2x}{p^2(r)}\int_g^\infty \frac{dt}{f(t)} \tag{4.18}$$

$$\begin{aligned} -\frac{4c(R^2 - r^2)}{R^2} = -\frac{8cx^2}{R^2} - \frac{g_{xx}}{p(r)f(g)} + \frac{4x\dot{p}(r)g_x}{p^2(r)f(g)} + \frac{g_n^2 f'(g)}{p(r)f^2(g)} \\ \frac{2\dot{p}(r)}{p^2(r)}\int_g^\infty \frac{dt}{f(t)} - \frac{4x^2\ddot{p}(r)}{p^2(r)}\int_g^\infty \frac{dt}{f(t)} \\ \frac{8x^2\dot{p}^2(r)}{p^3(r)}\int_g^\infty \frac{dt}{f(t)} \end{aligned} \tag{4.19}$$

where dot denotes differentiation with respect to r^2. With the help of (4.17) and (4.18), (4.19) becomes

$$\begin{aligned} \frac{g_{xx}}{p(r)f(g)} = -\frac{8cx^2}{R^2} + \frac{4c(R^2 - r^2)}{R^2} + \frac{16cx^2\dot{p}(r)(R^2 - r^2)}{R^2 p(r)} + \frac{4cx^2}{R^2} \\ \times f'(g)cp(r)\frac{(R^2 - r^2)^2}{R^2}\left[2 - \frac{\dot{p}(r)(R^2 - r)^2}{p(r)}\right]^2 \\ - \frac{2}{p^2(r)}(2x^2\ddot{p}(r) + \dot{p}(r))\int_g^\infty \frac{dt}{f(t)}\,. \end{aligned}$$

Summing over all x_k and using (4.11) it reduces to

$$\begin{aligned}(4.20)\quad \frac{\Delta g}{p(r)f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\} - \frac{16(R^2-r^2)cr^2\dot{p}(r)}{R^2}\lambda \\ - \frac{2c(R^2-r^2)^2}{R^2}\left\{\frac{2r^2\ddot{p}(r)+n\dot{p}(r)}{p(r)} - \frac{2r^2\dot{p}^2(r)}{p^2(r)}(1+\lambda)\right\}.\end{aligned}$$

We now consider the following cases:

Case I. Choose $p(r)$ such that $\dot{p}(r)/p(r)((2r^2\dot{p}(r)/p(r)) - n/(1+\lambda)) = 0$.

(i) If $\dot{p} = 0$ or $p = \alpha$ where α is an arbitrary positive constant then (4.20) becomes

$$(4.21)\qquad \frac{\Delta g}{\alpha f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\}.$$

If, $4\lambda \leqq n-2$ it follows that $\Delta g \leqq 4nc\alpha f(g)$ and if C is given by (4.15), we have

$$(4.22)\qquad \Delta g \leqq \alpha f(g).$$

If $4\lambda > n-2$ the right hand of (4.21) attains maximum for $R = r$ and the value of (4.16) for C again leads to (4.22). Since $\dot{g}(0) = 0$ and increases to ∞ as $r \to R$ the proof of (4.12) will follow from Osserman's lemma [8].

REMARK. If $\alpha = 1$ the left hand inequality (9) of Theorem 1 of Nehari [6] becomes a particular case of this result.

(ii) If $2r^2\dot{p}(r)/p(r) - (n/1+\lambda) = 0$ or $p = r^{n/1+\lambda}\beta$ where β is an arbitrary positive constant then (4.20) gives

$$\frac{\Delta g}{\beta r^{n/1+\lambda}f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\}.$$

If C is given by the values (4.15) and (4.16), we have

$$\Delta g \leqq \beta r^{n/1+\lambda}f(g).$$

Now the proof of (4.13) will follow from Osserman's lemma [8].

Case II. Assume $p(r)$ to satisfy

$$2r^2p(r)(\ddot{p}r) + np(r)\dot{p}(r) - 2r^2(1+\lambda)\dot{p}^2(r) = 0$$

or $p(r) = \gamma r^{n-2/\lambda}$ where γ is an arbitrary positive constant. Then (4.20) reduces to

$$\frac{\Delta g}{\gamma r^{n-2/\lambda}f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\}.$$

Now if C takes the values (4.15) and (4.16) respectively, we have

$$\Delta g \leqq \gamma r^{n-2/\lambda} f(g)$$

and (4.14) is proved with the help of Osserman's lemma [8].

We derive the following corollary:

COROLLARY 4.2. *If ω satisfies the equation*

$$\Delta\omega = \beta r^{n/1+\lambda}\omega^{1+(1/\lambda)} \qquad (\lambda > 0,\ n \geqq 2)$$

where β is an arbitrary constant, then

$$\omega \leqq \left(\frac{\lambda R^2}{c(\lambda)\beta r^{n/1+\lambda}(R^2 - r^2)^2}\right)^{\lambda}. \tag{4.23}$$

Also the behaviour of ω is such that

$$\overline{\lim_{r\to 0}}\left(\frac{\log\omega}{\log 1/r}\right) \leqq \frac{n\lambda}{1+\lambda}\ .$$

Indeed, setting $f(t) = t^{1+(1/\lambda)}$ in (4.13), we have (4.23), where $\omega = u$. Taking logarithm on both sides, we have, from (4.23)

$$\log\omega \leqq \lambda\log\frac{\lambda R^2}{\beta c(\lambda)(R^2 - r^2)^2} + \frac{n\lambda}{1+\lambda}\log\frac{1}{r}\ .$$

Dividing by $\log 1/r$ and letting $r \to 0$

$$\overline{\lim_{r\to 0}}\left(\frac{\log\omega}{\log 1/r}\right) \leqq \frac{n\lambda}{1+\lambda}\ .$$

A similar result could also be proved about the solutions of the equation

$$\Delta\omega = \gamma r^{n-2/\lambda}\omega^{1+(1/\lambda)}\ .$$

The next theorem concerns the lower bounds for the maximum of the solutions of (4.1).

THEOREM 4.3. *Let $f(\omega)$ satisfy the conditions of theorem* 4.2 *with* (4.11) *replaced by*

$$f'(\omega)\int_{\omega}^{\infty}\frac{dt}{f(t)} = 1 + \lambda\ , \qquad (\lambda > 0)\ . \tag{4.11$'$}$$

If

$$v(r) = \operatorname*{Sup}_{Q\in S_r}\omega(Q) \tag{G$'$}$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r and which satisfy (4.1) *then*

$$(4.24)\qquad \int_v^\infty \frac{dt}{f(t)} \leqq \frac{\kappa(R^2 - r^2)}{2n}$$

if $p(r) = \kappa$ *where* κ *is an arbitrary positive constant,*

$$(4.25)\qquad \int_v^\infty \frac{dt}{f(t)} \leqq \frac{\delta r^{n-2/\lambda-1}(R^2 - r^2)}{2n} \quad \left(n > 2,\ \lambda > 1,\ n < \frac{4\lambda}{1+\lambda}\right)$$

provided $p(r) = \delta r^{n-2/\lambda-1}$ $(\delta > 0)$.

$$(4.26)\qquad \int_v^\infty \frac{dt}{f(t)} \leqq \frac{\mu r^{1/\lambda}(R^2 - r^2)}{6} \qquad (n = 3)$$

in case $p(r) = \mu r^{1/\lambda}$ $(\mu > 0)$. *However, in 2-dimensional case*

$$(4.27)\qquad \int_v^\infty \frac{dt}{f(t)} \leqq \frac{\nu r^l(R^2 - r^2)}{4}$$

where $p(r) = \nu r^l$, ν *and* l *being arbitrary positive constants.*

Proof. Consider the function $h = h(r)$ defined by

$$(4.28)\qquad \frac{\rho^2 - r^2}{2n} = \frac{1}{p(r)}\int_h^\infty \frac{dt}{f(t)} \qquad (\rho > R > r)$$

where $p(r)$ is positive, monotonically increasing and twice continuously differentiable. Clearly, h belongs to the class C^2 in D_r. Differentiating (5.28) twice with respect to $x = x_k$ we obtain

$$-\frac{n}{x} = -\frac{h_x}{f(h)p(r)} - \frac{2x\dot{p}(r)}{p^2(r)}\int_h^\infty \frac{dt}{f(t)}$$

$$(4.29)\qquad -\frac{1}{n} = -\frac{h_{xx}}{f(h)p(r)} + \frac{4xh_x\dot{p}(r)}{f(h)p^2(r)} + \frac{h_x^2 f'(h)}{p(r)f^2(h)} - \frac{2\dot{p}}{p^2}\int_h^\infty \frac{dt}{f(t)}$$
$$- \frac{4x^2\ddot{p}(r)}{p^2(r)}\int_h^\infty \frac{dt}{f(t)} + \frac{8x^2\dot{p}(r)}{p^3(r)}\int_h^\infty \frac{dt}{f(t)}\,.$$

Using (4.29) and summing over all x_k, we obtain

$$\frac{\Delta h}{f(h)p(r)} = 1 + \frac{4r^2\dot{p}(r)}{np(r)} + r^2p(r)f'(h)\left[\frac{\rho^2 - r^2}{n}\frac{\dot{p}}{p} - \frac{1}{n}\right]^2$$
$$- \frac{2r^2\ddot{p} + n\dot{p}}{p} \times \frac{\rho^2 - r^2}{n}\,.$$

Since $f' > 0$ we obtain with the help of (4.11)′

$$(4.30)\qquad \frac{\Delta h}{f(h)p(r)} \geqq 1 - \frac{4r^2\dot{p}}{np}\lambda - \frac{\rho^2 - r^2}{n}\left[\frac{n\dot{p} + 2r^2\ddot{p}}{p} - (1+\lambda)\frac{2r^2\dot{p}^2}{p^2}\right].$$

Now we consider the following cases.

Case I. Choose p such that $\dot{p} = 0$ or, $p = \kappa$ where κ is an arbitrary positive constant. Hence (4.30) reduces to

$$\Delta h \geqq \kappa f(h) . \tag{4.31}$$

Consequently $(G)'$ implies

$$h(r) \leqq v(r) .$$

Since we can take ρ arbitrarily close to R, we have

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\kappa(R^2 - r^2)}{2n} .$$

Case II. Assume $p(r)$ to be such that

$$n\dot{p}(r)p(r) + 2r^2 p(r)\ddot{p}(r) - 2\lambda r^2 \dot{p}^2(r) = 0$$

or $p = \delta r^{n-2/\lambda-1}$ where δ is an arbitrary positive constant, $n > 2$, $\lambda > 1$ and such that $n < (4\lambda/1 + \lambda)$. Hence (4.30) becomes

$$\Delta h \geqq \left\{1 - \frac{2\lambda(n-2)}{n(\lambda - 1)}\right\}\delta r^{n-2/\lambda-1} f(h) .$$

Using $(G)'$ and arguing as above, we obtain

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\delta r^{n-2/\lambda-1}(R^2 - r^2)}{2n} .$$

Case III. Choose p to satisfy

$$np(r)\dot{p}(r) + 2r^2 p(r)\ddot{p}(r) - (1 + \lambda)2r^2\dot{p}^2(r) = 0$$

or $p = \mu r^{1/\lambda}$ where μ is an arbitrary positive constant and $n = 3$. Hence (4.30) gives

$$\Delta h \geqq \frac{\mu}{3} r^{1/\lambda} f(h) .$$

Using the same argument as above, we have

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\mu r^{1/\lambda}(R^2 - r^2)}{6} .$$

Case IV. Assume p to be such that $2r^2 p\ddot{p} + np\dot{p} - 2r^2\dot{p}^2 = 0$ or $p = \nu r^l$ where ν and l are arbitrary positive constants. Consequently

$$\Delta h \geqq \nu(1 - l\lambda)r^l f(h) .$$

And, as above we conclude

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\nu r^l(R^2 - r^2)}{4} .$$

This completes the proof of the theorem.

We derive the following corollaries:

COROLLARY 4.3. *In case of a function* ω *regular in* D_r *and which satisfies the differential equation*

$$\Delta\omega = \delta r^{n-2/\lambda-1}\left\{1 - \frac{2\lambda(n-2)}{n(\lambda+1)}\right\}\omega^{1+(1/\lambda)}$$

where δ *is an arbitrary positive constant,* $n > 2$, $\lambda > 1$ *and such that* $n < (4\lambda/1 + \lambda)$ *we have*

$$\left(\frac{2n\lambda}{\delta r^{n-2/\lambda-1}(R^2 - r^2)}\right)^\lambda \leqq \omega .$$

And also the behaviour of ω *is such that*

$$\overline{\lim_{r\to 0}}\left(\frac{\log \omega}{\log 1/r}\right) \geqq \lambda\frac{n-2}{\lambda - 1} .$$

Indeed, setting $f(t) = t^{1+(1/\lambda)}$ in (4.25), where $v = \omega$, we obtain

$$\omega^{1/\lambda} \geqq \frac{2n\lambda}{\delta r^{n-2/\lambda-1}(R^2 - r^2)} .$$

Taking logarithm on both sides, we get

$$\log \omega \geqq \lambda \log \frac{2n\lambda}{\delta(R^2 - r^2)} + \lambda\frac{n-2}{\lambda-1}\log\frac{1}{r} .$$

Dividing by $\log 1/r$ and taking the limit

$$\overline{\lim_{r\to 0}}\left(\frac{\log \omega}{\log 1/r}\right) \geqq \lambda\frac{n-2}{\lambda - 1} .$$

COROLLARY 4.4. *If* $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ *is a* 3*-dimensional Laplace operator and* ω *satisfies the equation*

$$\Delta\omega = \frac{\mu}{3} r^{1/\lambda}\omega^{1+(1/\lambda)}$$

we have

$$\omega \geqq \left(\frac{6}{\mu r^{1/\lambda}(R^2 - r^2)}\right)^\lambda$$

and

$$\overline{\lim_{r\to 0}}\left(\frac{\log \omega}{\log 1/r}\right) \geqq 1 .$$

COROLLARY 4.5. *If the function* ω *is regular in* D_r *and satisfies the differential equation*

$$\Delta\omega = \delta(1 - l\lambda)r^l\omega^{1+(1/\lambda)} \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

we have

$$\left(\frac{4}{\delta r^l(R^2 - r^2)}\right)^\lambda \leqq \omega$$

and also the behaviour of ω *is such that*

$$\overline{\lim_{r\to 0}}\left(\frac{\log \omega}{\log 1/r}\right) \geqq l\lambda .$$

The proof of Corollaries 4.4 and 4.5 is exactly the same as that of 4.3.

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Received January 20, 1966, in revised form January 27, 1967, and July 22, 1969.

This work is a portion of the author's doctoral dissertation written under the direction of Professor Z. Nehari at Carnegie Institute of Technology in 1963, and was supported by the United States Air Force under contract No. AF20A.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON $|C, 1|$ SUMMABILITY FACTORS OF FOURIER SERIES AT A GIVEN POINT

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Let $f(x)$ be a function integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and periodic with period 2π. Let its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
$$\equiv \sum_{n=0}^{\infty} A_n(x) .$$

Whittaker proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha \qquad (\alpha > 0)$$

is summable $|A|$ almost everywhere. Prasad improved this result by showing that the series

$$\sum_{n=n_0}^{\infty} A_n(x) \Big/ \left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon} \qquad (\log^k n_0 > 0)$$

is summable $|A|$ almost everywhere.

In this note, the author is interested particularly in the $|C, 1|$ summability factors of the Fourier series at a given point x_0.

Write

$$\varphi(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0) ,$$
$$\Phi(t) = \int_0^t |\varphi(u)| \, du .$$

The author establishes the following theorems.

THEOREM 1. **If**

$$\Phi(t) = O(t) \qquad (t \to +0) ,$$

then the series

$$\sum_{n=1}^{\infty} A_n(x_0)/n^\alpha$$

is summable $|C, 1|$ for every $\alpha > 0$.

THEOREM 2. **If**

$$\Phi(t) = O\left\{\frac{t}{\prod_{\mu=1}^{k} \log^\mu \frac{1}{t}}\right\}$$

as $t \to +0$, then the series

$$\sum_{n=n_0}^{\infty} \frac{A_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}}$$

is summable $|C, 1|$ for every $\varepsilon > 0$.

A series $\sum a_n$ is said to be absolutely summable (A) or summable $|A|$, if the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is of bounded variation in the interval $\langle 0, 1 \rangle$. Let σ_n^α denote the nth Cesàro mean of order α of the series $\sum a_n$, i.e.,

$$\sigma_n^\alpha = \frac{1}{(\alpha)_n} \sum_{k=0}^{n} (\alpha)_k a_{n-k}, \ (\alpha)_k = \Gamma(k+\alpha+1)/\Gamma(k+1)\Gamma(\alpha+1) \ .$$

If the series

$$\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha|$$

converges, then we say that the series $\sum a_n$ is absolutely summable (C, α) or summable $|C, \alpha|$. It is known that [2] *if a series is summable $|C|$, it is also summable $|A|$, but not conversely.*

2. Suppose that $f(x)$ is a function integrable in the sense of Lebesgue and periodic with period 2π. Let its Fourier series be

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &\equiv \sum A_n(x) \ . \end{aligned}$$

Whittaker [4] proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha \qquad (\alpha > 0)$$

is summable $|A|$ almost everywhere. Prasad [4] improved this result by showing that the series

$$\sum_{n=n_0}^{\infty} A_n(x) \bigg/ \left(\prod_{\mu=1}^{k-1} \log^\mu n \right) (\log^k n)^{1+\varepsilon} (\log^k n_0 > 0) \ ,$$

where $\log^k n = \log(\log^{k-1} n)$, $\log^2 = \log(\log n)$, is summable $|A|$ almost everywhere.

Let (λ_n) be a convex and bounded sequence, Chow [1] demonstrated that the series

$$\sum A_n(x) \lambda_n$$

is summable $|C, 1|$ almost everywhere, if the series $\sum n^{-1} \lambda_n$ converges.

In this note, we are interested particularly in the $|C, 1|$ summability factors of the Fourier series at a given point. For a fixed point x_0, we write

$$\varphi(t) = \varphi_{x_0}(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0) \ ,$$

and

$$\Phi(t) = \int_0^t |\varphi(u)|\, du \, .$$

We are going to establish the following

THEOREM 1. *If*

(i) $$\Phi(t) = O(t)$$

as $t \to +0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{A_n(x_0)}{n^\alpha}$$

is summable $|C, 1|$ *for every* $\alpha > 0$.

3. The following lemmas are required.

LEMMA 1 [3]. *Let* $\alpha > -1$ *and let* τ_n^α *be the nth Cesàro mean of order* α *of the sequence* $\{na_n\}$, *then*

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) \, .$$

LEMMA 2. *Write*

$$S_n(t) = \sum_{k=0}^{n} (n+2-k) \cos (n+2-k)t \, ,$$

then

$$S_n(t) = O\begin{cases} nt^{-1} & (nt \geqq 1) \, , \\ n^2 & (\textit{for all } t) \, . \end{cases}$$

In fact, we have

$$\begin{aligned} S_n(t) &= I\left\{\frac{d}{dt} e^{i(n+2)t} \sum_{k=0}^{n} e^{-ikt}\right\} \\ &= I\left\{\frac{d}{dt}\left(\frac{e^{i(n+2)t}}{1-e^{-it}} - \frac{e^{it}}{1-e^{-it}}\right)\right\} \\ &= I\left\{(n+2)\frac{ie^{i(n+2)t}}{1-e^{-it}} - \frac{ie^{i(n+2)t}}{(1-e^{-it})^2} \right. \\ &\qquad \left. - \frac{ie^{it}}{1-e^{-it}} + \frac{i}{(1-e^{-it})^2}\right\} \\ &= O(nt^{-1}) + O(t^{-2}) \\ &= O(nt^{-1}) \, , \end{aligned}$$

if $nt \leqq 1$. This proves the lemma. From this lemma, we can easily derive the following

LEMMA 3.

$$\left|\frac{1}{n+1}\left\{\sum_{\nu=1}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\}\right| \leqq \begin{cases} \dfrac{A}{th^\alpha} + \dfrac{A}{nt^{2-\alpha}} & (t \geqq 1)\,, \\ An^{1-\alpha} & (\textit{for all } t)\,. \end{cases}$$

By Lemma 2, for $nt \geqq 1$, we write

$$\begin{aligned}\frac{1}{n+1}\left\{\sum_{\nu=1}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\} &= \frac{1}{n+1}\left\{\sum_{\nu=1}^{[t^{-1}]-1} + \sum_{\nu=[t^{-1}]+1}^{n}\right\} + O\left(\frac{1}{nt^{2-\alpha}}\right) \\ &= \frac{1}{n}O\left(\sum_{\nu=1}^{[t^{-1}]}\nu^{1-\alpha}\right) + \frac{1}{nt}O\left(\sum_{\nu=1}^{n}\frac{1}{\nu^\alpha}\right) \\ &\quad + O\left(\frac{1}{nt^{2-\alpha}}\right) \\ &= O\left(\frac{1}{nt^{2-\alpha}}\right) + O\left(\frac{1}{tn^\alpha}\right)\,,\end{aligned}$$

and for all t,

$$\begin{aligned}\frac{1}{n+1}\left\{\sum_{\nu=1}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\} &= \frac{1}{n+1}O\left\{\sum_{\nu=1}^{n}\nu^2\frac{1}{\nu^{1+\alpha}}\right\} \\ &= \frac{1}{n+1}O\left\{\sum_{\nu=1}^{n}\nu^{1-\alpha}\right\} \\ &= O(n^{1-\alpha})\,.\end{aligned}$$

This proves the lemma.

4. We have

$$A_n(x_0) = \frac{2}{\pi}\int_0^\pi \varphi(t)\cos ntdt\,.$$

Let $\tau_n(x_0)$ be the nth Cesàro mean of first order of the sequence $\{nA_n(x_0)/n^\alpha\}$, then

$$\frac{\pi}{2}\tau_n(x_0) = \int_0^\pi \varphi(t)\frac{1}{n+1}\sum_{\nu=0}^{n}\frac{(\nu+2)\cos(\nu+2)t}{(\nu+2)^\alpha}dt\,.$$

Abel's transformation gives

$$\begin{aligned}\frac{\pi}{2}\tau_n(x_0) &= \int_0^\pi \varphi(t)\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} S_\nu(t)\Delta\frac{1}{(\nu+2)^\alpha}\right\}dt \\ &\quad + \int_0^\pi \varphi(t)\frac{1}{n+1}\cdot\frac{S_n(t)}{(n+3)^\alpha}dt \\ &= I_{1n} + I_{2n}\,,\end{aligned}$$

say. Thus, on writing

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{3n} + I_{4n} ,$$

say, we see that

$$I_{3n} = O\left(n^{1-\alpha}\int_0^{1/n} |\varphi|\, dt\right) = O(n^{-\alpha}) ,$$

by condition (i) of the theorem.

$$I_{4n} = O\left\{\frac{1}{n^\alpha}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\} + O\left\{\frac{1}{n}\int_{1/n}^{\pi} \frac{|\varphi|}{t^{2-\alpha}} dt\right\} .$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt = O(1) + O\left\{\int_{1/n}^{\pi} \frac{dt}{t}\right\} = O(\log n) ,$$

and

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t^{2-\alpha}} dt \leqq n^{1-\alpha}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = O(n^{1-\alpha} \log n) .$$

It follows that

$$I_{4n} = O\{\log n/n^\alpha\} .$$

As before, we write

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n} ,$$

say. Then,

$$I_{5n} = O\left(n^{1-\alpha}\int_0^{1/n} |\varphi|\, dt\right) = O(n^{-\alpha}) .$$

And

$$I_{6n} = O\left\{n^{-\alpha}\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\} = O\{\log n/n^\alpha\} ,$$

by the similar arguments as in the estimation of the integral I_{4n}. By Lemma 1, we have to establish the convergence of $\sum |\tau_n(x_0)|/n$. And from the above analysis, it concludes that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|\tau_n(x_0)|}{n} &\leqq \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\{|I_{3n}| + |I_{4n}| + |I_{5n}| + |I_{61}|\} \\ &= O\left\{\sum_{n=1}^{\infty} \frac{\log n}{n^{1+\alpha}}\right\} = O(1) .\end{aligned}$$

This proves Theorem 1.

5. Let $\tau_n(x_0)$ be the nth Cesàro mean of first order of the sequence

$$\left\{nA_n(x_0)\Big/\left(\prod_{\mu=1}^{k-1}\log^\mu n\right)(\log^k n)^{1+\varepsilon}\right\} \qquad (\varepsilon > 0)\ ,$$

where k is a positive integer. Abel's transformation gives

$$\begin{aligned}\frac{\pi}{2}\tau_n(x_0) &= \int_0^\pi \varphi(t)\frac{1}{n+1}\left\{\sum_{\nu=0}^n S_\nu(t)\Delta\frac{1}{\left\{\prod_{\mu=1}^{k-1}\log^\mu(\nu+2)\right\}\{\log^k(\nu+2)\}^{1+\varepsilon}}dt\right.\\ &\quad + \int_0^\pi \varphi(t)\frac{1}{n+1}\cdot\frac{S_n(t)}{\left\{\prod_{\mu=1}^{k-1}\log^\mu(n+3)\right\}\{\log^k(n+3)\}^{1+\varepsilon}}dt\\ &= I_{1n} + I_{2n}\ ,\end{aligned}$$

say. As before, we write

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^\pi = I_{3n} + I_{4n}\ ,$$

say, and

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^\pi = I_{5n} + I_{6n}\ ,$$

say. Since, for $\nu \geqq n_0$,

$$\left|\Delta\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^\mu \nu\right)(\log^k \nu)^{1+\varepsilon}}\right| \leqq \frac{A}{\nu\left(\prod_{\mu=1}^{k-1}\log^\mu \nu\right)(\log^k \nu)^{1+\varepsilon}}\ ,$$

we obtain

$$\left|\frac{1}{n+1}\left\{\sum_{\nu=0}^n S_\nu(t)\Delta\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^\mu(\nu+2)\right)(\log^k(\nu+2))^{1+\varepsilon}}\right.\right|$$

$$\leqq \begin{cases}\dfrac{A}{t\left(\prod_{\mu=0}^{k-1}\log^\mu n\right)(\log^k n)^{1+\varepsilon}} + \dfrac{A}{t^2\left(\prod_{\mu=1}^{k-1}\log^\mu \frac{1}{t}\right)\left(\log^k \frac{1}{t}\right)^{1+\varepsilon}} & (nt \geqq 1)\ ,\\[2ex] \dfrac{An}{\left(\prod_{\mu=1}^{k-1}\log^\mu n\right)(\log^k n)^{1+\varepsilon}} & (\text{for all } t)\ .\end{cases}$$

Now, if

$$\Phi(t) = O\left\{\frac{t}{\left(\prod_{\mu=1}^{k}\log^\mu \frac{1}{t}\right)}\right\}$$

as $t \to +0$, then

$$I_{3n} = O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\mathrm{long}^{k} n)^{1+\varepsilon}} \int_0^{1/n} |\varphi|\, dt\right\}$$

$$= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\}.$$

$$I_{4n} = O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\}$$

$$+ O\left\{\frac{1}{n}\int_{1/n}^{\pi} \frac{|\varphi|}{t^2\left(\prod_{\mu=1}^{k-1} \frac{1}{t}\right)\left(\log^{k} \frac{1}{t}\right)^{1+\varepsilon}}\right\}.$$

But since

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt$$

$$= O(1) + O\left\{\int_{1/n}^{\pi} \frac{dt}{t\left(\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}\right)}\right\}$$

$$= O(1) + O\{\log^{k+1} n\},$$

and

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t^2\left(\prod_{\mu=1}^{k-1} \log^{\mu} \frac{1}{t}\right)\left(\log^{k} \frac{1}{t}\right)^{1+\varepsilon}} dt = O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\}$$

$$= O\left\{\frac{n \log^{k+1} n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\},$$

we obtain

$$I_{4n} = O\left\{\frac{\log^{k+1} n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\}.$$

Finally,

$$I_{5n} = O\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \int_0^{1/n} |\varphi|\, dt\right\}$$

$$= O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\},$$

$$I_{6n} = O\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \int_{1/n}^{\pi} \frac{|\varphi|}{t} dt\right\}$$
$$= O\left\{\frac{\log^{k+1} n}{\left(\prod_{\mu=1}^{k=1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{|\tau_n(x_0)|}{n} = O\left\{\sum_{n=n_0}^{\infty} \frac{\log^{k+1} n}{n\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}}\right\} + O(1)$$
$$= O(1) .$$

Hence, we establish

THEOREM 2. *If*

(ii) $$\Phi(t) = O\left\{\frac{t}{\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}}\right\}$$

as $t \to +0$, *then the series*

$$\sum_{n=n_0}^{\infty} \frac{A_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \qquad (\log^{k} n_0 > 0)$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

6. For the conjugate series

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x) ,$$

we can derive two analogous theorems. Write, for a fixed $x = x_0$,

$$\Psi(t) = \int_0^t |\psi(u)|\, du \equiv \int_0^t |f(x_0 + u) - f(x_0 - u)|\, du .$$

We have the following

THEOREM 3. *If*

(iii) $$\Psi(t) = O(t)$$

as $t \to +0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{B_n(x_0)}{n^{\alpha}}$$

is summable $|C, 1|$ *for every* $\alpha > 0$.

THEOREM 4. *If*

(iv)
$$\Psi(t) = O\left\{\frac{t}{\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}}\right\}$$

as $t \to +0$, *then the series*

$$\sum_{n=n_0}^{\infty} \frac{B_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^{k} n)^{1+\varepsilon}} \qquad (\log^{k} n_0 > 0)$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

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Received February 28, 1969. This paper was written and completed during the author's stay at Berkeley in 1968.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

HOMOTOPY GROUPS OF PL-EMBEDDING SPACES

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Let N be a compact PL-n-manifold, and let M be a PL-m-manifold without boundary. Two of the major problems in PL-topology are to determine conditions such that (1) any continuous map of N into M can be homotoped to a PL-embedding, and (2) two homotopic PL-embeddings are PL-isotopic.

If $C(N, M)$ is the space of continuous maps of N into M with the compact open topology, and if $PL(N, M)$ is the subspace of PL-embeddings, one can consider the map $i_\#: \Pi_0(PL(N, M)) \to \Pi_0(C(N, M))$ induced by inclusion. If (1) is true, then $i_\#$ is onto; if (2) is true, then $i_\#$ is one-to-one. In this paper, we investigate the higher homotopy groups of $PL(N, M)$ and $C(N, M)$.

Irwin has shown that if N is a closed manifold, $m \geqq n + 3$, then sufficient conditions for (1) are that N is $(2n - m)$—connected and M is $(2n - m + 1)$—connected. By raising the connectivities of N and M by one, Zeeman [7] proved (2).

By using Proposition 1 of Morlet [4] and Irwin [3], one can easily show the following theorem by using techniques similar to the proof of Theorem 2 below.

THEOREM 1. *Let N be a closed $(2n + s + 1 - m)$—connected PL-n-manifold and let M be a $(2n + s + 2 - m)$—connected PL-m-manifold without boundary, $m \geqq n + 3$. The homomorphism $i_\#$: $\Pi_s(PL(N, M)) \to \Pi_s(C(N, M))$ induced by inclusion is an isomorphism; if the connectivities of N and M are lowered by one, then $i_\#$ is onto.*

An analogous theorem in the differential case has been proved by J. P. Dax [1], [2].

If N has a nonempty boundary, then Dancis, Hudson and Tindell (independently and unpublished) have shown that if N has a k-dimensional spine with $m \geqq \{n + 3,\ n + k\}$, this is a sufficient condition for (1). If $m \geqq \{n + 3,\ n + k + 1\}$, they obtain (2). We generalize.

THEOREM 2. *Let N be a compact PL-n-manifold with k-spine K, $k < n$, and let M be a PL-m-manifold without boundary. If $m \geqq n + k + s + 1$, the homomorphism $i_\#$: $\Pi_s(PL(N, M)) \to \Pi_s(C(N, M))$ induced by inclusion is an isomorphism; if $m \geqq n + k + s$, $i_\#$ is onto.*

Note that the codimension 3 restriction is eliminated. In § 3, we obtain some consequences of this theorem and its proof.

The author wishes to express his gratitude to N. Max who read a preliminary version of this paper and suggested some corrections.

In this paper, we shall consider $PL(N, M)$ and $C(C, M)$ as Δ-sets (-i.e., as semisimplicial complexes in which the degeneracy maps are ignored). In § 1, we list the basic definitions and results on Δ-sets which we shall use. One may use either Rourke and Sanderson [6] or Morlet [5]. [Morlet uses the terminology "quasisimplicial" set.]

We shall assume familiarity with either [1] or [7] and shall use terminology therein with one exception. When referring to piecewise linear maps or manifolds, we shall always use the prefix "*PL*-".

Let X and Y be polyhedra. In this paper p_1 and p_2 will always denote projections of $X \times Y$ onto the first and second factors respectively. An isotopy between X and Y will be represented as a family of embeddings $f_t: X \rightarrow Y$, $t \in I = [0, 1]$.

1. Δ-sets. Let Δ^n denote the standard n-simplex with ordered vertices $v_0, v_1, \cdots, v_n$. The i-th *face map* $\partial_i: \Delta^{n-1} \rightarrow \Delta^n$ is the order preserving simplicial embedding which omits v_i. Δ is the category whose objects are Δ^n, $n = 0, 1, \cdots$ and whose morphisms are generated by the face maps. A *Δ-set* (*Δ-group*) is a contravariant functor from Δ to the category of sets (groups). A *Δ-map* between Δ-sets (Δ-groups) is a natural transformation between the functors.

If X is a Δ-set, $X^k = X(\Delta^k)$ is the set of *k-simplexes* and the maps $\partial_i = X(\partial_i)$ are called *face maps*. We shall be interested in pointed Δ-sets in which we distinguish a simplex $*^k \in X^k$ for each k and designate $* \subset X$ as the sub-Δ-set of X consisting of these simplexes and maps ∂_i defined by $\partial_i *^k = *^{k-1}$.

With each ordered simplicial complex K, we associate a Δ-set, also designated by K, whose k-simplexes are order-preserving simplicial embeddings of Δ^k into K.

Let $\Lambda_{n,i} = \mathrm{Cl}\,(\mathrm{bdry}\ \Delta^n - \partial_i \Delta^{n-1})$. A Δ-set X is called a *Kan Δ-set* if every Δ-map $f: \Lambda_{n,i} \rightarrow X$ can be extended to a Δ-map $f_1: \Delta^n \rightarrow X$.

If X is a Kan Δ-set and P is a polyhedron, a *map* $f: P \rightarrow X$ is a Δ-map $f: K \rightarrow X$ where K is an ordered triangulation of P. $f_0, f_1: P \rightarrow X$ are *homotopic* if there is a map $F: P \times I \rightarrow X$ such that $F \mid P \times \{i\} = f_i$, $i = 0, 1$. $[P; X]$ denotes the set of homotopy classes. We shall need the following two propositions which are proved by Rourke and Sanderson.

PROPOSITION 1. *Any homotopy class in $[P; X]$ is represented by a Δ-map $f: K \rightarrow X$ where K is any ordered triangulation of P.*

PROPOSITION 2. *Let Q be a subpolyhedron of P and let*

$h: Q \times I \cup P \times \{0\} \to X$ be a Δ-map to a Kan Δ-set X; then h extends to a Δ-map $h': P \times I \to X$.

If X is a pointed Kan Δ-set, then the n-th *homotopy group* of X, $\Pi_n X = [I^n, \mathrm{bdry}\, I^n; X, *]$, the homotopy classes of Δ-maps of pairs, where I^n is the PL-n-cell.

$C(N, M)(PL(N, M))$ is made into a Δ-set by defining the k-simplexes to be maps (PL-embeddings) $f: N \times \Delta^k \to M \times \Delta^k$ such that $p_2 f = p_2$ and defining $\partial_i f = f \mid N \times \partial_i \Delta^k$.

PROPOSITION 3. *$C(N, M)$ and $PL(N, M)$ are Kan Δ-sets.*

Proof. Let $f: \Lambda_{n,i} \to PL(N, M)$ be a Δ-map. f can then be considered as a PL-embedding

$$f: N \times \Lambda_{n,i} \longrightarrow M \times \Lambda_{n,i}$$

such that $p_2 f = p_2$. Using the fact that the pair $(\Lambda_{n,i} \times I, \Lambda_{n,i} \times \{0\})$ is PL-homeomorphic to $(\Delta^n, \Lambda_{n,i})$, one can easily construct the desired extension.

2. Proof of Theorem 1. The following two propositions are generalizations to product spaces of the simplicial approximation and general position theorems. They can be proved similarly.

PROPOSITION 4. *Let M and Y be PL-manifolds and let $P \subseteqq Q$ be compact polyhedra. Suppose $f: Q \to M \times Y$ is a continuous map such that $f \mid P$ is PL. There exists a homotopy $h_t: M \times Y \to M \times Y$, $t \in I$, such that*

(i) *$p_2 h_t = p_2$ for $t \in I$;*

(ii) *$h_t f \mid P = f$ for $t \in I$;*

(iii) *$h_1 f: Q \to M \times Y$ is PL.*

PROPOSITION 5. *Let M and Y be PL-manifolds and let $P \subseteqq Q$ be compact polyhedra. Suppose $f: Q \to M \times Y$ is a PL-map such that $f \mid P$ is a PL-embedding. There exists a PL-homotopy $h_t: M \times Y \to M \times Y$, $t \in I$, such that*

(i) *$p_2 h_t = p_2$ for $t \in I$;*

(ii) *$h_t f \mid P = f$ for $t \in I$;*

(iii) *the singular set of $h_1 f$ has dimension $\leqq 2 \dim Q - \dim (M \times Y)$;*

(iv) *the branch set of $h_1 f$ has dimension $< 2 \dim Q - \dim (M \times Y)$.*

The following two constructions are needed frequently in the following propositions.

PROPOSITION 6. *Let N be a PL-n-manifold with k-spine K. Let*

P be a polyhedron in N such that $\dim P + \dim K + 1 \leqq \dim N$. *There exists a PL-isotopy H_t of N, $t \in I$, such that $H_0 =$ identity and $H_1(N) \cap P = \emptyset$.*

Proof. By general position, we can find a PL-ambient isotopy L_t of N so that $L_1K \cap P = \emptyset$. Let N' be a regular neighborhood of L_1K in N such that $N' \cap P = \emptyset$. Note that L_1K is also a spine of N. Hence, by the uniqueness theorem of regular neighborhoods, there is a PL-isotopy H_t of $N, t \in I$, such that $H_0 =$ identity and $H_1(N) = N'$.

CONSTRUCTION α. Let I^s_+ be a PL-cell in the interior of I^s and let U be a neighborhood of $\mathrm{Cl}\,(I^s - I^s_+)$ in I^s. Let U_0, U_1 be regular neighborhoods of $\mathrm{Cl}\,(I^s - I^s_+)$ in I^s such that $U_0 \subseteqq \mathrm{int}\, U_1$ and $U_1 \subseteqq U$. Let $\varphi: S^{s-1} \times I \to \mathrm{Cl}\,(U_1 - U_0)$ be a PL-homeomorphism such that $\varphi(S^{s-1} \times \{i\}) = \mathrm{bdry}\, U_i \cap \mathrm{int}\, I^s$, $i = 0, 1$.

PROPOSITION 7. *Let N, K, M be as in Theorem 2 with $m \geqq n + k + s$. Let $f: N \times I^s \to M \times I^s$ be a PL-map such that $p_2 f = p_2$ and such that there exists a neighborhood U of* $\mathrm{Cl}\,(I^s - I^s_+)$ *such that $f \mid N \times U$ is a PL-embedding, then there exists a PL-homotopy $f_t: N \times I^s \to M \times I^s$ and a neighborhood V of* $\mathrm{Cl}\,(I^s - I^s_+)$ *in I^s such that*

(i) $f_0 = f$, $p_2 f_t = p_2$, $t \in I$;

(ii) $f_t \mid V = f$, $t \in I$;

(iii) $f_1: N \times I^s \to M \times I^s$ *is a PL-embedding.*

Proof. By Proposition 5, we can assume that the singular set T of f has dimension $\leqq 2(n + s) - (m + s)$, the branch set $B \subset T$ of f has dimension $< 2(n + s) - (m + s)$, and that $f \mid K \times I^s$ is a PL-embedding. By Proposition 6, there is a PL-isotopy H_t of N such that $H_0 =$ identity and $H_1(N) \cap p_1 B = \emptyset$. Hence there is no loss of generality in assuming that $f \mid p_1^{-1}(H_1(N)) \times I^s$ is a PL-embedding.

Let U_0, U_1 and φ be as in construction α. Define $F_t: N \times I^s \to N \times I^s$, $t \in I$, by

$$F_t(x, y) = \begin{cases} (H_t(x), y) & y \in \mathrm{Cl}\,(I^s - U_1) \\ (x, y) & y \in U_0 \\ (H_{tt_0}(x), y) & y \in \mathrm{Cl}\,(U_1 - U_0),\ y = \varphi(y_0, t_0). \end{cases}$$

Let $f_t = fF_t$ and $V = U_0$.

The following is the theorem of Dancis, Hudson and Tindell mentioned in the introduction. We include the proof for completeness.

PROPOSITION 8. *Let N, K, M be as in Theorem 2 with $m \geqq n + k$. There exists a PL-embedding $f: N \to M$.*

Proof. Let $f'\colon N \to M$ be a continuous map and approximate f' by a PL-map f'' such that f''/K is a PL-embedding and f'' is in general position. Let $B \subset S$ be the branch and singular set of f'' respectively. By Proposition 6, there is a PL-isotopy H_t, $t \in I$, of N such that $H_1(N) \cap S = S \cap K$. Let $f = f''H_1$.

REMARK. We shall make $PL(N, M)$ and $C(N, M)$ into pointed Δ-sets by defining the basepoint complex $*$ as follows. Let $*^s(x, y) = (f(x), y)$, $x \in N$, $y \in \Delta^s$ where f is defined in Proposition 8. The face operators are defined naturally.

The proof of the following proposition is well known.

PROPOSITION 9. Let N, M, K *be as in Theorem* 2 *with* $m \geqq n + k$. *Let* $g\colon N \times I^s \to M \times I^s$ *represent an s-simplex in* $PL(N, M)(C(N, M))$ *such that*

$$g \mid N \times \operatorname{bdry} I^s = *^s \mid N \times \operatorname{bdry} I^s ,$$

g is homotopic rel bdry I^s *in* $PL(N, M)(C(N, M))$ *to* $g'\colon N \times I^s \to M \times I^s$ *such that for some neighborhood* U *of* $\operatorname{Cl}(I^s - I^s +)$ *in* I^s, $g' \mid N \times U = *^s \mid N \times U$.

PROPOSITION 10. *Let* N, M, K *be as in Theorem* 2 *with* $m \geqq n + k + s + 1$ *and let* $F_t\colon N \times I^s \to M \times I^s$ *be a PL-homotopy such that*

(i) F_i *are PL-embeddings*, $i = 0, 1$;

(ii) $p_2F_t = p_2$, $t \in I$:

(iii) *there exists a neighborhood* U *of* $\operatorname{Cl}(I^s - I^s_+)$ *in* I^s *such that* $F_t \mid N \times U = *^s$.

Then there exists a PL-isotopy $G_t\colon N \times I_s \to M \times I^s$ *such that*

(i) $G_i = F_i$ *for* $i = 0, 1$;

(ii) $p_2G_t = p_2$, $t \in I$:

(iii) *there exists a neighborhood* V *of* $\operatorname{Cl}(I^s - I^s_+)$ *in* I^s *such that* $G_t \mid N \times V = *^s$.

Proof. Note that there is no loss of generality in assuming that there is an $\varepsilon > 0$ so that F_t are PL-embeddings, $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$. However, now this is a restatement of Proposition 7.

The proof of Theorem 2 now follows easily from the above propositions.

3. Applications. One of the immediate consequences of Theorem 2 is a partial generalization of Hudson's "concordance implies isotopy" theorem [2]. (See also Proposition 1 of [4].)

COROLLARY 1. *Let N be a compact PL-n-manifold with k-spine K, $k < n$, and let M be a PL-m-manifold without boundary. Let $f: N \times I^s \to M \times I^s$ be a PL-embedding such that $p_2 f \mid N \times \mathrm{bdry}\, I^s = p_2$. Then if $m \geqq n + k + s$, there exists a PL-embedding $F: N \times I^s \to M \times I^s$ such that $F \mid N \times \mathrm{bdry}\, I^s = f$ and $p_2 F = p_2$. If $m \geqq n + k + s + 1$, f and F can be chosen to be isotopic* rel $N \times \mathrm{bdry}\, I^s$.

Let X be an s-dimensional polyhedron and let $p: E \to X$ and $q: F \to X$ be PL-fiber bundles with fibers N and M respectively with structure groups $\mathrm{Aut}\,(N)$ and $\mathrm{Aut}\,(M)$ where

(i) N is a PL-n-manifold with k-spine, $k < n$;

(ii) M is a PL-m-manifold without boundary;

(iii) $\mathrm{Aut}\,(N)$ and $\mathrm{Aut}\,(M)$ are the groups of PL-automorphisms of N and M, respectively.

By triangulating X and by using the propositions above together with induction on the dimension of the simplexes of X, one can easily prove the following.

COROLLARY 2. *If $f: E \to F$ is a continuous bundle map* (-i.e., $qf = p$) *and $m \geqq n + k + s$, then f is homotopic through bundles maps to a PL-bundle map which is an embedding of E into F. If $m \geqq n + k + s + 1$; any two PL-bundle embeddings of E into F are isotopic through bundle maps.*

A PL_m-bundle is a PL-bundle $q: F \to X$ whose fiber is Euclidean m-space R^m and whose structural group is the PL-automorphisms of R^m mod the origin.

COROLLARY 3. *Let N be a PL-n-manifold with k-spine, $k < n$; let $p: E \to X^s$ be a PL-fiber bundle with N as fiber and* $\mathrm{Aut}\,(N)$ *as structural group. If $m \geqq n + k + s$, then for any PL_m-bundle $q: F \to X$, there exists a PL-bundle map $f: E \to F$ which is an embedding. If $m \geqq n + k + s + 1$, then any such two PL-bundle embeddings are isotopic through bundle maps.*

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Received April 11, 1969. Research supported in part by National Science Foundation Grant GP-8615.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

INTEGRATION WITH RESPECT TO VECTOR MEASURES

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The purpose of this paper is to develop a theory of integration with respect to measures into a locally convex, Hausdorff linear topological space, using a linear functional approach.

Section 1 presents some basic facts about such measures, chiefly through the study of their p-semi-variations (Definition 1.2). Devices of this sort have been considered by other authors (see [1], [4]), but chiefly to give expressions for the norms of linear operators defined by vector measures. We consider the continuity properties of the p-semi-variation, and define regularity in terms of the p-semi-variation.

The integration theory is developed in § 2. Although the integral is defined in terms of linear functionals, it is in no sense a weak integral. The dominated (strong) convergence theorem is proven under the additional assumption that the limit function is integrable, and it is shown that this is true whenever the range space of the measure is sequentially complete.

In § 3 integral representations of weakly compact operators from $C(S)$, $C_0(T)$, $C_c(T)$ and $C(T)_\beta$ into a locally convex, Hausdorff space are given. We used these representations to show that the above spaces satisfy a strengthened rersion of the Dunford-Pettis Property-specifically that a weakly compact operator on these spaces maps weakly Cauchy sequences into convergent sequences, without any assumption about the completeness of the range of the operator.

1. Throughout the first two sections (S, Σ) denotes a measurable space, $(X, \mathscr{T})$ a complex, locally convex Hausdorff linear topological space with dual X^* and μ an additive set function from Σ into X. If $x^* \in X^*$ and p is a semi-norm on X we will write $x^* \leqq p$ whenever $|x^*(x)| \leqq p(x)$ for all $x \in X$. The following theorem, stated here without proof, was first proved by Pettis [6] for normed spaces and Grothendieck [5] for locally convex spaces.

THEOREM 1.1. *If μ is countably additive in the weak topology, then μ is countably additive in $\mathscr{T}$.*

DEFINITION 1.2. If p is a semi-norm on X, then the *p-semivariation* of μ is the function from Σ into the extended reals defined by $\|\mu\|_p(E) = \sup_{x^* \leqq p} v(x^*\mu, E)$, where $v(x^*\mu, \cdot)$ is the scalar variation of

$x^*\mu$.

It follows immediately that $\|\mu\|_p(\cdot)$ is monotone, subadditive and that $p[\mu(E)] \leqq \|\mu\|_p(E) \leqq 4 \sup_{F\subset E} p[\mu(F)]$ for each $E \in \Sigma$. If μ is a measure then $\|\mu\|_p(\cdot)$ is countably subadditive and real valued, since the range of μ is bounded.

THEOREM 1.3. *If μ is a measure, p a continuous semi-norm and (E_n) a convergent sequence in Σ, then* $\|\mu\|_p(\lim_n E_n) = \lim_n \|\mu\|_p(E_n)$.

Proof. We first establish a special case. Suppose (E_n) is a decreasing sequence in Σ with empty intersection and that, for some $\varepsilon > 0$, $\|\mu\|_p(E_n) > \varepsilon$ for each n. Let $n_1 = 1$. For some $x^* \leqq p$ and $n_2 > n_1$, $v(x^*\mu, E_{n_1}) > \varepsilon$ and $v(x^*\mu, E_{n_2}) < \varepsilon/2$. Then also $4 \sup_{F\subset E_{n_1}\backslash E_{n_2}} p[\mu(F)] \geqq v(x^*\mu, E_{n_1}\backslash E_{n_2}) > \varepsilon/2$, so for some $F_1 \subset E_{n_1}\backslash E_{n_2}$ we have

$$p[\mu(F_1)] > \frac{\varepsilon}{8}\,.$$

Continuing in this manner there is an increasing sequence (n_k) of positive integers and a sequence (F_k) in Σ such that $F_k \subset E_{n_k}\backslash E_{n_{k+1}}$ and $p[\mu(F_k)] > \varepsilon/8$ for each k. This contradicts the countable additivity of μ since the F_k's are pairwise disjoint.

If (E_n) is in Σ and has limit E, then $\|\mu\|_p(E) = \lim_n \|\mu\|_p(E_n)$ since the inequality

$$\big|\|\mu\|_p(E) - \|\mu\|_p(E_n)\big| \leqq \|\mu\|_p[\textstyle\bigcup_{k\geqq n}(E\backslash E_k)] + \|\mu\|_p[\textstyle\bigcup_{k\geqq n}(E_k\backslash E)]$$

holds.

COROLLARY 1.4. *If μ is a measure and (E_n) is a convergent sequence in Σ, then* $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.

Proof. Let $E = \lim_n E_n$. The corollary follows since

$$\lim_n(E\backslash E_n) = \lim_n(E_n\backslash E) = \varnothing$$

and $p[\mu(E) - \mu(E_n)] < \|\mu\|_p(E\backslash E_n) + \|\mu\|_p(E_n\backslash E)$ for each n and semi-norm p.

DEFINITION 1.5. Suppose S is a topological space. μ is *regular* (in $\mathscr{T}$) if, for each $E \in \Sigma$ $\varepsilon > 0$ and continuous semi-norm p on X, there is a relatively compact set K in Σ whose closure is contained in E and a set G in Σ whose interior contains E such that

$$\|\mu\|_p(G\backslash K) < \varepsilon\,.$$

If μ is bounded and regular, then $x^*\mu$ is bounded and regular for each $x^* \in X^*$. This implies that $x^*\mu$ is countably additive for each $x^* \in X^*$, which in turn implies that μ is a measure. Also, regularity in the weak topology is equivalent to the regularity of $x^*\mu$ for each $x^* \in X^*$.

THEOREM 1.6. *If μ is regular in the weak topology and μ is a measure, then μ is regular in $\mathscr{T}$.*

Proof. If μ is not inner regular there is an $E \in \Sigma$, a positive ε and a $\mathscr{T}$-continuous semi-norm p such that $\|\mu\|_p(E\backslash K) > \varepsilon$ for each relatively compact K in Σ with $\bar{K} \subset E$. Let $K_1 = \varnothing$. There is an $x_1 \leqq p$ and a relatively compact set K_2 in Σ such that

$$K_1 \subset K_2,\ \bar{K}_2 \subset E,\ v(x_1\mu, E\backslash K_1) > \varepsilon$$

and $v(x_1\mu, E\backslash K_2) < \varepsilon/2$. Since $\|\mu\|_p(E\backslash K_2) > \varepsilon$ we may continue this process and obtain a sequence (x_n) of functionals dominated by p and an increasing sequence (K_n) of relatively compact sets in Σ such that $\bar{K}_n \subset E$, $v(x_n\mu, E\backslash K_n) > \varepsilon$ and $v(x_n\mu, E\backslash K_{n+1}) < \varepsilon/2$ for each n. Let $K = \bigcup_{n \geqq 1} K_n$. Since $\lim_n(K\backslash K_n) = \varnothing$ there is an m such that

$$\|\mu\|_p(K\backslash K_m) < \frac{\varepsilon}{2}\,.$$

But then $v(x_m\mu, E\backslash K_m) \leqq v(x_m\mu, E\backslash K_{m+1}) + \|\mu\|_p(K\backslash K_m) < \varepsilon$. The outer regularity of μ is proven similarly.

2. Throughout this section μ is a fixed measure from Σ into $(X, \mathscr{T})$ and $\mathbb{C}$ the complex numbers.

DEFINITION 2.1. A function $f\colon S \to \mathbb{C}$ is *μ-integrable* if
(1) f is $x^*\mu$-integrable for each $x^* \in X^*$, and
(2) for each $E \in \Sigma$ there is an element of X, denoted by

$$\int_E f(t)\mu(dt)\ ,$$

such that $x^*\int_E f(t)\mu(dt) = \int_E f(t)x^*\mu(dt)$ for each $x^* \in X^*$.

Since $(X, \mathscr{T})$ is Hausdorff the integral is well-defined. No assumption is made about the completeness of $(X, \mathscr{T})$. $\mathscr{T}$ enters into the definition of the integral only in that it determines X^*, so that in any of the results of this section $\mathscr{T}$ may be replaced by any topology in the Mackey spectrum of $(X, \mathscr{T})$.

It is easy to see that the integral has the following properties:

(1) The integral is linear.

(2) Every simple function $\sum_{i\leqq n}a_i\chi_{E_i}$ is μ-integrable and

$$\int_E\Big(\sum_{i\leqq n}a_i\chi_{E_i}\Big)(t)\mu(dt)=\sum_{i\leqq n}a_i\mu(E\cap E_i)$$

for $E\in\Sigma$.

(3) If f is bounded and μ-integrable, then

$$p\Big[\int_E f(t)\mu(dt)\Big]\leqq\|\mu\|_p(E)\cdot\sup_{s\in S}|f(s)|$$

for each $E\in\Sigma$ and continuous semi-norm p.

(4) If f is μ-integrable and T is a continuous linear operator from X into a locally convex Hausdorff space Y, then f is $T\mu$-integrable and $\int_E f(t)T\mu(dt)=T\int_E f(t)\mu(dt)$ for $E\in\Sigma$.

THEOREM 2.2. (1) *If f is μ-integrable, then the set function on Σ defined by $\Phi(E)=\int_E f(t)\mu(dt)$ is a measure,*

$$\|\Phi\|_p(E)=\sup_{x^*\leqq p}\int_E|f(t)|v(x^*\mu,dt)\ \textit{and}\ \lim_{\|\mu\|_p(E)\to 0}\|\Phi\|_p(E)=0$$

for each continuous semi-norm p.

(2) *Let (f_n) be a sequence of μ-integrable functions which converge pointwise to f on S and g be a μ-integrable function such that $|f_n|\leqq|g|$ for each n. f is μ-integrable if X is sequentially complete. If f is μ-integrable then*

$$\int_E f(t)\mu(dt)=\lim_n\int_E f_n(t)\mu(dt)$$

uniformly with respect to $E\in\Sigma$.

Proof. The set function Φ in (1) is a measure by Pettis's Theorem, and the expression for its p-semi-variation is correct since

$$v(x^*\Phi,E)=\int_E|f(t)|v(x^*\mu,dt)$$

for each x^*. Clearly $\|\Phi\|_p(E)=0$ whenever $\|\mu\|_p(E)=0$, so in light of Theorem 1.3 the usual proof by contradiction that zero-zero and $\varepsilon-\delta$ absolute continuity are equivalent in finite measure spaces establishes that $\lim_{\|\mu\|_p(E)\to 0}\|\Phi\|_p(E)=0$.

To prove (2) we first show that

$$\Big(\int_E f_n(t)\mu(dt)\Big)$$

is Cauchy uniformly with respect to $E \in \Sigma$. Let p be a continuous seminorm, $\varepsilon > 0$,

$$\Phi(F) = \int_F g(t)\mu(dt)$$

and $E_n = \{s \in S: |f(s) - f_n(s)| \geqq \varepsilon\}$. If $E \in \Sigma$ and $x^* \leqq p$, then f is $x^*\mu$-integrable by the dominated convergence theorem for scalar measures and for each n

$$\left|\int_E (f - f_n)(t)x^*\mu(dt)\right| \leqq \varepsilon \| \mu \|_p(E \backslash E_n) + 2\| \Phi \|_p(E \cap E_n) .$$

Thus

$$p\left[\int_E f_n(t)\mu(dt) - \int_E f_m(t)\mu(dt)\right]$$
$$\leqq 2\varepsilon\| \mu \|_p(S) + 2\| \Phi \|_p(E_n) + 2\| \Phi \|_p(E_m)$$

for all n and m. The sequence is Cauchy since $\lim_n \| \Phi \|_p(E_n) = 0$.

The first assertion of (2) follows from this Cauchy condition and the dominated convergence theorem, and the second is true since

$$\left(\int_E f_n(t)\mu(dt)\right)$$

is uniformly Cauchy in $\mathscr{T}$ with respect to E and converges weakly to $\int_E f(t)\mu(dt)$.

It follows that every bounded measurable function is μ-integrable if X is sequentially complete. The last two theorems of this section give characterizations of μ-integrable functions when X is sequentially complete.

LEMMA 2.3. *Let λ be a complex valued measure on (S, Σ) and (f_n) a sequence of λ-integrable functions such that*

(1) *(f_n) converges to f pointwise on S, and*

(2) *$\left(\int_E f_n(t)\lambda(dt)\right)$ is Cauchy for each $E \in \Sigma$. Then f is λ-integrable and $\int_E f(t)\lambda(dt) = \lim_n \int_E f_n(t)\lambda(dt)$ uniformly with respect to $E \in \Sigma$.*

Proof. The proof follows the standard argument. For each m define λ_m on Σ by $\lambda_m(E) = \int_E f_m(t)\lambda(dt)$. Each λ_m is $v(\lambda)$-continuous and by (2) $\lim_m \lambda_m(E)$ exists for each $E \in \Sigma$. By the Vitali-Hahn-Saks Theorem $\lim_{v(\lambda, E) \to 0} \lambda_m(E) = 0$ uniformly in m. Let $\varepsilon > 0$ and

$$E_n = \{s \in S: |f_n(s) - f(s)| \geqq \varepsilon\}$$

for each n. By (1) $\lim_n E_n = \emptyset$, so $\sup_{n \geqq n_0} \sup_m v(\lambda_m, E_n) < \varepsilon$ for some n_0. Also, for $n \geqq n_0$,

$$\begin{aligned}
&\int |f(t) - f_n(t)| v(\lambda, dt) \\
&\leqq \varepsilon v(\lambda, S \backslash E_n) + \liminf_m \int_{E_n} |f_m(t) - f_n(t)| v(\lambda, dt) \\
&\leqq \varepsilon [v(\lambda, S) + 2] .
\end{aligned}$$

This inequality establishes the lemma.

THEOREM 2.4. *Suppose X is sequentially complete and f is a complex valued function on S. The following are equivalent:*

(1) *f is μ-integrable.*

(2) *There is a sequence (f_n) of bounded measurable functions which converges pointwise to f and for which $\left(\int_E f_n(t)\mu(dt)\right)$ is Cauchy uniformly with respect to $E \in \Sigma$.*

(3) *There is a sequence (f_n) of simple functions which converges pointwise to f and for which $\left(\int_E f_n(t)\mu(dt)\right)$ is Cauchy for each $E \in \Sigma$.*

Proof. For each n let $E_n = \{s \in S: |f(s)| \leqq n\}$ and $f_n = f\chi_{E_n}$. If f is μ-integrable then (f_n) satisfies condition (2) by Theorem 2.2. (2) clearly implies (3). To see that (3) implies integrability, let $E \in \Sigma$ and $x_E = \lim_n \int_E f_n(t)\mu(dt)$. For $x^* \in X^*$, an application of Lemma 2.3 with $\lambda = x^*\mu$ shows that

$$x^*(x_E) = \lim_n \int_E f_n(t) x^*\mu(dt) = \int_E f(t) x^*\mu(dt) .$$

Notice that if f and (f_n) satisfy condition (2) or (3) of Theorem 2.4, then, by Lemma 2.3,

$$\int_E f(t)\mu(dt) = \lim_n \int_E f_n(t)\mu(dt)$$

for each $E \in \Sigma$. Also, a reformation of (3) with the word Cauchy replaced by convergent is equivalent to (1) without sequential completeness.

We next consider the case in which X is normed. Here we need only one semi-variation of μ, that with $p(x) = \|x\|$. The dual of X^* under its natural norm topology will be denoted by X^{**}.

DEFINITION 2.5. Suppose X is a normed space. A function $f: S \to \mathbb{C}$ has a *generalized integral with respect to μ* if f is $x^*\mu$-

integrable for each $x^* \in X^*$. If f is such a function, then $\int_E f d\mu$ is the linear form on X^* defined by $\left(\int_E f d\mu\right)x^* = \int_E f(t)x^*\mu(dt)$.

If f is μ-integrable, then $\int_E f du$ is the image of $\int_E f(t)\mu(dt)$ under the natural map from X into X^{**}. $\int_E f du$ is always in X^{**}, since it is the pointwise limit of a sequence of the integrals of simple functions.

THEOREM 2.6. *Suppose X is a Banach space and $f: S \to ¢$ has a generalized integral. The following are equivalent:*

(1) *f is μ-integrable.*

(2) *The set function Φ from Σ into X^{**} defined by*

$$\Phi(E) = \int_E f d\mu$$

is measure in the norm topology on X^{**}.

(3) $\lim_{\|\mu\|(E)\to 0}\left\|\int_E f d\mu\right\| = 0.$

Proof. (1) implies (3) by Theorem 2.2. (2) is immediate from (3). For each n let $F_n = \{s \in S: |f(s)| \leqq n\}$ and $f_n = f\chi_{E_n}$. For n and m positive integers and $E \in \Sigma$,

$$\left\|\int_E f_m(t)\mu(dt) - \int_E f_m(t)\mu(dt)\right\| \leqq \|\Phi\|(S\backslash E_n) + \|\Phi\|(S\backslash E_m)\ ,$$

so that (f_n) satisfies condition (2) of Theorem 2.4 if Φ is a measure.

3. Below S is a compact Hausdorff space, Σ the Borel sets of S, $C(S)$ the Banach space (under supremum norm) of continuous complex valued functions on S and $(X, \mathscr{T})$ a locally convex, Hausdorff linear topological space.

THEOREM 3.1. *Let $A: C(S) \to X$ be a weakly compact linear operator. There is a measure $\mu: \Sigma \to X$ such that*

(1) *μ is regular,*

(2) *the closed absolutely convex hull of $\mu[\Sigma]$ is weakly compact,*

(3) *every bounded Borel function on S is μ-integrable,*

(4) *$Af = \int f(t)\mu(dt)$ for $f \in C(S)$, and*

(5) *$A^*x^* = x^*\mu$ for $x^* \in X^*$.*

Conditions (1) *and* (4) *define μ uniquely.* $\|A\| = \|\mu\|(S)$ *whenever X is normed. If μ is a measure on Σ which satisfies* (1), (2), *and*

(3), *then* (4) *defined a weakly compact operator which satisfies* (5). *If X is complete, then* (2) *and* (3) *follow from* (1).

Proof. Since the dual of $C(S)$ may be identified with the bounded regular Borel measures on Σ, the equation

$$g^\wedge(\lambda) = \int g(t)\lambda(dt)$$

defines an element of $C(S)^{**}$ for each bounded Borel function g. Since A is weakly compact, A^{**}, the algebraic adjoint of A^*, maps $C(S)^{**}$ into X. For $E \in \Sigma$ let $\mu(E) = A^{**}(\chi_E^\wedge)$. For each $x^* \in X^*$, $x^*\mu = A^*x^*$ is a regular measure, so μ is a regular measure. Since A^{**} maps the unit ball of $C(S)^{**}$ into a weakly compact subset of X, condition (3) is satisfied. If $\sum_{i \leqq n} a_i \chi_{E_i}$ is a simple Borel function, then

$$A^{**}\Big[\Big(\sum_{i\leqq n} a_i\chi_{E_i}\Big)^\wedge\Big] = \sum_{i\leqq n} a_i A^{**}(\chi_{E_i}^\wedge) = \int\Big(\sum_{i\leqq n} a_i \chi_{E_i}\Big)(t)\mu(dt) .$$

Thus $x^*(A^{**}g^\wedge) = \int g(t)x^*\mu(dt)$ holds for each bounded Borel function g and $x^* \in X^*$. Finally, $\|A\| = \|A^*\| = \|\mu\|(E)$ if X is normed.

Conversely, suppose μ satisfies (1), (2), (3). The operator A defined by (4) is continuous since $p[Af] \leqq \|f\| \, \|\mu\|_p(S)$ for each continuous semi-norm p on X. Also, by the regularity of μ, $A^*x^* = x^*\mu$ for $x^* \in X^*$. Let U be the polar of the closed, absolutely convex hull of $\nu[\Sigma]$. U is a neighborhood of zero in the Mackey topology on X^*, and, for $x^* \in U$, $\|A^*x^*\| \leqq 4$. Thus A^* is continuous with the Mackey topology on X^* and the norm topology on $C(S)^*$-this implies that A is weakly compact.

If X is complete and μ is regular, then $Af = \int f(t)\mu(dt)$ defines a continuous linear operator for $C(S)$ into X such that $A^*x^* = x^*\mu$ for $x^* \in X^*$. To see that (2) holds, it is sufficient to show that A is weakly compact-equivalently, that A^* maps equicontinuous sets into weakly relatively compact sets. Let V be an open neighborhood of zero in X generated by a semi-norm p. For $x^* \in V^\circ$ and $E \in \Sigma$,

$$|x^*\mu(E)| \leqq \|\mu\|_p(E) ,$$

so the countable additivity of $A^*[V^\circ]$ is uniform. This together with norm boundedness implies that $A^*[V^\circ]$ is relatively weakly compact.

In [1] Bartle, Dunford and Schwartz have given a similar integral representation for weakly compact operators from $C(S)$ into a Banach space. Grothendieck [5] has noted that there is a one-to-one correspondence between the weakly compact operators from $C(S)$ into a complete, locally convex, Hausdorff space X and the X-valued measures on the Baire sets of S, although he did not give an integral representation of such operators.

Let T be a locally compact Hausdorff space. $C_0(T)[C_c(T)]$ is the Banach space under supremum norm of complex valued functions on T which vanish at infinity [have compact support]. $C(T)_\beta$ is the space of bounded, continuous, complex valued functions on T topologized by the semi-norms $p_\varphi(f) = \sup_{t \in T} | \varphi(t)f(t) |$, where $\varphi \in C_0(T)$. A weakly compact operator $A: C_0(T) \to X$ can be represented by integration with respect to an X-valued measure since A can be extended to a weakly compact operator on the space of continuous functions over the one point compactification of T. Weakly compact operators on $C_c(T)$ have such a representation since they can be extended to $C_0(T)$. The bounded sets of $C(T)_\beta$ are precisely the uniformly bounded sets and each element of $C(T)_\beta^*$ can be identified with a bounded regular Borel measure on T [2], so the proof of Theorem 3.1 generalizes immediately for $C(S)_\beta$.

COROLLARY 3.2. *Let A be a weakly compact operator from one of $C(S)$, $C_0(T)$, $C_c(T)$ or $C(T)_\beta$ into X. A maps weakly Cauchy sequences into convergent sequences.*

Proof. If (f_n) is weakly Cauchy in any of the above spaces, then (f_n) is uniformly bounded and converges pointwise to a bounded Borel function f. Let μ be the measure determining A. f is μ-integrable and, by Theorem 2.1, $\int f(t)\mu(dt) = \lim_n Af_n$.

This is proven in [5] for $C(S)$ under the assumption that X is complete.

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Received February 10, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

$\mathscr{L}-2$ SUBSPACES OF GRASSMANN PRODUCT SPACES

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The subspaces of the second order Grassmann product space consisting of products of a fixed irreducible length k and zero are interesting not only for their own sake and their usefulness when determining the structure of linear transformations on the product space into itself which preserve the irreducible length k, but also because they are isomorphic to subspaces of skew-symmetric matrices of fixed rank $2k$. The structure of these subspaces and the corresponding preservers are known for $k=1$, when the underlying field F is algebraically closed. This paper gives a complete characterization of these subspaces when $k=2$ and F is algebraically closed. When F is not algebraically closed, these subspaces can be different.

Let $\mathscr{U}$ be an n-dimensional vector space over an *algebraically closed* field F. Let $\bigwedge^2 \mathscr{U}$ denote the $\binom{n}{2}$-dimensional space spanned by all Grassmann products $x_1 \wedge x_2$, $x_i \in F$. A vector $f \in \bigwedge^2 \mathscr{U}$ is said to have *irreducible length* k if it can be written as a sum of k, and *not less than* k, nonzero pure (decomposable) products in $\bigwedge^2 \mathscr{U}$. Let $\mathscr{L}_k$ denote the set of all vectors of irreducible length k in $\bigwedge^2 \mathscr{U}$, and $f \in \mathscr{L}_k$ if and only if $\mathscr{L}(f) = k$. A subspace of $\bigwedge^2 \mathscr{U}$ whose nonzero members are in $\mathscr{L}_k$ is called an $\mathscr{L}-k$ *subspace.*

An $\mathscr{L}-2$ subspace H is a (1, 1)-*type subspace* if there exist fixed nonzero vectors $x \neq y$ such that each nonzero $f \in H$ can be written $f = x \wedge x_f + y \wedge y_f$. A basis of a (1, 1)-type subspace is called a (1, 1) basis. When $\dim \mathscr{U} = 4$, every $\mathscr{L}$-2 subspace has dimension one ([4], Th. 10).

It is shown here that (i) for $\dim \mathscr{U} = n \geqq 5$, there always exists an $\mathscr{L}-2$ subspace of (1, 1)-type and dimension two; (ii) the 2-dimensional $\mathscr{L}-2$ subspaces are of (1, 1)-type; (iii) every $\mathscr{L}-2$ subspace of dimension at least four is of (1, 1)-type; (iv) the $\mathscr{L}-2$ subspaces have dimension at most $(n-3)$ when $n \geqq 6$; and this maximum dimension is attained. Also the 3-dimensional $\mathscr{L}-2$ subspaces are characterized, and these are the most varied.

From [4], Theorem 5, each $f \in \mathscr{L}_k$ can be uniquely associated with a $2k$-dimensional subspace $[f]$ of $\mathscr{U}$. The pair $\{f_1, f_2\}$ is said to be a P_m-pair in $\mathscr{L}_2$ if $[f_1] + [f_2]$ has dimension m; and the set $\{f_1, \cdots, f_k\}$ in $\mathscr{L}_2$ is pairwise-P_m if each pair is a P_m-pair, for $i \neq j$.

THEOREM 1. *Let* $\dim \mathscr{U} = n \geqq 5$. *Then there always exists a*

(1, 1)-type $\mathscr{L}-2$ subspace of dimension two.

Proof. For $n=5$, $u_1, \cdots, u_5$ independent in $\mathscr{U}$, the subspace $\langle u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_5 + u_1 \wedge u_3\rangle$ is a (1, 1)-type $\mathscr{L}-2$ subspace of dimension two. For $n=6$, $u_1, \cdots, u_6$ independent in $\mathscr{U}$, the subspace $\langle u_1 \wedge u_2 + u_3 \wedge u_4, u_1 \wedge u_5 + u_3 \wedge u_6\rangle$ is a (1, 1)-type $\mathscr{L}-2$ subspace of dimension two.

THEOREM 2. *Every 2-dimensional $\mathscr{L}-2$ subspace is a (1, 1)-type subspace.*

The theorem follows from the following Lemmas 1 to 4.

LEMMA 1. *Let f_1 and f_2 be a P_7-pair in $\mathscr{L}_2$, a, b be nonzero in F. Then $\mathscr{L}(af_1 + bf_2) = 3$.*

Proof. Let $[f_1] \cap [f_2] = \langle x_1\rangle$. By Lemma 9 of [4], we can choose a basis $\{x_1, \cdots, x_4\}$ of $[f_1]$ such that $f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$ and a basis $\{x_1, x_5, x_6, x_7\}$ such that $f_2 = x_1 \wedge x_5 + x_6 \wedge x_7$, with $[f_1] + [f_2] = \langle x_1, \cdots, x_7\rangle$. Then $z = af_1 + bf_2 = x_1 \wedge (ax_2 + bx_5) + ax_3 \wedge x_4 + bx_6 \wedge x_7$ and $\mathscr{L}(z) = 3$ by Theorem 7 of [4].

LEMMA 2. *Let f_1, f_2 be a basis of a 2-dimensional $\mathscr{L}-2$ subspace. Then $\{f_1, f_2\}$ is a P_k-pair where k is either 5 or 6.*

Proof. Each of $[f_1]$ and $[f_2]$ has dimension four. It is easy to see that k cannot be 4 (Theorem 10 of [4]). By Lemma 1, we conclude $k \neq 7$. If $k = 8$, Theorem 6 of [4] implies that $\mathscr{L}(f_1 + f_2) = 4$. Hence k is either 5 or 6.

DEFINITION. $f_1, f_2 \in \mathscr{L}_2$ can be expressed in (1, 1)-form if $\{f_1, f_2\}$ have representations $f_i = x \wedge u_i + y \wedge v_i$, $i = 1, 2$ and $\langle x, y\rangle$ is a fixed 2-dimensional subspace of $\mathscr{U}$.

LEMMA 3. *Let $\{f_1, f_2\}$ be a P_5-pair and a basis for an $\mathscr{L}-2$ subspace. Then $\{f_1, f_2\}$ have representations*

$$f_1 = y_4 \wedge u_1 + u_2 \wedge u_3 ,$$
$$f_2 = y_5 \wedge u_2 + u_1 \wedge u_3 ,$$

where $\{u_1, u_2, u_3, y_4, y_5\}$ is some basis of $[f_1] + [f_2]$.

Proof. Let $\mathscr{U}_0 = [f_1] \cap [f_2]$. By Lemma 9 of [4], there are representations

$$f_1 = x_1 \wedge v_1 + v_2 \wedge v_3 ,$$
$$f_2 = x_2 \wedge w_1 + w_2 \wedge w_3 ,$$

where $\langle v_1, v_2, v_3\rangle = \langle w_1, w_2, w_3\rangle = \mathscr{U}_0$. If v_1, w_1 are dependent then some combination of f_1 and f_2 has irreducible length $\leqq 1$. Hence they are independent. Moreover $\langle v_1, w_1\rangle \cap \langle v_2, v_3\rangle$ and $\langle v_1, w_1\rangle \cap \langle w_2, w_3\rangle$ are both nonnull, and hence, without loss of generality, both v_2 and w_2 are in $\langle v_1, w_1\rangle$. Thus $v_2 = av_1 + bw_1$ and $w_2 = cv_1 + dw_1$. Clearly $b \neq 0$, $c \neq 0$. Finally

$$w_3 = pv_1 + qw_1 + rv_3,\ r \neq 0 .$$

Setting $y_4 = br^{-1}c^{-1}(x_1 - av_3)$, $y_5 = x_2 - dw_3 + cqv_1$, $u_1 = b^{-1}rcv_1$, $u_2 = w_1$, $u_3 = bv_3$, we obtain the desired representations.

COROLLARY 1. *Let $\{f_1, f_2\}$ be a P_5-pair and $\langle f_1, f_2\rangle$ a 2-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2\}$ can be expressed in* (1, 1)*-form.*

LEMMA 4. *Let $\{f_1, f_2\}$ be a P_6-pair and $\langle f_1, f_2\rangle$ a 2-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2\}$ can be expressed in* (1, 1)*-form.*

Proof. By Lemma 9 of [4], there are representations

$$f_1 = x_1 \wedge u + v \wedge w , \qquad f_2 = x_1 \wedge u' + v' \wedge w' ,$$

where $\langle x_1\rangle \subset [f_1] \cap [f_2]$ and $\langle u, v, w\rangle$, $\langle u', v', w'\rangle$ are contained in

$$([f_1] + [f_2] - \langle x_1\rangle) .$$

If $\langle v, w\rangle \cap \langle v', w'\rangle = 0$, some linear combination of f_1, f_2 has irreducible length 3. If $\langle v, w\rangle = \langle v', w'\rangle$ some linear combination of f_1, f_2 has irreducible length $\leqq 1$. The result follows.

Lemma 2 implies the following lemma.

LEMMA 5. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, \cdots, f_k\}$ be an independent subset of H. Then*

(i) $3 \geqq [f_i] \cap [f_j] \geqq 2$ *for* $1 \leqq i < j \leqq k$;

(ii) $\dim \sum_{i=1}^{k-1} [f_i] \leqq \dim \sum_{i=1}^{k} [f_i] \leqq \dim \sum_{i=1}^{k-1} [f_i] + 2$.

Corollary 1 implies:

LEMMA 6. *Let $\{f_1, f_2, f_3\}$ be pairwise-P_6 and generate a 3-dimensional $\mathscr{L} - 2$ subspace. Then $\{f_1, f_2, f_3\}$ is a* (1, 1) *basis for $\langle f_1, f_2, f_3\rangle$ if $[f_3] \supset [f_1] \cap [f_2]$.*

1. $\dim \mathscr{U} = 5$. It is not difficult to see that when $\dim \mathscr{U} = 5$, the basis of any $\mathscr{L} - 2$ subspace must consist of pairwise-P_5 vectors.

THEOREM 3. *Let* $\dim \mathscr{U} = 5$, *H an $\mathscr{L} - 2$ subspace. Let* $\{f_1, \cdots, f_k\}$ *be independent in H. Then* $k \leqq 3$.

Proof. Let $\{u_1, \cdots, u_5\}$ be a basis of $\mathscr{U}$. Then each f_l, $1 \leqq l \leqq k$, has the form $f_l = \sum a^l_{ij} u_i \wedge u_j (1 \leqq i < j \leqq 5)$, $a_{ij} \in F$. (*) Consider the vector $f = \sum_{i=1}^k \beta_i f_i$, $\beta_i \in F$ not all zero. Now $\mathscr{L}(z) \leqq 1$ if $k \geqq 4$ for some $\{\beta_i\}$ not all zero since the following is true. $f = \sum_{i=1}^k \beta_i f_i = \sum p(i_1, i_2) u_{i_1} \wedge u_{i_2} (1 \leqq i_1 < i_2 \leqq 5)$ where $p(k_{\sigma(1)}, k_{\sigma(2)}) = \operatorname{sgn} \sigma p(k_1, k_2)$, σ a permutation of $\{1, 2\}$, and $\{k_i\}$ are arbitrary integers $1 \leqq k_i \leqq 5$. Thus, using (*), it follows that $\{p(i_1, i_2)\}$ are linear homogeneous functions of $\{\beta_1, \cdots, \beta_k\}$. Then the quadratic p-relations

$$\sum_{\mu=0}^{r} (-1)^\mu p(i_1, \cdots, i_{r-1}, j_\mu) p(j_0, \cdots, j_{\mu-1}, j_{\mu+1}, \cdots, j_r) = 0$$

for all sequences $(i_1, \cdots, i_{r-1})$, $(j_0, \cdots, j_r)$ of integers taken from $\{1, \cdots, n\}$ define (for $n = 5$, $r = 2$ in this case) *five* nontrivial equations, which are in fact quadratic homogeneous equations in the indeterminates $\beta_1, \cdots, \beta_k$ in F. Moreover, of these five, exactly *three* are independent (see [3], pp. 289, 312). Hence, if $k \geqq 4$, then there exists a nontrivial solution for the five equations (see [6], chapter 11). For these values of $\beta_1, \cdots, \beta_k$ (not all zero), $\mathscr{L}(f) \leqq 1$. Hence $k < 4$. The following three vectors generate an $\mathscr{L} - 2$ subspace of dimension three:

$$\begin{aligned} f_1 &= u_4 \wedge u_1 + u_3 \wedge u_2 , \\ f_2 &= u_5 \wedge u_2 + u_3 \wedge u_1 , \\ f_3 &= (u_4 + u_5) \wedge u_3 + u_2 \wedge u_1 . \end{aligned}$$

The following theorem is true for all n.

THEOREM 4. *Let* $\dim \mathscr{U} = n$. *Let* $\{f_1, \cdots, f_k\}$ *be a* (1, 1) *basis for an $\mathscr{L} - 2$ subspace. Then* $k \leqq n - 3$.

Moreover, when $n \geqq 5$, there always exists a (1, 1)*-type $\mathscr{L} - 2$ subspace of dimension* $(n - 3)$.

Proof. Suppose $k = n - 2$. Each f_i can be written $f_i = u_1 \wedge y_i + u_2 \wedge z_i$, $1 \leqq i \leqq n - 2$, where $\langle u_1, u_2, y_1, \cdots, y_{n-2}, z_1, \cdots, z_{n-2} \rangle \subseteqq \mathscr{U}$. Now $\{u_1, u_2, y_1, \cdots, y_{n-2}\}$ must be independent for, if not, some linear combination of $\{f_i\}$ has irreducible length $\leqq 1$. Hence $\mathscr{U} = \langle u_1, u_2, y_1, \cdots, y_{n-2} \rangle$. Thus $z_j = \sum_{i=1}^{n-2} \alpha_{ij} y_i + \beta_j u_1$, $1 \leqq j \leqq n - 2$. If $\beta_j \neq 0$, write

$$f_j = u_1 \wedge (y_j - \beta_j u_2) + u_2 \wedge \left(\sum_{j=1}^{n-2} \alpha_{ij} y_i \right) .$$

Hence, without loss of generality, we can assume $\{z_i\}$ is dependent on $\{y_i\}$. Using a similar argument, $\{y_i\}$ is dependent on $\{z_i\}$. Hence $\langle y_1, \cdots, y_{n-2}\rangle = \langle z_1, \cdots, z_{n-2}\rangle$. Hence, for some $\{\alpha_i\} \in F$, not all zero, we have $\sum_{i=1}^{n-2}\alpha_i y_i = \lambda\sum_{i=1}^{n-2}\alpha_i z_i = y$ for some $0 \neq \lambda \in F$; and $f = \sum_{i=1}^{n-2}\alpha_i f_i$ has irreducible length $\leqq 1$. Hence $k \leqq n - 3$.

Now let $f_i = u_1 \wedge u_{i+2} + u_2 \wedge u_{i+3}$ for $i = 1, \cdots, (n-3)$, where $\langle u_1, \cdots, u_n\rangle = \mathscr{U}$. Then $\{f_i\}$ generate an $\mathscr{L} - 2$ subspace of dimension $(n - 3)$.

COROLLARY 2. *Let* $\dim \mathscr{U} = 5, H$ *an* $\mathscr{L} - 2$ *subspace of* $(1, 1)$-*type. Then, if* $\dim H > 1, \dim H = 2$.

We pause here to introduce some notation.

DEFINITION 1. For subsets S, T of $\mathscr{U}$, $[S; T] = \langle S \cup T\rangle - \langle T\rangle$. In the case where $S = \{x_1, \cdots, x_s\}$ and $T = \{x_{s+1}, \cdots, x_k\}$, we use the convention $[S; T] = [x_1, \cdots, x_s; x_{s+1}, \cdots, x_k]$. Note that in this case if $y \in [S; T]$, then $y = \sum_{i=1}^{k}\alpha_i x_i, \alpha_i \in F$, and *at least one* of $\alpha_1, \cdots, \alpha_s$ is nonzero.

DEFINITION 2. For subsets S, T of $\mathscr{U}$, $S \wedge T = \{x \wedge y : x \in S$ and $y \in T\}$. In the case where S is the singleton $\{x\}$, we shall write $S \wedge T$ as $x \wedge T$. Similarly for T. Also, if S is the space $\langle x_1, \cdots, x_k\rangle$, then we shall regard S as a set and write $S \wedge T$ as $[x_1, \cdots, x_k] \wedge T$. Similarly for T.

The three-dimensional $\mathscr{L} - 2$ ***subspace when*** $\dim \mathscr{U} = 5$. In this context, a basis $\{f_1, f_2, f_3\}$ of an $\mathscr{L} - 2$ subspace H is necessarily pairwise P_5. It is not a $(1, 1)$ basis. However, either there exists a three-dimensional subspace $\mathscr{U}_0$ of $\mathscr{U}$ contained in each $[f_i]$, or there exists a exists a five-dimensional subspace $\mathscr{W} \subseteqq \mathscr{U}$ which contains each $[f_i]$ (see [1], p. 14). In fact, $\mathscr{W} = \mathscr{U}$. Moreover, since $\dim \mathscr{U} = 5$, $\dim [f_1] \cap [f_2] = 3$, and $\dim [f_3] = 4$, then $\dim \bigcap_{i=1}^{3} [f_i] \geqq 2$. Consequently this intersection has dimension two or three.

THEOREM 5. *Let* $\dim \mathscr{U} = 5$. *Let* $\{f_1, f_2, f_3\}$ *be a basis for an* $\mathscr{L} - 2$ *subspace* H *such that* $[f_i] \supset \mathscr{U}_0, i = 1, 2, 3,$ *where* $\mathscr{U}_0$ *is a three-dimensional subspace of* $\mathscr{U}$. *Then* $\mathscr{U}$ *has a basis* $\{u_1, u_2, u_3, x_4, x_5\}$ *such that there are representations*

$$f_1 = x_4 \wedge u_1 + u_2 \wedge u_3 ,$$
$$f_2 = x_5 \wedge u_2 + u_1 \wedge u_3 ,$$
$$f_3 = y \wedge u_3 + u_2 \wedge u_1 ,$$

where $y \in [x_4; x_5 \cdot u_1, u_2] \cap [x_5; x_4, u_1, u_2]$.

Proof. $\mathscr{U}$ has a basis $\{w_1, w_2, w_3, y_4, y_5\}$ such that $\mathscr{U}_0 = \langle w_1, w_2, w_3\rangle$ and there are representations $f_1 = y_4\wedge w_1 + w_2\wedge w_3$, $f_2 = y_5\wedge w_2 + w_1\wedge w_3$ (see Lemma 3). Now there exists $y' \in [f_3]$ such that $y' \notin \mathscr{U}_0$ and $y' \in [y_4, y_5; w_1, w_2, w_3]$. Since $\{f_1, f_2, f_3\}$ is pairwise-P_5, it is easy to see $y' \in [y_4; y_5, w_1, w_2, w_3] \cap [y_5; y_4, w_1, w_2, w_3]$. Hence f_3 has a representation

$$f_3 = y' \wedge u + v \wedge w; \mathscr{U}_0 = \langle u, v, w\rangle ,$$

(see [4], Lemma 9). Now if $u \in \langle w_1, w_2\rangle$, it is possible to find representations of f_1, f_2, f_3 such that they form a (1, 1) basis for H. This contradicts Corollary 2. Hence $u \notin \langle w_1, w_2\rangle$, but $u \in [w_3; w_1, w_2]$. In fact, without loss of generality, we can take $u = w_3 + cw_1 + c'w_2$.

Now $\langle w_2, u\rangle$, $\langle w_1, u\rangle$, $\langle v, w\rangle$ intersect pairwise in dimension at least one. Also $u \notin \langle v, w\rangle$. Therefore we may suppose $v \in [w_2; u]$, $w \in [w_1; u]$. We set

$$v = aw_2 + a'u,\ w = bw_1 + b'u .$$

Then

$$f_3 = (y' + ab'w_2 - a'bw_1) \wedge u + \gamma w_2 \wedge w_1,\ 0 \neq \gamma \in F .$$

Let

$$\begin{aligned}&\alpha^2 = \gamma ,\\ &w_2 = \alpha^{-1}u_2,\ w_1 = \alpha^{-1}u_1,\ u = \alpha u_3 .\end{aligned}$$

Then

$$\begin{aligned}f_1 &= (y_4 - cw_2) \wedge \alpha^{-1}u + u_2 \wedge u_3 ,\\ f_2 &= (y_5 - c'w) \wedge \alpha^{-1}u_2 + u_1 \wedge u_3 ,\\ f_3 &= x \wedge \alpha u_3 + u_2 \wedge u_1 .\end{aligned}$$

We have the result on setting $x_4 = \alpha^{-1}(y_4 - cw_2)$, $x_5 = \alpha^{-1}(y_5 - c'w_1)$, $y = \alpha x$, and noting that $y \in [x_4; x_5, u_1, u_2] \cap [x_5; x_4, u_1, u_2]$.

THEOREM 6. *Let* $\dim \mathscr{U} = 5$. *Let* $\{f_1, f_2, f_3\}$ *be a basis for an* $\mathscr{L} - 2$ *subspace* H *such that* $\dim \bigcap_{i=1}^{3} [f_i] = 2$. *Then* $\mathscr{U}$ *has a basis* $\{u_1, u_2, u_3, x_4, x_5\}$ *such that* f_1, f_2, f_3 *have representations given by either* (i) *or* (ii) *below.*

(i) $f_1 = x_4\wedge u_1 + u_2\wedge u_3$, $f_2 = x_5\wedge u_2 + u_1\wedge u_3$, $f_3 = u\wedge y + u_3\wedge y'$, $y, y' \in [x_4, x_5; u_1, u_2, u_3]$, $u \in \langle u_1, u_2\rangle$,

(ii) f_1, f_2 *as in* (i). *With* $u \in \langle u_1, u_2\rangle$, $u' \in \langle u_1, u_2, u_3\rangle$, $f_3 = \gamma u\wedge u' + y \wedge y'$, $y, y' \in [x_4, x_5; u_1, u_2, u_3]$, $0 \neq \gamma \in F$.

Proof. The proof involves a suitable choice of a basis of $\mathscr{U}$, as in the proof of Theorem 5, and the use of the following lemma.

LEMMA 7. *Let* $f \in \mathscr{L}_2$ *and* $\langle u_1, u_2\rangle$ *any two-dimensional subspace* *of* $[f]$. *Then either*

(i) *there exist* $v, w \in [f]$ *such that* $f = \gamma u_1 \wedge u_2 + v \wedge w$, $0 \neq \gamma \in F$,

or (ii) *there exist* $v', w' \in [f]$ *such that* $f = u_1 \wedge v' + u_2 \wedge w'$.

Proof. Let $\{u_1, \cdots, u_4\}$ be any basis of $[f]$. By Lemma 9 of [4], f has a representation $f = u_1 \wedge u + v \wedge w$, where $\langle u, v, w\rangle = \langle u_2, u_3, u_4\rangle$. If $\underline{u_1 \wedge u_2 \wedge f = 0}$, then $\langle u_1, u_2\rangle \cap \langle v, w\rangle \neq 0$, and it is easy to see $u_2 \in \langle v, w\rangle$ since $u_1 \notin \langle u, v, w\rangle$. If $u_1 \wedge u_2 \wedge f \neq 0$, then $\langle u_1, u_2, v, w\rangle = [f]$, and $u = au_1 + bu_2 + cv + dw$ with $b \neq 0$. Then $f = bu_1 \wedge u_2 + [u_1 \wedge (cv + dw) + v \wedge w]$. By Corollary 8 of [4] and since $\mathscr{L}(f) = 2$, the term in square brackets has irreducible length one.

We can in fact replace the basis $\{f_1, f_2, f_3\}$ in Theorem 3 by the basis $\{f_1 + f_2, f_2, f_3\}$. Then $[f_1 + f_2] \cap [f_2] \cap [f_3]$ has dimension two. We obtain:

THEOREM 7. *Let* $\dim \mathscr{U} = 5$, *H an* $\mathscr{L} - 2$ *subspace of dimension three. Then H has a basis which is either of type* (i) *or type* (ii) *in Theorem* 6.

Examples of such bases are the following:

EXAMPLE 1. $f_1 = x_4 \wedge u_1 + u_2 \wedge u_3$, $f_2 = x_5 \wedge u_2 + u_1 \wedge u_3$,
$f_3 = u_2 \wedge x_4 + u_3 \wedge x_5$.

EXAMPLE 2. f_1, f_2 as in Example 1. $f_3 = u_2 \wedge (u_1 + u_3) + x_4 \wedge x_5$.

2. $\dim \mathscr{U} = 6$.

The three-dimensional $\mathscr{L} - 2$ *subspaces.* If H is an $\mathscr{L} - 2$ subspace with a basis $\{f_1, f_2, f_3\}$ and $\dim \mathscr{U} = 6$, then $\dim \sum_{i=1}^{3} [f_i] = 5$ or 6. The first case was discussed in § 1. We show that, in the second case, H has a basis of pairwise-P_6 vectors, and there are three possibilities for such a basis.

Suppose $\dim \sum_{i=1}^{3} [f_i] = 6$. Now each pair in $\{f_1, f_2, f_3\}$ is either a P_5-or a P_6-pair. Thus either $\{f_1, f_2, f_3\}$ is pairwise-P_5 or at least one pair is a P_6-pair. The first case is then reduced to the second.

THEOREM 8. *Let H be an* $\mathscr{L} - 2$ *subspace, and let* $\{f_1, f_2, f_3\}$ *be pairwise-*P_5, *independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 6$. *Then* $(\sum_{i=1}^{3} [f_i])$ *has a basis* $\{u_1, u_2, u_3, x_4, x_5, x_6\}$ *such that there are representations*

$$f_1 = x_4 \wedge u_1 + u_2 \wedge u_3 ,$$

$$f_2 = x_5 \wedge u_2 + u_1 \wedge u_3 ,$$
$$f_3 = x_6 \wedge u + v \wedge u_3 ,$$
$$\langle u, v\rangle = \langle u_1, u_2\rangle,\ u \notin \langle u_1\rangle,\ u \notin \langle u_2\rangle .$$

Proof. There exists a three-dimensional subspace $\mathscr{U}_0$ of $\mathscr{U}$ contained in each $[f_i]$ (see [1], p. 14). The proof is similar to that of Theorem 5. We choose a basis $\{u_1, u_2, v_3, y_4, y_5, y_6\}$ of $\sum_{i=1}^{3}[f_i]$ in order to obtain representations $f_1 = y_4 \wedge u_1 + u_2 \wedge v_3$, $f_2 = y_5 \wedge u_2 + u_1 \wedge v_3$, $f_3 = y_6 \wedge w_1 + w_2 \wedge w_3$, and $\langle w_1, w_2, w_3\rangle = \langle u_1, u_2, u_3\rangle = \mathscr{U}_0$. Without loss of generality, we can assume $w_2 \in \langle u_1, u_2\rangle$. Then $w_1 \in \langle u_1, u_2\rangle$, for, if not, $\langle u_1, u_2, w_1\rangle = \mathscr{U}_0$ and $(f_1 + f_2 + f_3)$ has irreducible length 3 (see [4], Th. 7). Moreover $u \notin \langle u_1\rangle$ and $u \notin \langle u_2\rangle$ (see proof of Lemma 3). Thus $\langle w_1, w_2\rangle = \langle u_1, u_2\rangle$ and $w_3 = \lambda(v_3 + \bar{u})$ for some $0 \neq \lambda \in F$ and $\bar{u} \in \langle u_1, u_2\rangle$. Then $f_1 = y_4' \wedge u_1 + u_2 \wedge (v_3 + \bar{u})$, $f_2 = y_5' \wedge u_2 + u_1 \wedge (v_3 + \bar{u})$, and $f_3 = y_6 \wedge w_1 + \lambda w_2 \wedge (v_3 + \bar{u})$. The appropriate choice of new basis vectors gives the required representations.

COROLLARY 3. *Let H be an $\mathscr{L} - 2$ subspace, and let $\{f_1, f_2, f_3\}$ be pairwise-P_5, independent in H such that* $\dim \sum_{i=1}^{3}[f_i] = 6$. *Then $\{f_1, f_2, f_3\}$ is a $(1, 1)$ basis for $\langle f_1, f_2, f_3\rangle$.*

Proof. Choose a suitable representation of f_3.

LEMMA 8. *Let $\{f_1, f_2, f_3\}$ be a $(1, 1)$ basis of an $\mathscr{L} - 2$ subspace satisfying* (i) $\dim \sum_{i=1}^{3}[f_i] = 6$, (ii) *$\{f_1, f_2\}$ is a P_6-pair. Then $\{f_1, f_2\}$ can be extended to a $(1, 1)$ basis of pairwise-P_6 vectors of $\langle f_1, f_2, f_3\rangle$.*

Proof. We choose a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ of $\sum_{i=1}^{3}[f_i]$ so that

$$f_1 = u_1 \wedge x_3 + u_2 \wedge x_4,\ f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$$

(Lemma 4). Also $f = u_1 \wedge y + u_2 \wedge y'$, and we can take

$$\langle y, y'\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle$$

([4], Lemma 9). Let $y = u + \sum_{i=3}^{6}\alpha_i x_i$, $y' = u' + \sum_{i=3}^{6}\beta_i x_i$ where $\{u, u'\} \in \langle u_2\rangle$. We can choose $\lambda, \mu \in F$ such that

$$\begin{vmatrix} \alpha_3 + \lambda & \alpha_4 \\ \beta_3 & \beta_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha_5 + \mu & \alpha_6 \\ \beta_5 & \beta_6 \end{vmatrix}$$

are both nonzero. Then $g_3 = (\lambda f_1 + \mu f_2 + f_3)$ extends $\{f_1, f_2\}$ to a basis of $\langle f_1, f_2, f_3\rangle$ and $[g_3] \cap \langle x_3, x_4\rangle = 0$, $[g_3] \cap \langle x_5, x_6\rangle = 0$.

In Lemma 8, we can in fact take

$$f_1 = u_1 \wedge x_3 + u_2 \wedge x_4 \,,$$
$$f_2 = u_1 \wedge x_5 + u_2 \wedge x_6 \,,$$
$$f_3 = u_1 \wedge y + u_2 \wedge y', \langle y, y'\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle$$

and does not intersect each $[f_i]$, $i \neq 3$.

THEOREM 9. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be pairwise-P_5, independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 6$. *Then $\langle f_1, f_2, f_3\rangle$ has a* (1, 1) *basis of pairwise-P_6 vectors.*

Proof. Using the representations of f_1, f_2, f_3 obtained in Theorem 8 and Corollary 3, we take $g_1 = (f_1 + f_3)$. Then $\{g_1, f_2, f_3\}$ is a (1, 1) basis $\{g_1, f_2\}$ a P_6-pair, and $[g_1] \cap [f_2] \cap [f_3] = \langle u_1, u_2\rangle$. The result follows by Lemma 8.

COROLLARY 4. *Let $\{f_1, f_2, f_3\}$ be a* (1, 1) *basis for an $\mathscr{L} - 2$ subspace such that* $\sum_{i=1}^{3} [f_i] = 6$. *Then there exist a* (1, 1) *basis of pairwise-P_6 vectors for $\langle f_1, f_2, f_3\rangle$.*

THEOREM 10. *Let H be an $\mathscr{L} - 2$ subspace,* $\dim H \geqq 3$. *Let $\{f_1, f_2, f_3\}$ be independent in H such that* (i) $\dim \sum_{i=1}^{3} [f_i] = 6$, (ii) $\bigcap_{i=1}^{3} [f_i] = 0$. *Then $\{f_1, f_2, f_3\}$ are pairwise-P_6 and for any basis $\{u_1, u_2\}$ of $[f_1] \cap [f_2]$, $(\sum_{i=1}^{3} [f_i])$ has a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ such that $\{f_1, f_2, f_3\}$ have representations $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4, f_2 = u_1 \wedge x_5 + u_2 \wedge x_6, f_3 = x_3 \wedge w_1 + x_4 \wedge w_2 = x_5 \wedge v_1 + x_6 \wedge v_2, \langle w_1, w_2\rangle = \langle x_5, x_6\rangle, \langle v_1, v_2\rangle = \langle x_3, x_4\rangle$.*

Proof. If $\{f_1, f_2, f_3\}$ were not pairwise-P_6, we would have a contradiction of (ii). Since $\{f_1, f_2\}$ is a P_6-pair, the choice of representations of f_1, f_2 is immediate (Lemma 4). Let

$$[f_3] = \langle x_3', x_4', z_1, z_2\rangle, x_3' \in [x_3; u_1, u_2,], x_4' \in [x_4; u_1, u_2] \,.$$

It is not difficult to show we can represent $f_3 = x_3' \wedge w_1 + x_4' \wedge w_2$, where $\langle w_1, w_2, x_4'\rangle = \langle x_4', z_1, z_2\rangle$, and thus $\{w_1, w_2\} \in [z_1, z_2; x_4']$, and $f_1 = u_1 \wedge x_3' + u_2 \wedge x_4'$ (using Lemma 9 of [4] and proof of Lemma 4).

In a similar fashion, *without altering* u_1 or u_2, we can choose

$$x_5' \in [x_5; u_1, u_2], x_6' \in [x_6; u_1, x_2], \langle u_5', x_6'\rangle = \langle z_1, z_2\rangle \,,$$

so that $f_2 = u_1 \wedge x_5' + u_2 \wedge x_6', f_3 = x_5' \wedge v_1 + x_6' \wedge v_2$, where $\langle v_1, v_2, x_6'\rangle = \langle x_6', x_3', x_4'\rangle$. Thus $\{v_1, v_2\} \in [x_3', x_4'; x_6']$. From above, f_3 is also $x_3' \wedge w_1 + x_4' \wedge w_2$, and $\{w_1, w_2\} \in [x_5', x_6'; x_4']$. With respect to the independent set $\{x_i' \wedge x_j'\}$, the coefficient of $x_3' \wedge x_4'$ is zero in the second expression obtained for f_3, and the coefficient of $x_5' \wedge x_6'$ is zero in the first. It follows that neither term appears in f_3. We have the result on placing x_i for x'_i, $i = 3, \cdots, 6$.

LEMMA 9. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H satisfying*

(i) $\dim \sum_{i=1}^{3} [f_i] = 6$,

(ii) $\{f_1, f_2\}$ *is a P_6-pair,*

(iii) $\dim \bigcap_{i=1}^{3} [f_i] = 1$.

Then there exists $g_3 \in \langle f_1, f_2, f_3\rangle$ such that $\{f_1, f_2, g_3\}$ is a basis of pairwise-P_6 vectors for $\langle f_1, f_2, f_3\rangle$ and $\bigcap_{i=1}^{3} [f_i] = [g_3] \cap [f_1] \cap [f_2]$.

Proof. There are representations $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, and $\sum_{i=1}^{3} [f_i] = \langle u_1, u_2, x_3, \cdots, x_6\rangle$. Let $\bigcap_{i=1}^{3} [f_i] = \langle u\rangle$. Then $u \in \langle u_1, u_2\rangle$. Without loss of generality, we can take $u = u_1$. By Lemma 9 of [4], $f_3 = u_1 \wedge w + w' \wedge v$, $\langle w, w', v\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle$. If $\{f_1, f_2, f_3\}$ are pairwise-P_6, we have the result.

Case 1. Suppose $\{f_1, f_3\}$ is a P_6-pair and $\{f_2, f_3\}$ is a P_5-pair. Then we can take $f_3 = u_1 \wedge w + x_4 \wedge v'$ (use Lemma 6 and (iii)), where

$$\langle w, v_4, v'\rangle \subset \langle u_2, x_3, \cdots, x_6\rangle .$$

Let $[f_2] \cap [f_3] = \langle u_1, y, y'\rangle$. Then $\{y, y'\} \in [x_5, x_6; u_2]$. Therefore

$$f_3 = u_1 \wedge w + x_4 \wedge v',\ w \in [x_5, x_6; u_2, x_4],\ v' \in [x_5, x_6; u_2] .$$

Let $v' = ax_5 + bx_6 + cu_2$. Choose $\gamma \neq 0$ such that $\gamma + c \neq 0$. Let $g_3 = f_3 + \gamma f_1$. Then $\{g_3, f_1\}$ and $\{f_2, g_3\}$ are P_6-pairs.

Case 2. Suppose $\{f_1, f_3\}$, $\{f_2, f_3\}$ are both P_5-pairs. This and (iii) imply $\dim([f_1] \cap [f_3]) + ([f_2] \cap [f_3]) = 5$, which exceeds the dimension of $[f_3]$. Hence this case is not possible.

LEMMA 10. *If $f \in \mathscr{L}_2$ and $f \in x_1 \wedge [x_2, x_3, x_4] + [x_4; x_2] \wedge [x_3; x_2]$ where $[f] = \langle x_1, \cdots, x_4\rangle$, then $f \in x_1 \wedge [x_2] + [x_4; x_1, x_2] \wedge [x_3; x_1, x_2]$.*

Proof. Apply Lemma 7 to $\langle x_1, x_2\rangle$ and notice that the coefficient of $x_4 \wedge x_3$ is nonzero in f.

THEOREM 11. *Let H be an $\mathscr{L} - 2$ subspace, $\dim H \geqq 3$. Let $\{f_1, f_2, f_3\}$ be pairwise-P_6 and independent in H satisfying*

(i) $\dim \sum_{i=1}^{3} [f_i] = 6$,

(ii) $\dim \bigcap_{i=1}^{3} [f_i] = 1$.

Then for $\langle u_1\rangle = \bigcap_{i=1}^{3} [f_i]$ and any vector u_2 such that $\langle u_1, u_2\rangle = [f_1] \cap [f_2]$, there exists a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ such that $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = u_1 \wedge y + x_4 \wedge x_6$, where $y \in \langle u_2, x_3, \cdots, x_6\rangle$,

$y \notin \langle u_1, x_3, x_5\rangle$, $y \notin [f_i]$, $i = 1, 2$. Furthermore, there exists g_3 such that $\langle f_1, f_2, g_3\rangle = \langle f_1, f_2, f_3\rangle$ and $g_3 = u_1 \wedge u_2 + v \wedge w$, $v \in [x_4; u_1, u_2]$, $w \in [x_6; u_1, u_2]$ and $g_3 = v' \wedge w' + \gamma x_4 \wedge x_6$, $0 \neq \gamma \in F$, $v' \in [u_1; x_4, x_6]$, $w' \in [u_2; x_4, x_6]$.

Proof. The proof involves choosing a suitable basis of $\sum_{i=1}^{3} [f_i]$ and the use of Lemma 6 and 7. To obtain the form of g_3, we use Lemma 10.

LEMMA 11. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H such that*

(i) $\dim \sum_{i=1}^{3} [f_i] = 6$,

(ii) *$\{f_1, f_2\}$ is a P_6-pair,*

(iii) $\dim \bigcap_{i=1}^{3} [f_i] = 2$;

then $\{f_1, f_2\}$ can be extended to a basis of pairwise-P_6 vectors for $\langle f_1, f_2, f_3\rangle$.

Proof. By a suitable choice of basis vectors for $\sum_{i=1}^{3} [f_i]$, and the application of Lemma 7, we have two possible cases. One case implies $\{f_1, f_2, f_3\}$ is a (1, 1) basis and the result follows by Lemma 8. This case is when either $\{f_1, f_3\}$ or $\{f_2, f_3\}$ is a P_6-pair. Thus, the other possible case is when both $\{f_1, f_3\}$ and $\{f_2, f_3\}$ are P_5-pairs. Then $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$ with $\sum_{i=1}^{3} [f_i] = \langle u_1, u_2, x_3, \cdots, x_6\rangle$. By Lemma 7, f_3 is either $u_1 \wedge v + u_2 \wedge w$ or $u_1 \wedge u_2 + v' \wedge w'$. The first case implies $\{f_1, f_2, f_3\}$ is $a(1, 1)$ basis and Lemma 8 applies. In the second case, we can take $v' \in [f_1]$, $w' \in [f_2]$; i.e., $v' \in [x_3, x_4; u_1, u_2]$, $w' \in [x_5, x_6; u_1, u_2]$. In fact, we can take $v' \in [x_3; x_4, u_1, u_2]$, and $v' = x_3 + au_1 + bu_2 + cx_4$. Now $w' = dx_5 + a'u_1 + b'u_2 + c'x_4$. We then show $c' - cd = 0$, by considering the determinant of (a_{ij}), where a_{ij} is defined as follows. Let $z = f_1 + f_2 + f_3$. We can express

$$z = w_1 \wedge w_2 + w_3 \wedge w_4 + w_5 \wedge w_6 .$$

For $i = 1, 2$, a_{ij} is the coefficient of u_i in w_j. For $i = 3, \cdots, 6$, a_{ij} is the coefficient of x_i in w_j. This determinant is $\pm(c' - cd)$. If it is nonzero, $\mathscr{L}(z) = 3$. Hence it must equal zero. Then a suitable choice of basis vectors of $\sum_{i=1}^{3} [f_i]$ will allow us to assume that $c = 0$ in v' and $c' = 0$ in w'. Then $g_3 = (f_3 - f_1 + f_2)$ will extend $\{f_1, f_2\}$ to a pair wise-P_6 basis for $\langle f_1, f_2, f_3\rangle$.

We have sufficient reason now to assert the following theorem.

THEOREM 12. *Let $\{f_1, f_2, f_3\}$ generate a three-dimensional $\mathscr{L} - 2$ subspace H, and $\dim \sum_{i=1}^{3} [f_i] = 6$. Then H has a basis of pairwise-P_6 vectors $\{g_1, g_2, g_3\}$ which either form a (1, 1) basis of H or have intersection $\bigcap_{i=1}^{3} [g_i]$ with dimension 0 or 1. Moreover, if $\{f_1, f_2\}$ is a*

P_6-pair, then this pair can be extended to a basis of pairwise-P_6 vectors of H.

EXAMPLES. H is generated by $\{f_1, f_2, f_3\}$ where

(i) $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4, f_2 = u_1 \wedge x_5 + u_2 \wedge x_6,$
$f_3 = u_1 \wedge (u_2 + x_3 + x_5) + x_4 \wedge x_6;$

(ii) f_1, f_2 as in (i), $f_3 = u_1 \wedge x_4 + u_2 \wedge x_5;$

(iii) f_1, f_2 as in (i), $f_3 = x_3 \wedge x_5 + x_4 \wedge x_6.$

The maximal $\mathscr{L} - 2$ subspaces, $\dim \mathscr{U} = 6$. We shall now obtain this main theorem:

THEOREM 13. *Let H be an $\mathscr{L} - 2$ subspace and* $\dim \mathscr{U} = 6$. *Then* $\dim H \leqq 3$.

We prove this theorem by a series of lemmas, which show $\dim H \not> 3$, in fact, $\dim H \neq 4$. We take two three-dimensional $\mathscr{L} - 2$ subspaces $\langle f_1, f_2, f_3\rangle$ and $\langle f_1, f_2, f_4\rangle$ and show their sum is not an $\mathscr{L} - 2$ subspace. Theorem 12 allows us to take $\{f_1, f_2, f_3\}$ and $\{f_1, f_2, f_4\}$ to be pairwise-P_6, and there are 6 cases to consider since $\dim \bigcap_{i=1}^3 [f_i] = 0, 1, 2$ and a similar intersection property holds for the second set.

The following results are true for any dimension n of $\mathscr{U}$ unless otherwise specified.

LEMMA 12. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent pairwise-P_6 in H satisfying*

(i) $\dim \sum_{i=1}^3 [f_i] = 6,$

(ii) $\bigcap_{i=1}^3 [f_i] = 0.$

If $f_4 \in \mathscr{L}_2$, independent of $\{f_1, f_2, f_3\}$, satisfying

(a) $\dim \sum_{i=1}^4 [f_i] = 6$

(b) *$\{f_1, f_2, f_4\}$ is pairwise-P_6*

(c) $\dim \bigcap_{i=1,2,4} [f_i] = 1,$

then $\langle f_1, \cdots, f_4\rangle$ is not an $\mathscr{L} - 2$ subspace.

Proof. By Lemma 10, $\sum_{i=1}^3 [f_i]$ has a basis $\{u_1, u_2, x_3, \cdots, x_6\}$ such that $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4, f_2 = u_1 \wedge x_5 + u_2 \wedge x_6, f_3 = x_5 \wedge z + x_6 \wedge z',$ $\langle z, z'\rangle = \langle x_3, x_4\rangle$. Let $\langle u\rangle = \bigcap_{1,2,4} [f_i]$. Then $u \in \langle u_1, u_2\rangle$. We can take $u_1 = u$.

By Theorem 11, there exists $g_3 \in \langle f_1, f_2, f_4\rangle$ such that $g_3 = v' \wedge w' + \gamma x_4 \wedge x_6$, $0 \neq \gamma \in F$ and $\langle f_1, f_2, g_3\rangle = \langle f_1, f_2, f_4\rangle$. Since $\{v', w', x_6, x_5, z, z'\}$ is independent and $\{x_4 + \alpha z', z\}$ is independent for some $\alpha \in F$, then $z = g_3 - \alpha f_3$ has irreducible length 3 for some α. Hence $\langle f_1, \cdots, f_4\rangle$ is not an $\mathscr{L} - 2$ subspace.

Since the proofs of the lemmas involving the other cases are similar to the proof of Lemma 8 in the sense that in each case, we exhibit a vector of irreducible length 3 or less than 2 *except* in the 0-0 case, which we can reduce to one of the other cases, we shall simply state the final lemma.

LEMMA 13. *Let H be an $\mathscr{L}-2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 6$. *If $f_4 \in \mathscr{L}_2$, independent $\{f_1, f_2, f_3\}$ such that* $\dim {}_{i=1}^{4} [f_i] = 6$, *then $\langle f_1, \cdots, f_4\rangle$ is not an $\mathscr{L}-2$ subspace.*

We have to check one more case before we obtain Theorem 13.

LEMMA 14. *Let H be an $\mathscr{L}-2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H,* $\dim \sum_{i=1}^{3} [f_i] = 5$. *If $f_4 \in \mathscr{L}_2, f_4 \notin \langle f_1, f_2, f_3\rangle$, and* $\dim \sum_{i=1}^{4} [f_i] = 6$, *then $\langle f_1, \cdots, f_4\rangle$ is not an $\mathscr{L}-2$ subspace.*

Proof. We note $\dim \sum_{1,2,4} [f_i] = 6$ and apply Lemma 13.

We have now:

LEMMA 15. *Let H be an $\mathscr{L}-2$ subspace. Let $\{f_1, \cdots, f_k\}$ be independent in H,* $\dim \sum_{i=1}^{k} [f_i] = 6$. *Then $k \leqq 3$. For $k = 3$, $\langle f_1, f_2, f_3\rangle$ has a basis of pairwise-P_6 vectors.*

Theorem 13 follows from Lemma 15

3. $\dim \mathscr{U} = 7$.

The three dimensional $\mathscr{L}-2$ subspaces.

THEOREM 14. *Let H be an $\mathscr{L}-2$ subspace of dimension $\geqq 3$. Let $\{f_1, f_2, f_3\}$ be independent in H such that* $\dim \sum_{i=1}^{3} [f_i] = 7$. *Then $\{f_1, f_2, f_3\}$ contains a P_6-pair, say $\{f_1, f_2\}$, which can be extended to a pairwise-P_6 basis $\{f_1, f_2, g_3\}$ of $\langle f_1, f_2, f_3\rangle$. Moreover, either this basis is a $(1, 1)$ basis or* $\dim ([f_1] \cap [f_2] \cap [g_3]) = 1$; *and any basis $\{u_1, u_2\}$ of $[f_1] \cap [f_2]$ can be extended to a basis $\{u_1, u_2, x_3, \cdots, x_7\}$ of $[f_1] + [f_2] + [g_3]$ such that $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4, f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$; and $g_3 = u_1 \wedge x_7 + u_2 \wedge v$, $v \in \langle u_2, x_3, \cdots, x_6\rangle$, $v \notin \langle u_2, x_4, x_6\rangle$, and $v \notin [f_1]$ and $v \notin [f_2]$ in the first case; $g_3 = u_1 \wedge x_7 + x_4 \wedge x_6$ in the second case.*

Proof. A consideration of the various intersections and sums of $[f_i]$, $i = 1, 2, 3$ shows $\dim \bigcap_{i=1}^{3} [f_i]$ is either 1 or 2, and that there are at least two P_6-pairs in $\{f_1, f_2, f_3\}$. In the *first* case this independent set is in fact pairwise-P_6. The *second* case implies $\{f_1, f_2, f_3\}$ is a $(1, 1)$ basis for $\langle f_1, f_2, f_3\rangle$. If this basis is not pairwise-P_6 but $\{f_1, f_2\}$ and $\{f_2, f_3\}$ are P_6-pairs, and $\{f_1, f_3\}$ a P_5-pair, we can choose a basis

$\{u_1, u_2, x_3, \cdots, x_7\}$ to give $f_1 = u_1 \wedge x_3 + u_2 \wedge x_4$, $f_2 = u_1 \wedge x_5 + u_2 \wedge x_6$, $f_3 = u_1 \wedge x_7 + u_2 \wedge v$, $v \in \langle u_2, x_3, \cdots, x_6 \rangle$. Then we can take $g_3 = f_2 + f_3$. To obtain the desired representations of $\{f_1, f_2, f_3\}$ in the first case, we use an argument similar to the ones used earlier to obtain basis representations.

The maximal $\mathscr{L} - 2$ *subspaces,* $\dim \mathscr{U} = 7$. We obtain the following theorem.

THEOREM 15. *Let H be an $\mathscr{L} - 2$ subspace, $\dim \mathscr{U} = 7$. Then* $\dim H \leqq 4$. *When* $\dim H = 4$, *H has a* $(1, 1)$ *basis, three of whose members are pairwise-P_6.*

The proof is contained in Lemmas 16, 17, and 18 which follow.

LEMMA 16. *Let $\{f_1, f_2, f_3\}$ be a* $(1, 1)$ *basis for the $\mathscr{L} - 2$ subspace $\langle f_1, f_2, f_3 \rangle$, such that* $\dim \sum_{i=1}^{3} [f_i] = 7$. *If $f_4 \in \mathscr{L}_2$, independent of $\{f_1, f_2, f_3\}$ such that*

(i) $\dim \sum_{i=1}^{4} [f_i] = 7$,

(ii) *$\langle f_1, \cdots, f_4 \rangle$ is an $\mathscr{L} - 2$ subspace, then $\langle f_1, \cdots, f_4 \rangle$ has a* $(1, 1)$ *basis, three of whose members are pairwise-P_6.*

Proof. By Theorem 14, $\{f_1, f_2, f_3\}$ can be assumed to be pairwise-P_6 with the representations given. Then it is easy to see that some pair in $\{f_1, f_2, f_3\}$, say $\{f_1, f_2\}$, is such that $\dim \sum_{i=1,2,4} [f_i] = 7$, and $\{f_1, f_2, f_4\}$ can be assumed pairwise-P_6. The two cases given in Theorem 14, apply to $\{f_1, f_2, f_4\}$. One case gives the desired result immediately. We can eliminate the other case by showing the presence of a vector in $\mathscr{L}_3$ in $\langle f_1, \cdots, f_4 \rangle$; in fact we can take the vector $f_1 + f_2 + f_3 + \alpha f_4$ for some suitable $0 \neq \alpha \in F$.

LEMMA 17. *Let H be an $\mathscr{L} - 2$ subspace. Let $\{f_1, f_2, f_3\}$ be independent in H,* $\dim \sum_{i=1}^{3} [f_i] = 7$. *If $f_4 \in \mathscr{L}_2$, $f_4 \notin \langle f_1, f_2, f_3 \rangle$ such that*

(i) $\dim \sum_{i=1}^{4} [f_i] = 7$,

(ii) *$\langle f_1, \cdots, f_4 \rangle$ is an $\mathscr{L} - 2$ subspace,*

then $\langle f_1, \cdots, f_4 \rangle$ has a $(1, 1)$ *basis, three of whose members are pairwise-P_6.*

Proof. In view of Theorem 14 and Lemma 16, it is sufficient to eliminate the case $\dim \bigcap_{i=1}^{3} [f_i] = 1$. We use a similar procedure as in the proof of Lemma 16, and the representations of $\{f_i\}$ in Theorem 14. We have two cases: (a) $\bigcap_{i=1,2,4} [f_i] = \langle u_1 \rangle$, (b) $\bigcap_{i=1,2,4} [f_i] = \langle u_2 \rangle$. In (a), $\langle f_1, \cdots, f_4 \rangle$ contains a vector of irreducible length one. In (b), $\langle f_1, \cdots, f_4 \rangle$ contains a vector or irreducible length at least three.

In addition to these two lemmas, we note that if H is an $\mathscr{L}-2$ subspace, $\{f_1, f_2, f_3\}$ independent in H and (i) $\dim \sum_{i=1}^{3}[f_i] = 6$, then $\{f_i\}$ can be taken to be pairwise-P_6 (Lemma 15) and if $f_4 \notin \langle f_1, f_2, f_3\rangle$, $\dim \sum_{i=1}^{4}[f_i] = 7$, then $\dim \sum_{i=1,2,4}[f_i] = 7$; (ii) $\sum_{i=1}^{3}[f_i] = 5$, and if $f_4 \notin \langle f_1, f_2, f_3\rangle$, $\dim \sum_{i=1}^{4}[f_i] = 7$, then $\dim \sum_{i=2}^{4}[f_i] = 7$. Hence both these cases reduce to the case considered in Lemma 17.

LEMMA 18. *Let H be an $\mathscr{L}-2$ subspace, and $\{f_1, \cdots, f_4\}$ be independent in H, $\dim \sum_{i=1}^{4}[f_i] = 7$. If $f_5 \in \mathscr{L}_2$, $f_5 \notin \langle f_1, \cdots, f_4\rangle$, and $\dim \sum_{i=1}^{5}[f_i] = 7$, then $\langle f_1, \cdots, f_5\rangle$ is not an $\mathscr{L}-2$ subspace.*

Proof. Apply Lemma 17 to $\{f_1, \cdots, f_4\}$ and $\{f_2, \cdots, f_4\}$ taking $\{f_1, f_2, f_3\}$ pairwise-P_6. Then $\langle f_1, \cdots, f_5\rangle$ has a (1, 1) basis, contradicting Theorem 4.

4. The main results.

LEMMA 19. *If H is an $\mathscr{L}-2$ subspace and $\{f_1, f_2, f_3\}$ is independent in H, $\dim \sum_{i=1}^{3}[f_i] = 8$, then $\{f_1, f_2, f_3\}$ is a (1, 1), pairwise-P_6 basis of $\langle f_1, f_2, f_3\rangle$, and we can represent*

$$\begin{aligned} f_1 &= u_1 \wedge x_3 + u_2 \wedge x_4 , \\ f_2 &= u_1 \wedge x_5 + u_2 \wedge x_6 , \\ f_3 &= u_1 \wedge x_7 + u_2 \wedge x_8 ; \\ \sum_{i=1}^{3}[f_i] &= \langle u_1, u_2, x_3, \cdots, x_8\rangle . \end{aligned}$$

If $f_4 \in \mathscr{L}_2$, $f_4 \notin \langle f_1, f_2, f_3\rangle$, and $\langle f_1, \cdots, f_4\rangle$ is an $\mathscr{L}-2$ subspace, then $\{f_1, \cdots, f_4\}$ is a (1, 1) basis for $\langle f_1, \cdots, f_4\rangle$.

Proof. The first part is not difficult to see. Using Lemma 5 we obtain $\dim [f_4] \cap \langle u_1, u_2\rangle \geqq 1$. This intersection will have dimension 2, and f_4 forms a P_6-pair with one of $\{f_1, f_2, f_3\}$ since $\dim [f_4] = 4$.

Lemma 19 is extremely *important* as the second part states that presence of a 3-subset $\{f_1, f_2, f_3\}$ of any basis of an $\mathscr{L}-2$ subspace H such that $\dim \sum_{i=1}^{3}[f_i] = 8$ will guarantee that the basis will be a (1, 1) basis. We know that if $\dim \mathscr{L} \geqq 8$, then in any basis of H, we can find a 3-subset $\{g_1, g_2, g_3\}$ such that $\dim \sum_{i=1}^{3}[g_i] = 6, 7$ or 8. It is by now a more or less routine, and somewhat tedious, procedure to show the existence of a 3-subset $\{f_1, f_2, f_3\}$ in such a basis of H for $\dim \mathscr{U} = 8$, and then by induction for $\dim \mathscr{U} \geqq 9$. We shall simply state the main result and remark here that Theorem 4 provides the value of the maximal dimension of a (1, 1) basis.

THEOREM 16. *Let $\dim \mathscr{U} = n \geqq 6$. If H is an $\mathscr{L}-2$ subspace,* *then* $\dim H \leqq n - 3$. *If* $\dim H \geqq 4$, *then* H *has a* $(1, 1)$ *basis, and is hence a* $(1, 1)$-*type subspace.*

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Received April 14, 1969, and in revised form April 14, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ORTHOGONAL GROUPS OF POSITIVE DEFINITE MULTILINEAR FUNCTIONALS

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Let V be a finite dimensional vector space over the real numbers R and let $T: V \to V$ be a linear transformation. If $\varphi: \times_1^m V \to R$ is a real multilinear functional and

$$\varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots, x_m),$$

$x_1, \cdots, x_m \in V$, T is called an isometry with respect to φ. We say φ is positive definite if $\varphi(x, \cdots, x) > 0$ for all nonzero $x \in V$. In this paper we prove that if φ is positive definite and T is an isometry with respect to φ, then all eigenvalues of T have modulus one and all elementary divisors of T over the complex numbers are linear.

Let V be an n-dimensional vector space over the real numbers R. Let $T: V \to V$ be a linear transformation of V. The following theorem [1, Th. 3] is easy to prove:

THEOREM 1. *There exists a positive definite symmetric quadratic form $\varphi: V \times V \to R$ such that*

$$\varphi(Tx, Ty) = \varphi(x, y),\ x, y \in V \tag{1}$$

if and only if

(2)
1. *all eigenvalues of T have modulus* 1;
2. *all elementary divisors of T over the complex numbers C are linear.*

Moreover, if T satisfies (2), *then there is a positive definite symmetric φ such that* (1) *holds.*

Theorem 1 can also be expressed in matrix theoretic terms. If A is a real $n \times n$ positive definite symmetric matrix and X is any automorph of A;

$$X^T AX = A\ , \tag{3}$$

then X satisfies (2); moreover, if an $n \times n$ matrix X satisfies (2), then there is a positive definite symmetric A such that (3) holds.

Let $\varphi: \times_1^m V \to R$ be a real multilinear functional. Let H be a subgroup of the symmetric group S_m. If

$$\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = \varphi(x_1, \cdots, x_m) \tag{4}$$

for all $\sigma \in H$ and all $x_i \in V$, $i = 1, \cdots, m$, then φ is said to be symmetric with respect to H. If

$$(5) \qquad \varphi(Tx_1, \cdots, Tx_m) = \varphi(x_1, \cdots x_m)$$

for all $x_1, \cdots, x_m \in V$, T is called an isometry of V with respect to φ. (Note that if $m > 2$, (5) has no matrix analogue). Let $\Omega_m(H, T)$ be the set of all φ satisfying (4) and (5). Clearly $\Omega_m(H, T)$ is a subspace of the vector space of all multilinear functionals symmetric with respect to H. We say φ is positive definite if

$$(6) \qquad \varphi(x, \cdots, x) > 0$$

for all nonzero x in V. The set of all positive definite φ in $\Omega_m(H, T)$ is denoted by $P_m(H, T)$. It is clear that $P_m(H, T)$ is a (possibly empty) convex cone in $\Omega_m(H, T)$.

The following result [1] was proved as a partial generalization of Theorem 1.

THEOREM 2. *Let T: $V \to V$ be linear. If $P_m(H, T)$ is nonempty, then*

(a) *m is even*

(b) *every eigenvalue γ of T has modulus* 1

(c) *elementary divisors of T corresponding to $\gamma = \pm 1$ are linear.*

Conversely, if m is even, all eigenvalues of T are ± 1, and all elementary divisors of T are linear, then $P_m(H, T)$ is nonempty.

We conjectured that if $P_m(H, T)$ is nonempty,then (c) can be replaced by (c') "all elementary divisors of T over the complex field are linear." This would provide a complete generalization of Theorem 2, and thus justify (6) as a definition of a positive definite multilinear functional. The purpose of this paper is to prove this conjecture.

THEOREM 3. *If $P_m(H, T)$ is nonempty, then*

(a) *m is even*

(b) *all eigenvalues of T have modulus* 1

(c′) *all elementary divisors of T over C are linear.*

Conversely, if (a), (b), *and* (c′) *hold, then $P_m(H, T)$ is nonempty.*

2. Proof of Theorem 3. Assume that $P_m(H, T)$ is nonempty. Parts (a) and (b) follow from Theorem 2. We now prove two lemmas.

LEMMA 1. *If γ is an eigenvalue of T and $(x - \gamma)^k$, $k > 1$, is a nonlinear elementary divisor of T corresponding to γ, then $\gamma^m \neq 1$ for any integer m.*

Proof. Since T is a real transformation, it has a real elementary divisor

$$[(x-\gamma)(x-\bar{\gamma})]^k . \tag{7}$$

(By Theorem 2, γ cannot be real in this case.) Let W be the invariant subspace of T determined by (7), and let S be the restriction of T to W. Then S is an isometry of W with respect to φ, and hence S^r is also an isometry for any integer r. Now if $\gamma^r = 1$, then all eigenvalues of S^r are 1, and hence Theorem 2 implies that all elementary divisors of S^r are linear. Therefore, S^r is the identity on W, and thus, the elementary divisors of S are linear, a contradiction.

LEMMA 2. *If Theorem 3 is true for the case $H = S_m$, then it is true for any subgroup H of S_m.*

Proof. Let H be a subgroup of S_m and let $\varphi \in P_m(H, T)$. For each $\sigma \in S_m$, define

$$\varphi_\sigma(x_1, \cdots, x_m) = \varphi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) , \tag{8}$$

$x_1, \cdots, x_m \in V$. In general, φ_σ is not symmetric with respect to H, but φ_σ is positive definite and T is an isometry with respect to φ_σ. Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma . \tag{9}$$

Clearly ψ is positive definite, and T is an isometry with respect to ψ. Moreover, for any $\tau \in S_m$, and $x_1, \cdots, x_m \in V$,

$$\begin{aligned}
\psi(x_{\tau(1)}, \cdots, x_{\tau(m)}) &= \sum_{\sigma \in S_m} \varphi_\sigma(x_{\tau(1)}, \cdots, x_{\tau(m)}) \\
&= \sum_{\sigma \in S_m} \varphi(x_{\tau\sigma(1)}, \cdots, x_{\tau\sigma(m)}) \\
&= \sum_{\mu \in S_m} \varphi(x_{\mu(1)}, \cdots, x_{\mu(m)}) \\
&= \sum_{\mu \in S_m} \varphi_\mu(x_1, \cdots, x_m) \\
&= \psi(x_1, \cdots, x_m) .
\end{aligned}$$

Thus $\psi \in P_m(S_m, T)$, and hence the elementary divisors of T are linear. This proves Lemma 2.

We may assume henceforth that $H = S_m$ and abbreviate $P_m(S_m, T)$ to P_m. If P_m is nonempty, and T has a nonlinear elementary divisor over C corresponding to the eigenvalue $\gamma = a + ib$ ($b \neq 0$), then there exist four linearly independent vectors $v_1, \cdots, v_4$ in V such that

$$
\begin{aligned}
Tv_1 &= av_1 - bv_2 \\
Tv_2 &= bv_1 + av_2 \\
Tx_3 &= v_2 + av_3 - bv_4 \\
Tv_4 &= bv_3 + av_4 \,.
\end{aligned} \tag{10}
$$

Let $\bar{V}$ be the extension of V to an n-dimensional space over C, i.e., $\bar{V}$ consists of all vectors of the form $x + iy$, $x, y \in V$. By linear extension, we regard T as a linear transformation of $\bar{V}$, and by multilinear extension, φ becomes a complex valued multilinear functional on $\times_1^m \bar{V}$. Equation (5) still holds in $\bar{V}$, but φ is no longer positive definite. Set

$$
\begin{aligned}
e_1 &= v_1 + iv_2,\ e_2 = v_1 - iv_2 \\
e_3 &= v_3 + iv_4,\ e_4 = v_3 - iv_4 \,.
\end{aligned} \tag{11}
$$

From (10) and (11),

$$
\begin{aligned}
Te_1 &= \gamma e_1 \,, \quad Te_2 = \bar{\gamma} e_2 \\
Te_3 &= \gamma e_3 + v_2,\ Te_4 = \bar{\gamma} e_4 + v_2 \,.
\end{aligned} \tag{12}
$$

By Lemma 1, γ is not a root of unity; thus,

$$
\begin{aligned}
\varphi(e_1, \cdots, e_1, e_2, \cdots e_2) &= \varphi(Te_1, \cdots, Te_1, Te_2, \cdots, Te_2) \\
&= \gamma^k \bar{\gamma}^{m-k} \varphi(e_1, \cdots e_1, e_2, \cdots, e_2) \\
&= 0 \,,
\end{aligned} \tag{13}
$$

unless $k = m - k$, where k is the number of times e_1 occurs in (13). With $r = m/2$, we set

$$\varphi(e_1, \overset{r}{\cdots}, e_1, e_2, \overset{r}{\cdots}, e_2) = \nu \,.$$

Now $\nu \neq 0$; otherwise

$$
\begin{aligned}
\varphi(v_1, \cdots, v_1) &= 2^{-m} \varphi(e_1 + e_2, \cdots, e_1 + e_2) \\
&= 0 \,,
\end{aligned} \tag{14}
$$

contradicting (6). (Note that we are using the assumption that φ is symmetric with respect to S_m; this gives us a convenient way of sorting expressions such as those on the right side of (14).)

Let $\mu = \varphi(v_1, \cdots, v_1, e_3)$. Using (13) and (14), we compute,

$$
\begin{aligned}
\mu &= 2^{-m+1} \varphi(e_1 + e_2, \cdots, e_1 + e_2, e_3) \\
&= 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots, \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + v_2) \\
&= 2^{-m+1} \varphi\left(\gamma e_1 + \bar{\gamma} e_2, \cdots \gamma e_1 + \bar{\gamma} e_2, \gamma e_3 + \frac{e_1 - e_2}{2i}\right) \\
&= -2^{-m} i \binom{m-1}{r} (\bar{\gamma} - \gamma) \nu + \gamma 2^{-m+1} \varphi(\gamma e_1 + \bar{\gamma} e_2, \cdots \gamma e_1 + \bar{\gamma} e_2, e_3)
\end{aligned}
$$

$$= -2^{-m}i\binom{m-1}{r}(\bar{\gamma}-\gamma)\nu + \gamma 2^{-m+1}$$
$$\varphi\left(\gamma^2 e_1 + \bar{\gamma}^2 e_2, \cdots, \gamma^2 e_1 + \bar{\gamma}^2 e_2, \gamma e_3 + \frac{e_1 - e_2}{2i}\right)$$
$$= -2^{-m}i\binom{m-1}{r}(2\bar{\gamma}-\gamma-\gamma^3)\nu + \gamma^2 2^{-m+1}$$
$$\varphi(\gamma^2 e_1 + \bar{\gamma}^2 e_2, \cdots, \gamma^2 e_1 + \bar{\gamma}^2 e_2, e_3)\ .$$

Continuing this procedure, we obtain for any positive integer s

$$\mu = -2^{-m}i\binom{m-1}{r}\left(s\bar{\gamma} - \sum_{j=0}^{s-1}\gamma^{2j+1}\right)\nu + \gamma^s 2^{-m+1} \tag{15}$$
$$\varphi(\gamma^s e_1 + \bar{\gamma}^s e_2, \cdots \gamma^s e_1 + \bar{\gamma}^s e_2, e_3)\ .$$

Let

$$f(z) = z\varphi(ze_1 + \bar{z}e_2, \cdots, ze_1 + \bar{z}e_2, e_3)\ ,$$

where z is a complex variable. Then f is a continuous function of z on the complex plane, and hence f is bounded on the unit circle. Moreover, since γ is not a root of unity (in particular, $\gamma \neq \pm 1$),

$$\sum_{j=0}^{s-1}\gamma^{2j-1}$$

is also bounded as s becomes large. Thus, letting s approach infinity in (15) forces μ to become infinite, a contradiction. This proves Theorem 3 in one direction.

Now suppose all eigenvalues of T are 1 in absolute value and all elementary divisors of T are linear over C. Let 1 (p times), -1 (q times) and $\gamma_j, \bar{\gamma}_j = a_j \pm ib_j, |\gamma_j| = 1, j = 1, \cdots, t$, be the eigenvalues of T. Then there is a basis of V, $v_1, \cdots, v_p, u_1, \cdots, u_q, x_1, y_1, \cdots x_t, y_t$ such that

$$\begin{aligned} Tv_j &= v_j,\ j = 1, \cdots, p \\ Tu_j &= -u_j,\ j = 1, \cdots, q \\ Tx_j &= a_j x_j - b_j y_j,\ j = 1, \cdots, t \\ Ty_j &= b_j x_j + a_j y_j,\ j = 1, \cdots, t\ . \end{aligned} \tag{16}$$

Set

$$\begin{aligned} w_j &= x_j + iy_j \\ \bar{w}_j &= x_j - iy_j,\ j = 1, \cdots, t\ . \end{aligned}$$

Then $v_1, \cdots, v_p, u_1, \cdots, u_q, w_1, \bar{w}_1, \cdots, w_t, \bar{w}_t$ form a basis of $\bar{V}$ of eigenvectors of T. Let $f_1, \cdots, f_p, g_1, \cdots, g_q, h_1, k_1, \cdots, h_t, k_t$ be the corresponding dual basis. If $l_1, \cdots, l_m$ are linear functionals on a space V, then $l_1 \cdots l_m$ is the m-linear functional on $\times_1^m V$ such that

$$l_1 \cdots l_m(x_1, \cdots, x_m) = \prod_{i=1}^{m} l_i(x_i) .$$

Define φ as follows:

$$\varphi = \sum_{j=1}^{p} f_j^m + \sum_{j=1}^{q} g_j^m + \sum_{j=1}^{t} [(h_j k_j)^r + (\bar{h}_j \bar{k}_j)^r] , \tag{17}$$

where $r = m/2$ and $\bar{f}(v) = \overline{f(v)}$. Now $\bar{h}_j$ and $\bar{k}_j$ are not linear on the complex space $\bar{V}$, but they are complex valued linear functionals on V, i.e., they are linear functionals on V but are not in the dual space of V. Thus φ is a real multilinear functional on V. Set

$$\psi = \sum_{\sigma \in S_m} \varphi_\sigma .$$

We assert that $\psi \in P_m(H, T)$. Clearly ψ is symmetric with respect to S_m, and thus with respect to any subgroup H of S_m. It remains to show that ψ is positive definite and that T is an isometry with respect to ψ. It suffices to prove these last two properties for φ. Let

$$x = \sum_{j=1}^{p} \alpha_j v_j + \sum_{j=1}^{q} \beta_j u_j + \sum_{j=1}^{t} (\delta_j x_j + \lambda_j y_j)$$

be an arbitrary vector of V. Then from (17),

$$\varphi(x, \cdots, x) = \sum_{j=1}^{p} \alpha_j^m + \sum^{q} \beta_j^m + 2 \sum_{j=1}^{t} \left[\left(\frac{\delta_j}{2} \right)^2 + \left(\frac{\lambda_j}{2} \right)^2 \right]^r .$$

Since m is even and $\alpha_j, \beta_j, \delta_j, \lambda_j$ are all real, φ is positive definite. Now let z_k, $k = 1, \cdots, m$, be arbitrary vectors in V, with

$$z_k = \sum_{j=1}^{p} a_{kj} v_j + \sum_{j=1}^{q} b_{kj} u_j + \sum_{j=1}^{t} (c_{kj} x_j + d_{kj} y_j) . \tag{18}$$

Then

$$\begin{aligned} \varphi(z_1, \cdots, z_m) &= \sum_{j=1}^{p} \prod_{k=1}^{m} a_{kj} + \sum_{j=1}^{q} \prod_{k=1}^{m} b_{kj} \\ &\quad + \sum_{j=1}^{t} \prod_{k=1}^{r} \left(\frac{c_{2k-1,j}}{2} + \frac{d_{2k-1,j}}{2i} \right) \left(\frac{c_{2k,j}}{2} - \frac{d_{2k,j}}{2i} \right) \\ &\quad + \sum_{j=1}^{t} \prod_{k=1}^{r} \left(\frac{c_{2k-1,j}}{2} - \frac{d_{2k-1,j}}{2i} \right) \left(\frac{c_{2k,j}}{2} + \frac{d_{2k,j}}{2i} \right) . \end{aligned} \tag{19}$$

From (16)

$$\begin{aligned} Tz_k &= \sum_{j=1}^{p} a_{kj} v_j + \sum_{j=1}^{q} (-b_{kj}) u_j \\ &\quad + \sum_{j=1}^{t} (a_j c_{kj} + b_j d_{kj}) x_j + (a_j d_{kj} - b_j c_{kj}) y_j , \end{aligned} \tag{20}$$

$k = 1, \cdots, m$. Let

$$e_{kj} = a_j c_{kj} + b_j d_{kj}$$
$$f_{kj} = a_j d_{kj} - b_j c_{kj} \, .$$

Then from (19) and (20)

$$\begin{aligned} \varphi(Tz_1, \cdots, Tz_m) &= \sum_{j=1}^{p} \prod_{k=1}^{m} a_{kj} + \sum_{j=1}^{q} \prod_{k=1}^{m} (-b_{kj}) \\ &+ \sum_{j=1}^{t} \prod_{k=1}^{m} \left(\frac{e_{2k-1,j}}{2} + \frac{f_{2k-1,j}}{2i} \right) \left(\frac{e_{2k,j}}{2} - \frac{f_{2k,j}}{2i} \right) \\ &+ \sum_{j=1}^{t} \prod_{k=1}^{m} \left(\frac{e_{2k-1,j}}{2} - \frac{f_{2k-1,j}}{2} \right) \left(\frac{e_{2k,j}}{2} + \frac{f_{2k,j}}{2i} \right) . \end{aligned} \tag{21}$$

It is easily verified that

$$\begin{aligned} \frac{e_{kj}}{2} + \frac{f_{kj}}{2i} &= \bar{\gamma}_j \left(\frac{c_{kj}}{2} + \frac{d_{kj}}{2i} \right) \\ \frac{e_{kj}}{2} - \frac{f_{kj}}{2i} &= \gamma_j \left(\frac{c_{kj}}{2} - \frac{d_{kj}}{2i} \right) . \end{aligned} \tag{22}$$

Using (22) in (21) and the fact that $|\gamma_j| = 1$, we obtain

$$\varphi(Tz_1, \cdots, Tz_m) = \varphi(z_1, \cdots, z_m) \, .$$

This completes the proof of Theorem 3.

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Received July 8, 1960. This work was done while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D. C., 1968-70.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON THE GROWTH OF ENTIRE FUNCTIONS OF BOUNDED INDEX

W. J. PUGH AND S. M. SHAH

A class E of entire functions of zero order and with widely spaced zeros has been defined and it is proved that if $f \in E$ then $f', f'', \cdots \in E$. Furthermore f is of index one. This class includes many functions which are both of bounded index and arbitrarily slow growth. If f is any transcendental entire function then there is an entire function g of unbounded index with the same asymptotic behavior. When f is of infinite order then it is of unbounded index and we simply take $g = f$. When f is of finite order we give the construction for g.

DEFINITION 1. An entire function $f(z)$ is said to be of bounded index if there exists an integer M, independent of z, such that

$$\left|\frac{f^{(n)}(z)}{n!}\right| \leqq \max_{0 \leqq s \leqq M}\left\{\left|\frac{f^{(s)}(z)}{s!}\right|\right\}$$

for all n and all z. The least such integer M is called the index of $f(z)$.

Although functions of bounded index have been the object of a number of recent investigations (cf: [3], [5], [6], [7]-[9]), little is known about their properties, and most of the following natural questions seem to require further study.

I. *What are the growth properties of functions of bounded index*:

(a) can they increase arbitrarily rapidly,

(b) can they increase arbitrarily slowly,

(c) is it possible to derive the boundedness (or the unboundedness) of the index from the asymptotic properties of the logarithm of the maximum modulus of $f(z)$, i.e., $\log M(r, f)$?

II. *Classes of functions of bounded index*:

(a) find classes of functions of bounded index,

(b) is the sum (or product) of two functions of bounded index also of bounded index?

Question I(a) was settled by Shah [8] who proved that the growth of functions of bounded index is at most of the exponential type of order one. (See also Lepson [6].) Shah [8] and Lepson [6] have constructed functions of arbitrarily slow growth and of unbounded index.

In the present note we derive a simple answer to Question I(b) from the consideration of

Functions with widely spaced zeros. Let $f(z)$ be an entire function of genus zero, and let $\{a_j\}_{j=1}^{\infty}$ be the sequence of its zeros. We say that $f(z)$ has widely spaced zeros if the zeros $\{a_j\}$ are all simple and

$$|a_1| \geqq a = 5, \ |a_{n+1}| \geqq a^n |a_n| \qquad (n = 1, 2, 3, \cdots) .$$

Using this definition we prove

THEOREM 1. *Let $f(z)$ have widely spaced zeros. Then, for all z,*

$$|f^{(n)}(z)| < \max \{|f(z)|, |f'(z)|\} \qquad (n = 2, 3, 4, \cdots) .$$

COROLLARY 1.1. *Functions with widely spaced zeros are of bounded index.*

COROLLARY 1.2. *There exist functions of bounded index and of arbitrarily slow growth.*

Corollary 1.1 may also be considered as a contribution to Question II(a). Corollary 1.2 answers Question I(b). Other contributions, due to separate efforts of the present authors, will be found elsewhere. In [9] Shah proves that all solutions of certain classes of linear differential equations are of bounded index. In his doctoral dissertation, Pugh shows that the functions

$$F_\sigma(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{j^\sigma}\right) \qquad (\sigma > 8) ,$$

and

$$f_q(z) = \prod_{j=0}^{\infty} (1 - q^j z) \qquad \left(0 < q < \frac{1}{16}\right) ,$$

are of bounded index. As a contribution to II(b), Pugh [7] has shown that the sum of two functions of bounded index need not be of bounded index.

Our second result clarifies one aspect of Question I(c). We prove

THEOREM 2. *Let $f(z)$ be any transcendental entire function of finite order. It is always possible to find an entire function $g(z)$, of unbounded index such that*

$$\log M(r, f) \sim \log M(r, g) \qquad (r \longrightarrow \infty) .$$

Choosing $f(z)$ to be of bounded index, we see that it is always possible to find *functions of unbounded index with the same asymptotic behavior as* $f(z)$.

The authors gratefully acknowledge the help of Professor Albert Edrei who suggested the class of functions with widely spaced zeros, and indicated the connection between Theorem 2 and the results of [2].

1. Successive derivatives of functions with widely spaced zeros.

LEMMA 1. *Let* $f(z)$ *be an entire function with widely spaced zeros* $\{a_j\}_{j=1}^{\infty}$. *Let* $\{b_j\}_{j=1}^{\infty}$ $(|b_j| \leqq |b_{j+1}|)$, *be the zeros of* $f'(z)$.
Then

$$\frac{|a_{n+1}|}{b} < |b_n| \leqq |a_{n+1}| , \qquad (n \geqq 2,\, b = 1.6) , \tag{1.1}$$

and

$$\left(1 + \frac{2R + d}{a}\right)|a_1| < |b_1| \leqq |a_2|,\ (R = 2.4,\, d = 10^{-3},\, |a_1| \geqq a = 5) . \tag{1.2}$$

Proof. In §§ 1-3, we shall write $1.6 = b$, $2.4 = R$, $10^{-3} = d$, $1 + (2R + d)/a = 1.9602 = c$. Put

$$g_n(z) = \sum_{j=1}^{n} \frac{1}{z - a_j} , \qquad (n \geqq 1) ,$$

and

$$h_n(z) = \frac{f'(z)}{f(z)} - g_n(z) = \sum_{j=n+1}^{\infty} \frac{1}{z - a_j} . \tag{1.3}$$

Our proof of the lemma depends on obvious applications of Rouché's theorem [4, p. 254].

Let $z = re^{i\theta}$ and

$$|a_n| < r < |a_{n+1}| , \qquad (n \geqq 1) . \tag{1.4}$$

Clearly

$$\begin{aligned} \operatorname{Re}(zg_n(z)) &= \sum_{j=1}^{n} \frac{\operatorname{Re}(r^2 - z\bar{a}_j)}{|z - a_j|^2} \\ &\geqq \sum_{j=1}^{n} \frac{r}{r + |a_j|} \end{aligned}$$

and hence

$$| g_n(z) | \geqq \sum_{j=1}^{n} \frac{1}{r + | a_j |} .$$

In particular by the definition of widely spaced zeros we have

$$(1.5) \qquad | g_n(z) | \geqq \frac{n}{| a_{n+1} | + | a_n |} \geqq \frac{n}{| a_{n+1} |} \frac{25}{26} , \qquad (n \geqq 2) ,$$

$$(1.6) \qquad \left| g_n\left(\frac{| a_{n+1} |}{b} e^{i\theta}\right)\right| \geqq 2\left(\frac{| a_{n+1} |}{b} + | a_2 |\right)^{-1}$$

$$> \frac{3}{| a_{n+1} |} , \qquad (n \geqq 2) .$$

For $h_n(z)$ we have

$$\left| h_n\left(\frac{| a_{n+1} |}{b} e^{i\theta}\right)\right| \leqq \left(| a_{n+1} | - \frac{| a_{n+1} |}{b}\right)^{-1} + \left(| a_{n+2} | - \frac{| a_{n+1} |}{b}\right)^{-1} + \cdots$$

$$(1.7) \qquad < \frac{b}{b-1} \frac{1}{| a_{n+1} |} + \frac{1.25}{| a_{n+2} | - (| a_{n+1} | / b)}$$

$$< \frac{2.8}{| a_{n+1} |} \qquad (n \geqq 2) .$$

Now in the disc

$$(1.8) \qquad | z | \leqq \frac{| a_{n+1} |}{b} ,$$

$g_n(z)$ has n poles, and, by the theorem of Gauss-Lucas [10, p. 6], exactly $(n-1)$ zeros. The function $h_n(z)$ is regular in the disc (1.8), and by (1.6) and (1.7)

$$| g_n(z) | > | h_n(z) | , \qquad \left(n \geqq 2, | z | = \frac{| a_{n+1} |}{b}\right) .$$

Hence, by Rouché's theorem

$$g_n(z) + h_n(z) = \frac{f'(z)}{f(z)}$$

has exactly $(n-1)$ zeros in the disc (1.8).

We have thus proved

$$(1.9) \qquad \frac{| a_{n+1} |}{b} < | b_n | , \qquad (n \geqq 2) .$$

Similarly, for

$$(1.10) \qquad r = | z | = \gamma | a_n | , \qquad (1 < \gamma < 1.01, n \geqq 2)$$

we have

$$\begin{aligned} |h_n(z)| &< (|a_{n+1}| - \gamma |a_n|)^{-1} + (|a_{n+2}| - \gamma |a_n|)^{-1} + \cdots \\ &\leqq (|a_{n+1}| - \gamma |a_n|)^{-1} + (1.1)(|a_{n+2}| - \gamma |a_n|)^{-1} \\ &\leqq (\gamma |a_n| + |a_1|)^{-1} < |g_n(z)| . \end{aligned}$$

Again by Rouché's theorem $f'(z)/f(z)$ has exactly $(n-1)$ zeros in any disc with center at the origin and a radius r satisfying (1.10). Hence

$$|b_{n-1}| < \gamma |a_n| \qquad (n \geqq 2) ,$$

and letting $\gamma \rightarrow 1+$, we obtain

$$|b_{n-1}| \leqq |a_n| \qquad (n \geqq 2) . \tag{1.11}$$

The second of the inequalities (1.2) also follows from (1.11).

We complete the proof of the lemma by showing that

$$|z| \leqq c |a_1| \tag{1.12}$$

implies

$$\left| \frac{f'(z)}{f(z)} \right| > 0 . \tag{1.13}$$

Thus $f'(z)$ will have no zeros in the disc (1.12) and, therefore

$$c |a_1| < |b_1| ,$$

which is the first of the inequalities (1.2).

In order to verify (1.13) notice that (1.12) and the definition of widely spaced zeros imply

$$\begin{aligned} \left| \frac{f'(z)}{f(z)} \right| &\geqq \frac{1}{|a_1|} \left\{ \frac{1}{1+c} - \sum_2^\infty \frac{1}{a^{j(j-1)/2} - c} \right\} \\ &> 0 . \end{aligned}$$

This completes the proof of Lemma 1.

LEMMA 2. *If $f(z)$ has widely spaced zeros all the derivatives*

$$f'(z), f''(z), \cdots$$

have the same property.

Proof. It is sufficient to prove that if $f(z)$ has widely spaced zeros, the zeros of $f'(z)$ are also widely spaced. By (1.2)

$$9.801 \leqq c |a_1| < |b_1| . \tag{1.14}$$

By (1.1) and (1.2)

$$|b_n| \leqq |a_{n+1}|, \qquad (n \geqq 1)$$
$$\frac{1}{b}|a_{n+2}| < |b_{n+1}|, \qquad (n \geqq 1).$$

Hence

$$\left|\frac{b_{n+1}}{b_n}\right| > \frac{|a_{n+2}|}{b\,|a_{n+1}|} \geqq \frac{a^{n+1}}{b} > a^n \qquad (n \geqq 1). \tag{1.15}$$

The relations (1.14) and (1.15) show that the b's are widely spaced.

2. Minimum distance between a zero of $f(z)$ and a zero of $f'(z)$. The inequalities (1.1) do not preclude the possibility that $|a_{n+1} - b_n|$ be very small. In this section we show that

$$\inf_{\substack{1 \leqq j < \infty \\ 1 \leqq k < \infty}} |a_j - b_k| > 2R + d. \tag{2.1}$$

I. From now on, we denote the zeros of $f^{(k)}(z)$, in order of ascending moduli by $\{a_j^{(k)}\}_{j=1}^{\infty}$. By definition $a_n^{(0)} = a_n$ and $f^{(0)} \equiv f$.

II. We consider systematically the sets

$$D_k(\rho) = \bigcup_{j=1}^{\infty} \{z: |z - a_j^{(k)}| \leqq \rho\} \qquad (\rho > 0,\ k = 0, 1, \cdots).$$

LEMMA 3. *If $f(z)$ has widely spaced zeros, and if $z \notin D_0(R)$, then*

$$\left|\frac{f'(z)}{f(z)}\right| < 1,\ \left|\frac{f''(z)}{f(z)}\right| < 1. \tag{2.2}$$

Proof. The identities

$$\frac{d}{dz}\left(\frac{f'(z)}{f(z)}\right) = -\sum_{j=1}^{\infty}\frac{1}{(z-a_j)^2} = \frac{f''(z)}{f(z)} - \left(\frac{f'(z)}{f(z)}\right)^2$$

imply

$$\left|\frac{f''(z)}{f(z)}\right| \leqq \sum_{j=1}^{\infty}\frac{1}{|z-a_j|^2} + \left(\sum_{j=1}^{\infty}\frac{1}{|z-a_j|}\right)^2 \leqq 2\left(\sum_{j=1}^{\infty}\frac{1}{|z-a_j|}\right)^2.$$

Hence, the inequalities (2.2) follow from the single inequality

$$\sum_{j=1}^{\infty}\frac{1}{|z-a_j|} < \frac{\sqrt{2}}{2}. \tag{2.3}$$

If $z \notin D_0(R)$, and $|z| < |a_1|$, then

$$(2.4) \qquad |z - a_1| > R$$

and

$$(2.5) \qquad |z-a_j| \geqq |a_j| - |z| > |a_j| - |a_1| > |a_1|(a^{j-1}-1) > \frac{a^j}{2}, \ (j \geqq 2) .$$

Hence

$$\sum_{j=1}^{\infty} \frac{1}{|z - a_j|} < \frac{1}{R} + 2\sum_{j=2}^{\infty} \frac{1}{a^j} < \frac{\sqrt{2}}{2} ,$$

so that (2.3) holds if $|z| < |a_1|$.

In general, the relations

$$|a_n| \leqq |z| < |a_{n+1}| \qquad (n \geqq 1), \ z \notin D_0(R)$$

imply

$$(2.6) \qquad |z - a_j| \geqq |z| - |a_j| \geqq |a_n| - |a_{n-1}| > \frac{a^n}{2}$$

provided

$$(2.7) \qquad n \geqq 2 , \qquad j < n .$$

Similarly, for $j > n + 1$

$$(2.8) \qquad \begin{aligned} |z - a_j| &\geqq |a_j| - |a_{n+1}| > (a^{j-1} - 1)\,|a_{n+1}| \\ &> \frac{a^{j-1}}{2}\,|a_{n+1}| . \end{aligned}$$

Finally,

$$(2.9) \qquad \frac{1}{|z - a_n|} + \frac{1}{|z - a_{n+1}|} \leqq \frac{1}{R} + (\max\{|z - a_n|, |z - a_{n+1}|\})^{-1}$$

with

$$(2.10) \quad \max\{|z - a_n|, |z - a_{n+1}|\} \geqq \frac{|a_{n+1}| - |a_n|}{2} > \frac{(a^n - 1)|a_n|}{2} .$$

Combining (2.6), (2.8), (2.9) and (2.10), we find, for $n \geqq 2$,

$$(2.11) \qquad \begin{aligned} &\sum_{j=1}^{\infty} \frac{1}{|z - a_j|} < \frac{2(n-1)}{a^n} + \frac{1}{R} + \frac{2}{(a^n - 1)|a_n|} + \\ &\frac{2a}{|a_{n+1}|} \sum_{j=n+2}^{\infty} \frac{1}{a^j} < \frac{2(n-1)}{a^n} + \frac{1}{R} + \frac{2}{(a^n - 1)a} + \frac{2}{(a-1)a^{n+2}} . \end{aligned}$$

It is easily seen that (2.11) holds for $n = 1$ also and that (2.11) implies (2.3). Hence the lemma is proved.

LEMMA 4. *If $z \in D_0(2R + d)$, then $f'(z) \neq 0$.*

Proof. If $z \in D_0(2R + d)$, then for some n,

$$|z - a_n| \leqq 2R + d = 4.801 . \tag{2.12}$$

Hence, if $j < n$ and $n \geqq 2$,

$$\begin{aligned} |z - a_j| &\geqq |z| - |a_{n-1}| \geqq |a_n| - |a_{n-1}| - (2R + d) \\ &\geqq |a_n|\left(1 - \frac{1}{a} - \frac{2R + d}{a^2}\right) > \frac{6}{10}|a_n| . \end{aligned} \tag{2.13}$$

If $j > n$, then

$$\begin{aligned} |z - a_j| &\geqq |a_j| - |a_n| - (2R + d) \\ &> |a_j|\left(1 - \frac{1}{a} - \frac{2R + d}{a^2}\right) > \frac{6}{10}|a_j| . \end{aligned} \tag{2.14}$$

By (2.12), (2.13), and (2.14) we have, for $n \geqq 2$,

$$\begin{aligned} \left|\frac{f'(z)}{f(z)}\right| &\geqq \frac{1}{4.801} - \frac{10(n-1)}{6|a_n|} - \frac{10}{6}\sum_{j=n+1}^{\infty}\frac{1}{|a_j|} , \\ &\geqq \frac{1}{4.801} - \frac{1}{3}\frac{(n-1)}{a^{n(n-1)/2}} - \frac{5}{12}\frac{1}{a^{(n+1)n/2}} . \end{aligned} \tag{2.15}$$

Again, it is easily seen that (2.15) holds for $n = 1$ also. The expression on the right of (2.15) is positive and consequently in $D_0(2R + d)$, $f'(z) \neq 0$ unless $f(z) = 0$. On the other hand $f'(z) \neq 0$ if $f(z) = 0$ because all the zeros of $f(z)$ are simple. This completes the proof of Lemma 4.

3. **Proof of Theorem 1.** Because all the derivatives of $f(z)$ have widely spaced zeros, Lemmas 1 to 4 apply to all of the functions $f^{(k)}(z)$, $(k = 0, 1, 2, 3, \cdots)$. In particular Lemma 4 shows that the sets $D_{n-2}(R)$ and $D_{n-1}(R)$ are disjoint for $n \geqq 2$.

Hence, by Lemma 3, at least one of the two inequalities

$$\left|\frac{f^{(n)}(z)}{f^{(n-2)}(z)}\right| < 1, \quad \left|\frac{f^{(n)}(z)}{f^{(n-1)}(z)}\right| < 1 \qquad (n \geqq 2) \tag{3.1}$$

must hold.

Thus, for all z

$$|f^{(n)}(z)| < \max\{|f^{(n-1)}(z)|, |f^{(n-2)}(z)|\} \qquad (n = 2, 3, 4, \cdots) . \tag{3.2}$$

Theorem 1 follows from (3.2) by an obvious induction over n.

4. Proof of Theorem 2. In this section we assume familiarity with the most elementary results and notations of Nevanlinna's theory of meromorphic functions.

Let $f(z)$ be a given entire, nonrational function of finite order. A theorem of Edrei and Fuchs [2; p. 384 and p. 390, formula (3.5)] asserts the existence of an entire function $h(z)$ such that $h(0) = 1$ and

$$N\left(r, \frac{1}{h}\right) \sim \log M(r, h) \sim \log M(r, f) \qquad (r \to +\infty) . \tag{4.1}$$

We take $g(z)$ to be of the form

$$g(z) = h(z)P(z) , \tag{4.2}$$

where

$$P(z) = \prod_{j=1}^{\infty}\left(1 + \frac{z}{d_j}\right)^j . \tag{4.3}$$

The quantities d_j are positive and satisfy the following conditions:
(i) $d_1 > e^2$, $d_{j+1} > d_j^2$ $(j = 1, 2, 3, \cdots)$;
(ii) for $t \geqq d_j$,

$$\frac{j(j+1)}{2} < \left\{\frac{\log M(t, f)}{\log t}\right\}^{1/2} .$$

Since $f(z)$ is not rational

$$\frac{\log M(t, f)}{\log t} \longrightarrow +\infty \qquad (t \to +\infty) \tag{4.4}$$

and hence it is possible to satisfy condition (ii).

Putting

$$n(t) = n\left(t, \frac{1}{P}\right) ,$$

we see that

$$n(t) = 0 \ \ (0 \leqq t < d_1),\ n(t) = \frac{k(k+1)}{2} \quad (d_k \leqq t < d_{k+1}) . \tag{4.5}$$

Hence, if

$$d_k \leqq t < d_{k+1} \qquad (k \geqq 1) \tag{4.6}$$

(4.5) and condition (i) imply

$$n(t) < 2^k < \log d_k \leqq \log t < t^{1/2} \qquad (k \geqq 1) . \tag{4.7}$$

By (4.6), (i) and (4.5)

$$t^2 < d_{k+1}^2 < d_{k+2} ,$$

$$\frac{n(t^2)}{n(t)} \leqq 1 + \frac{2}{k} . \tag{4.8}$$

By (4.6), (ii), (4.5) and (4.4)

$$n(t) \log t < \log M(t, f)\left\{\frac{\log t}{\log M(t, f)}\right\}^{1/2} = o(\log M(t, f)) \qquad (t \to \infty) . \tag{4.9}$$

By (4.1), (4.2) and the elements of Nevanlinna's theory

$$\begin{aligned}(1 + o(1)) \log M(r, f) &= N\left(r, \frac{1}{h}\right) \leqq N\left(r, \frac{1}{g}\right) \\ &\leqq \log M(r, g) \leq \log M(r, h) + \log M(r, P) \\ &= \log M(r, f)\left\{1 + o(1) + \frac{\log M(r, P)}{\log M(r, f)}\right\} \qquad (r \to +\infty) .\end{aligned}$$

Hence, in order to obtain Theorem 2 it is sufficient to show that

$$\frac{\log M(r, P)}{\log M(r, f)} \longrightarrow 0 \qquad (r \to +\infty) \tag{4.10}$$

and to remark that $g(z)$ cannot be of bounded index because it has zeros of arbitrarily high multiplicity.

The relation (4.10) follows readily from the identity [1, p. 48]

$$\log M(r, P) = r \int_0^\infty \frac{n(t)}{t(t + r)} dt ,$$

which, in view of (4.7), (4.8) and (4.9), leads to

$$\begin{aligned}\log M(r, P) &< n(r) \log r + r \int_r^{r^2} \frac{n(r^2)}{t^2} dt + r \int_{r^2}^\infty t^{-3/2} dt \\ &= o(\log M(r, f)) \qquad (r \to +\infty) .\end{aligned}$$

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Received October 16, 1968, and in revised form November 11, 1969. The first author (now deceased) gratefully acknowledges support by the National Science Foundation under grant GP-7507.

The research of the second named author was supported by the National Science Foundation under grant GP-7544. He regrets to announce the death of Mr. W. J. Pugh on April 17, 1969.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

EXISTENCE OF TRICONNECTED GRAPHS WITH PRESCRIBED DEGREES

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Necessary and sufficient conditions for the existence of a p-connected (linear undirected) graph with prescribed degrees $d_1, d_2, \cdots, d_n$ are known for $p = 1, 2$. In this paper we solve this problem for $p = 3$.

Let $d_1, d_2, \cdots, d_n$ be positive integers and let $d_1 \leqq d_2 \leqq \cdots \leqq d_n$.

LEMMA. *If a triconnected graph G exists with degrees $d_1, d_2, \cdots, d_n$, then*

(1) $d_i \geqq 3$.

(2) *$d_1, d_2, \cdots, d_n$ is graphical*, i.e., *there exists a graph with these degrees.*

(3) $d_n + d_{n-1} \leqq m - n + 4$ *where* $2m = \sum_{i=1}^n d_i$.

(4) *If* $d_n + d_{n-1} = m - n + 4$, *then* $m \geqq 2n - 2$.

Proof. (1) and (2) are evident. To prove (3), let x_n, x_{n-1} be the vertices of G with degrees d_n and d_{n-1} respectively. Then the number of edges in $G - \{x_n, x_{n-1}\}$ is $m - (d_n + d_{n-1} - 1)$ or $m - (d_n + d_{n-1})$ according as x_n, x_{n-1} are adjacent or not adjacent in G. Also $G - \{x_n, x_{n-1}\}$ is connected, so (3) follows. If now $d_n + d_{n-1} = m - n + 4$, then

$$2m \geqq d_n + d_{n-1} + 3(n - 2) = m + 2n - 2 .$$

This completes the proof of the lemma.

THEOREM. *Conditions* (1) *to* (4) *of the lemma are necessary and sufficient for the existence of a triconnected graph with degrees $d_1, d_2, \cdots, d_n$.*

Proof. Necessity was proved in the lemma.

To prove sufficiency, first let conditions (1), (3) be satisfied and let $d_n + d_{n-1} = m - n + 4 = n + \lambda$ where $2 \leqq \lambda \leqq n - 2$. Let k be the number of d_i such that $1 \leqq i \leqq n - 2$ and $d_i = 3$. Then define

$$e_i = d_i - 2 \text{ for } i = k + 1, \cdots, n - 2 .$$

Then we have

$$\sum_{i=1}^{n-2} d_i = 2m - d_n - d_{n-1} = 3n + \lambda - 8 ,$$

$$\sum_{i=k+1}^{n-2} e_i = 3n + \lambda - 8 - 3k - 2(n - 2 - k) = n + \lambda - k - 4 .$$

Define now $\eta = n - 2 - \lambda$ and $\varepsilon = k - \eta$. Then $\eta \geqq 0$, and $\varepsilon \geqq 2$ since

$$\begin{aligned} 2m &\geqq m - n + 4 + 3k + 4(n - 2 - k) \\ &= m + 3n - k - 4 \end{aligned}$$

and so

$$\lambda = m - 2n + 4 \geqq n - k .$$

Write now

$$e_i = \begin{cases} 1 & \text{for } i = 1, 2, \cdots, \varepsilon , \\ 2 & \text{for } i = \varepsilon + 1, \cdots, k , \\ d_i - 2 & \text{for } i = k + 1, \cdots, n - 2 . \end{cases}$$

Then $\sum_{i=1}^{n-2} e_i = 2(n - 3)$ and so there exists a tree T with degrees $e_1, \cdots e_{n-2}$, attained by the vertices $x_1, \cdots, x_{n-2}$, say, in that order [2]. Take two more vertices x_{n-1} and x_n and join them. Also join each of x_{n-1}, x_n to x_i for $i = 1, \cdots, \varepsilon, k + 1, \cdots, n - 2$. Of the η vertices $x_{\varepsilon+1}, \cdots, x_k$, join $d_{n-1} - 1 - \varepsilon - n + 2 + k$ to x_{n-1} and the rest ($d_n - 1 - \varepsilon - n + 2 + k$ in number) to x_n. Note that

$$d_{n-1} - 1 - \varepsilon - n + 2 + k = d_{n-1} - \lambda - 1 \geqq 0 .$$

The graph we thus obtain has degrees $d_1, \cdots, d_n$ and is triconnected since any vertex of T with degree in T less than 3 is joined to either x_{n-1} or x_n.

Next let conditions (1), (2) be satisfied and let

$$d_n + d_{n-1} \leqq m - n + 3 .$$

Then $d_n < m - n + 2$, so there exists a biconnected graph G with degrees $d_1, d_2, \cdots, d_n$ [2]. If G is not triconnected, let x_i, x_j be two vertices such that $G - \{x_i, x_j\}$ is disconnected. Let $C_1, C_2, \cdots$ be the components of $G - \{x_i, x_j\}$. By (1), $|C_g| \geqq 2$ for $g = 1, 2, \cdots$. Also by hypothesis,

$$m - d_i - d_j \geqq n - 3 ,$$

so it follows that one of the components, say C_1, contains a cycle.

We first prove that there exists an edge (x, y) in C_1 and two chains μ_1, μ_1' of G connecting x and y such that $(x, y), \mu_1, \mu_1'$ are disjoint except for x and y, and μ_1 is contained in C_1. Since G is biconnected, there exists a chain connecting x_i and x_j with all intermediate vertices in C_2.

If now two vertices x, y with degree two in C_1 are adjacent and belong to a cycle of C_1, the required edge is (x, y). So we may take that no two vertices of degree two in C_1 can belong to a block (on more than two vertices) and be adjacent. Let B be any block of C_1 which is not an edge. If some cycle of B has a chord (x, y), then (x, y) is the required edge. Otherwise, by the results of [1], two vertices y, z of degree two in B will be adjacent to a vertex x of degree three in B. If w is another vertex of B adjacent to x, then there is a chain connecting w to y in $B - \{x\}$. This chain together with (x, w) may be taken as μ_1. To get μ_1', go from x to z along (x, z), from z to x_i or x_j (through another block of C_1 at z if necessary), then to y. Thus (x, y) is the required edge.

Let now (x, y) be an edge of C_1 chosen as explained above. If C_2 is a tree, take any edge (u, v) of C_2. Then (u, v) is a chord of a cycle of G. If C_2 is not a tree, choose an edge (u, v) of C_2 such that there are chains μ_2, μ_2' of G connecting u and v, (u, v), μ_2, μ_2' are disjoint except for u, v, and μ_2 is contained in C_2.

We define $f_G(s, t)$ to be the number of components of $G - \{s, t\}$. Now we will make a modification on G so that the degrees of the vertices are unaltered, $f(x_i, x_j)$ decreases and $f(s, t)$ does not increase for any two vertices s and t.

First we associate with x, a subset $A(x)$ of $\{x_i, x_j\}$ by the following rule. $x_i \in A(x)$ if and only if there is a chain ν connecting x to x_i with all intermediate vertices in C_1 such that ν is disjoint with (x, y) and μ_1 except for x. Similarly $A(y)$ is defined. If C_2 is a tree, put $A(u) = A(v) = \{x_i, x_j\}$. Otherwise $A(u)$, $A(v)$ are defined in a manner similar to that of $A(x)$ and $A(y)$. Now $A(x)$, $A(y)$ are made nonempty by a proper choice of μ_1, and $A(u)$, $A(v)$ are made nonempty by a proper choice of μ_2 (in case C_2 is not a tree).

Now suppress the edges (x, y), (u, v) and join x to one of u, v and y to the other as follows. Join x to u if $A(x) \neq A(u)$ and $A(y) \neq A(v)$ whenever such a choice is possible. Let the new graph thus obtained be H. To be specific we take that x is joined to u in H.

First we show that H is biconnected. Obviously $G_1 = G - (x, y)$ is biconnected. Now we show that (u, v) is a chord of a cycle of G_1. If C_2 is a tree, then the cycle is

$$(u, x) + \mu_1[x, y] + (y, v) + [v, \cdots, p_1] + (p_1, x_i) + (x_i, p_2) + [p_2, \cdots, u]$$

where p_1, p_2 are suitable pendant vertices of C_2. Otherwise the cycle is

$$\mu_2[u, v] + \mu_2'[v, u]$$

where if μ_2' contains the edge (x, y), then (x, y) is replaced by $\mu_1[x, y]$ and the resulting cycle is made elementary.

Trivially now $f_G(x_i, x_j) = f_H(x_i, x_j) + 1$. Next we will show that

$$f_G(s, t) \geqq f_H(s, t) \tag{5}$$

for any two vertices s and t. For this it is enough to show that x, y are connected and u, v are connected in $H - \{s, t\}$.

First let $s = x_i$. Now x, y, u, v belong to a cycle in $H - \{x_i\}$, so (5) follows. So we may take $\{s, t\} \cap \{x_i, x_j\} = \emptyset$.

Now let $s = x$. Then to prove (5) it is enough to show that u, v are connected in $H - \{x, t\}$ when $t \neq u$ and $t \neq v$. This is evident if C_2 is a tree or $t \notin \mu_2$. So let $t \in \mu_2$ and C_2 be not a tree. If $A(u) \cap A(v) \neq \emptyset$, there is a chain connecting u, v in $H - \{x, t\}$. So we take without loss of generality $A(u) = x_j$ and $A(v) = x_i$. If now $x_j \in A(y)$, then u, v are connected through x_j and y in $H - \{x, t\}$. So we take $A(y) = x_i$. If $x_j \in A(x)$, then y would not have been joined to v, so $A(x) = x_i$. Now in G, x_j is connected to some vertex z of μ_1 by a chain with all intermediate vertices belonging to C_1 but not to μ_1. Now we obtain a chain connecting u, v in $H - \{x, t\}$ by going from u to x_j, x_j to z, z to y along μ_1, y to x_i, and x_i to v. Thus we may take $\{s, t\} \cap \{x_i, x_j, x, y\} = \emptyset$.

Next let $s = u$. If $t \notin \mu_1$, then (5) is trivial, so let $t \in \mu_1$. Suppose first that C_2 is a tree. Then we obtain a chain connecting x, y in $H - \{u, t\}$ by going from x to x_i or x_j, then to v through a suitable pendant vertex of C_2 and then to y. If C_2 is not a tree, the situation is similar to that of the preceding paragraph. Thus we take $\{s, t\} \cap \{x_i, x_j, x, y, u, v\} = \emptyset$.

If none of s, t belongs to μ_1, then (5) is trivial. So let $s \in \mu_1$.

Suppose now that C_2 is a tree. Then for any fixed vertex t, there are chains in $H - \{s, t\}$ from one of u, v to both x_i and x_j, and a chain from the other (of the vertices u, v) to x_i or x_j. Hence u, v are connected and (5) follows.

Suppose next that C_2 is not a tree. Obviously we may take $s \in \mu_1$ and $t \in \mu_2$. If now $A(x) \cap A(y) \neq \emptyset$ or $A(u) \cap A(v) \neq \emptyset$, then again (5) follows. So we may take $A(x) = x_i$, $A(y) = x_j$, $A(u) = x_j$ $A(v) = x_i$. Now we obtain a chain connecting x, y in $H - \{s, t\}$ by going from x to u, u to x_j, x_j to y. This proves (5) completely.

Now by a repeated application of the above procedure we reduce the graph until finally $f(s, t) = 1$ for any two vertices. The final graph has degrees $d_1, d_2, \cdots, d_n$ and is triconnected and this completes the proof of the theorem.

Perhaps necessary and sufficient conditions, similar to the conditions (1) to (4) above, for the existence of a p-connected graph with prescribed degrees $d_1, d_2, \cdots, d_n$ can be obtained for all $p \geqq 3$, but the authors have not yet succeeded in this.

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Received August 13, 1969.

Indian Statistical Institute, Calcutta

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON THE MAXIMAL MONOTONICITY OF SUBDIFFERENTIAL MAPPINGS

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The subdifferential of a lower semicontinuous proper convex function on a Banach space is a maximal monotone operator, as well as a maximal cyclically monotone operator. This result was announced by the author in a previous paper, but the argument given there was incomplete; the result is proved here by a different method, which is simpler in the case of reflexive Banach spaces. At the same time, a new fact is established about the relationship between the subdifferential of a convex function and the subdifferential of its conjugate in the nonreflexive case.

Let E be a real Banach space with dual E^*. A *proper convex function* on E is a function f from E to $(-\infty, +\infty]$, not identically $+\infty$, such that

$$f((1-\lambda)x + \lambda y) \leqq (1-\lambda)f(x) + \lambda f(y)$$

whenever $x \in E$, $y \in E$ and $0 < \lambda < 1$. The *subdifferential* of such a function f is the (generally multivalued) mapping $\partial f: E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* \mid f(y) \geqq f(x) + \langle y - x, x^* \rangle, \forall y \in E\} ,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between E and E^*.

A multivalued mapping $T: E \to E^*$ is said to be a *monotone operator* if

$$\langle x_0 - x_1, x_0^* - x_1^* \rangle \geqq 0 \quad \text{whenever} \quad x_0^* \in T(x_0),\ x_1^* \in T(x_1) .$$

It is said to be a *cyclically monotone operator* if

$$\langle x_0 - x_1, x_0^* \rangle + \cdots + \langle x_{n-1} - x_n, x_{n-1}^* \rangle + \langle x_n - x_0, x_n^* \rangle \geqq 0$$
$$\text{whenever} \quad x_i^* \in T(x_i),\ i = 0, \cdots, n .$$

It is called a *maximal* monotone operator (resp. maximal cyclically monotone operator) if, in addition, its graph

$$G(T) = \{(x, x^*) \mid x^* \in T(x)\} \subset E \times E^*$$

is not properly contained in the graph of any other monotone (resp. cyclically monotone) operator $T': E \to E^*$.

This note is concerned with proving the following theorems.

THEOREM A. *If f is a lower semicontinuous proper convex function on E, then ∂f is a maximal monotone operator from E to E^*.*

THEOREM B. *Let $T: E \to E^*$ be a multivalued mapping. In order that there exist a lower semicontinuous proper convex function f on E such that $T = \partial f$, it is necessary and sufficient that T be a maximal cyclically monotone operator. Moreover, in this case T determines f uniquely up to an additive constant.*

These theorems have previously been stated by us in [4] as Theorem 4 and Theorem 3, respectively. However, a gap occurs in the proofs in [4], as has kindly been brought to our attention recently by H. Brézis. (It is not clear whether formula (4.7) in the proof of Theorem 3 of [4] will hold for ε sufficiently small, because x_i^* depends on ε and could conceivably increase unboundedly in norm as ε decreases to 0. The same oversight appears in the penultimate sentence of the proof of Theorem 4 of [4]). In view of this oversight, the proofs in [4] are incomplete; further arguments must be given before the maximality in Theorem A, the maximality in the necessary condition in Theorem B, and the uniqueness in Theorem B can be regarded as established. Such arguments will be given here.

2. Preliminary result. Let f be a lower semicontinuous proper convex function on E. (For proper convex functions, lower semicontinuity in the strong topology of E is the same as lower semicontinuity in the weak topology.) The conjugate of f is the function f^* on E^* defined by

$$f^*(x^*) = \sup \{\langle x, x^*\rangle - f(x) \mid x \in E\} . \tag{2.1}$$

It is known that f^* is a weak* lower semicontinuous (and hence strongly lower semicontinuous) proper convex function on E^*, and that

$$f(x) + f^*(x^*) - \langle x, x^*\rangle \geqq 0, \forall x \in E, \forall x^* \in E^* , \text{ with equality if and only if } x^* \in \partial f(x) \tag{2.2}$$

(see Moreau [3, § 6]). The subdifferential ∂f^*, which is a multivalued mapping from E^* to the bidual E^{**}, can be compared with the subdifferential ∂f from E to E^*, when E is regarded in the canonical way as a weak** dense subspace of E^{**} (the weak** topology being the weak topology induced on E^{**} by E^*). Facts about the relationship between ∂f^* and ∂f will be used below in proving Theorems A and B.

In terms of the conjugate f^{**} of f^*, which is the weak** lower semicontinuous proper convex function on E^{**} defined by

$$(2.3)\qquad f^{**}(x^{**}) = \sup\{\langle x^{**}, x^*\rangle - f^*(x^*) \mid x^* \in E^*\}\ ,$$

we have, as in (2.2),

$$(2.4)\qquad f^{**}(x^{**}) + f^*(x^*) - \langle x^{**}, x^*\rangle \geqq 0,\ \forall x^{**} \in E^{**},\ \forall x^* \in E^*\ , \\ \text{with equality if and only if } x^{**} \in \partial f^*(x^*)\ .$$

Moreover, the restriction of f^{**} to E is f (see [3, § 6]). Thus, if E is reflexive, we can identify f^{**} with f, and it follows from (2.2) and (2.4) that ∂f^* is just the "inverse" of ∂f, in other words one has $x \in \partial f^*(x^*)$ if and only if $x^* \in \partial f(x)$. If E is not reflexive, the relationship between ∂f^* and ∂f is more complicated, but ∂f^* and ∂f still completely determine each other, according to the following result.

PROPOSITION 1. *Let f be a lower semicontinuous proper convex function on E, and let $x^* \in E^*$ and $x^{**} \in E^{**}$. Then $x^{**} \in \partial f^*(x^*)$ if and only if there exists a net $\{x_i^* \mid i \in I\}$ in E^* converging to x^* in the strong topology and a bounded net $\{x_i \mid i \in I\}$ in E (with the same partially ordered index set I) converging to x^{**} in the weak** topology, such that $x_i^* \in \partial f(x_i)$ for every $i \in I$.*

Proof. The sufficiency of the condition is easy to prove. Given nets as described, we have

$$f(x_i) + f^*(x_i^*) = \langle x_i, x_i^*\rangle,\ \forall i \in I$$

by (2.2), where $f(x_i) = f^{**}(x_i)$. Then by the lower semicontinuity of f^* and f^{**} we have

$$\begin{aligned} f^{**}(x^{**}) + f^*(x^*) &\leqq \liminf\{f^{**}(x_i) + f^*(x_i^*)\} \\ &= \lim\langle x_i, x_i^*\rangle = \langle x^{**}, x^*\rangle\ . \end{aligned}$$

(The last equality makes use of the boundedness of the norms $\|x_i\|$, $i \in I$.) Thus $x^{**} \in \partial f^*(x^*)$ by (2.4).

To prove the necessity of the condition, we demonstrate first that, given any $x^{**} \in E^{**}$, there exists a *bounded* net $\{y_i \mid i \in I\}$ in E such that y_i converges to x^{**} in the weak** topology and

$$(2.5)\qquad \lim f(y_i) = f^{**}(x^{**})\ .$$

Consider $f + h_\alpha$, where α is a positive real number and h_α is the lower semicontinuous proper convex function on E defined by

$$(2.6)\qquad h_\alpha(x) = 0 \quad \text{if} \quad \|x\| \leqq \alpha,\ h_\alpha(x) = +\infty \quad \text{if} \quad \|x\| > \alpha\ .$$

Assuming that α is sufficiently large, there exist points x at which f and h_α are both finite and h_α is continuous (i.e., points x such that $f(x) < +\infty$ and $\|x\| < \alpha$). Then, by the formulas for conjugates of sums of convex functions (see Moreau [3, pp. 38, 56, 57] or Rockafellar [5, Th. 3]), we have $(f + h_\alpha)^* = f^* \Box h_\alpha^*$ (infimal convolution), and consequently

$$(f + h_\alpha)^{**} = (f^* \Box h_\alpha^*)^* = f^{**} + h_\alpha^{**} . \tag{2.7}$$

Moreover $h_\alpha^*(x^*) = \alpha \| x^* \|$ for ever $x^* \in E^*$, so that

$$\begin{aligned} h_\alpha^{**}(x^{**}) &= \sup \{\langle x^{**}, x^*\rangle - \alpha \| x^* \| \mid x^* \in E^*\} \\ &= \begin{cases} 0 & \text{if } \| x^{**} \| \leqq \alpha , \\ +\infty & \text{if } \| x^{**} \| > \alpha . \end{cases} \end{aligned}$$

Hence by (2.7), given any $x^{**} \in E^{**}$, we have

$$f^{**}(x^{**}) = (f + h_\alpha)^{**}(x^{**}) \tag{2.8}$$

for sufficiently large $\alpha > 0$. On the other hand, it is known that, for any lower semicontinuous proper convex function g on E, g^{**} is the greatest weak** lower semicontinuous function on E^{**} majorized by g on E (see [3, § 6]), so that

$$g^{**}(x^{**}) = \liminf_{y \to x^{**}} g(y) , \tag{2.9}$$

where the "lim inf" is taken over all nets in E converging to x^{**} in the weak** topology. Taking $g = f + h_\alpha$, we see from (2.8) and (2.9) that

$$f^{**}(x^{**}) = \liminf_{y \to x^{**}} [f(y) + h_\alpha(y)] ,$$

implying that (2.5) holds as desired for some net $\{y_i \mid i \in I\}$ in E such that y_i converges to x^{**} in the weak** topology and $\| y_i \| \leqq \alpha$ for every $i \in I$.

Now, given any $x^* \in E^*$ and $x^{**} \in \partial f^*(x^*)$, let $\{y_i \mid i \in I\}$ be a bounded net in E such that y_i converges to x^{**} in the weak** topology and (2.5) holds. Define $\varepsilon_i \geqq 0$ by

$$\varepsilon_i^2 = f(y_i) + f^*(x^*) - \langle y_i, x^*\rangle .$$

Note that $\lim \varepsilon_i = 0$ by (2.5) and (2.4). According to a lemma of Brøndsted and Rockafellar [1, p. 608], there exist for each $i \in I$ an $x_i \in E$ and an $x_i^* \in E^*$ such that

$$x_i^* \in \partial f(x_i),\ \| x_i - y_i \| \leqq \varepsilon_i,\ \| x_i^* - x^* \| \leqq \varepsilon_i .$$

The latter two conditions imply that the net $\{x_i^* \mid i \in I\}$ converges to x^* in the strong topology of E^*, while the net $\{x_i \mid i \in I\}$ is bounded and converges to x^{**} in the weak** topology of E^{**}. This completes the proof of Proposition 1.

3. Proofs of Theorems *A* and *B*. In the sequel, f denotes a lower semicontinuous proper convex function on E, and j denotes the continuous convex function E defined by $j(x) = (1/2)\|x\|^2$. We shall make use of the fact that, for each $x \in E$, $\partial f(x)$ is by definition a certain (possibly empty, possibly unbounded) weak* closed convex subset of E^*, whereas $\partial j(x)$ is (by the finiteness and continuity of j, see [3, p. 60]) a certain nonempty weak* compact convex subset of E^*. Furthermore

$$\partial(f + j) = \partial f(x) + \partial j(x), \ \forall x \in E \tag{3.1}$$

(see [3, p. 62] or [5, Th. 3]). The conjugate of j is given by $j^*(x^*) = (1/2)\|x^*\|^2$, and since

$$(f + j)^*(x^*) = (f^* \square j^*)(x^*) = \min_{y^* \in E^*} \{f^*(y^*) + j^*(x^* - y^*)\}$$

([3, § 9] or [5, Th. 3]) the conjugate function $(f + j)^*$ is finite and continuous throughout E^*.

Proof of Theorem A. Theorem A has already been established by Minty [2] in the case of convex functions which, like j, are everywhere finite and continuous. Applying Minty's result to the function $(f + j)^*$, we may conclude that $\partial(f + j)^*$ is a maximal monotone operator from E^* to E^{**}. We shall show this implies that ∂f is a maximal monotone operator from E to E^*.

Let T be a monotone operator from E to E^* such that the graph of T includes the graph of ∂f, i.e.,

$$T(x) \supset \partial f(x), \ \forall x \in E \ . \tag{3.2}$$

We must show that equality necessarily holds in (3.2).

The mapping $T + \partial j$ defined by

$$\begin{aligned}(T + \partial j)(x) &= T(x) + \partial j(x) \\ &= \{x_1^* + x_2^* \mid x_1^* \in T(x),\ x_2^* \in \partial j(x)\}\end{aligned}$$

is a monotone operator from E to E^*, since T and ∂j are, and by (3.1) and (3.2) we have

$$(T + \partial j)(x) \supset \partial(f + j)(x), \ \forall x \in E \ . \tag{3.3}$$

Let S be the multivalued mapping from E^* to E^{**} defined as follows: $x^{**} \in S(x^*)$ if and only if there exists a net $\{x_i^* \mid i \in I\}$ in E^* converging to x^* in the strong topology, and a bounded net $\{x_i \mid i \in I\}$ in E (with the same partially ordered index set I) converging to x^{**} in the weak** topology, such that

$$x_i^* \in (T + \partial j)(x_i), \ \forall i \in I \ .$$

It is readily verified that S is a monotone operator. (The boundedness of the nets $\{x_i \mid i \in I\}$ enters in here.) Moreover

$$(3.4) \qquad S(x^*) \supset \partial(f + j)^*(x^*),\ \forall x^* \in E^* ,$$

by (3.3) and Proposition 1. Since $\partial(f + j)^*$ is a maximal monotone operator, equality must actually hold in (3.4). This shows that one has $x \in \partial(f + j)^*(x^*)$ whenever $x \in E$ and $x \in S(x^*)$, hence in particular whenever $x^* \in (T + \partial j)(x)$. On the other hand, one always has $x^* \in \partial(f + j)(x)$ if $x \in \partial(f + j)^*(x^*)$ and $x \in E$. (This follows from applying (2.2) and (2.4) to $f + j$ in place of f.) Thus one has $x^* \in \partial(f + j)(x)$ if $x^* \in (T + \partial j)(x)$, implying by (3.3) and (3.1) that

$$(3.5) \qquad T(x) + \partial j(x) = \partial f(x) + \partial j(x),\ \forall x \in E .$$

We shall show now from (3.5) that actually

$$T(x) = \partial f(x),\ \forall x \in E ,$$

so that ∂f must be a maximal monotone operator as claimed. Suppose that $x \in E$ is such that the inclusion in (3.2) is proper. This will lead to a contradiction. Since $\partial f(x)$ is a weak* closed convex subset of E^*, there must exist some point of $T(x)$ which can be separated strictly from $\partial f(x)$ be a weak* closed hyperplane. Thus, for a certain $y \in E$, we have

$$\sup \{\langle y, x^*\rangle \mid x^* \in T(x)\} > \sup \{\langle y, x^*\rangle \mid x^* \in \partial f(x)\} .$$

But then

$$\begin{aligned} &\sup \{\langle y, z^*\rangle \mid z^* \in T(x) + \partial j(x)\} \\ &= \sup \{\langle y, x^*\rangle \mid x^* \in T(x)\} + \sup \{\langle y, y^*\rangle \mid y^* \in \partial j(x)\} \\ &> \sup \{\langle y, x^*\rangle \mid x^* \in \partial f(x)\} + \sup \{\langle y, y^*\rangle \mid y^* \in \partial j(x)\} \\ &= \sup \{\langle y, z^*\rangle \mid z^* \in \partial f(x) + \partial j(x)\} , \end{aligned}$$

inasmuch as $\partial j(x)$ is a nonempty bounded set, and this inequality is incompatible with (3.5).

Proof of Theorem B. Let g be a lower semicontinuous proper convex function on E such that

$$(3.6) \qquad \partial g(x) \supset \partial f(x),\ \forall x \in E .$$

As noted at the beginning of the proof Theorem 3 of [4], to prove Theorem B it suffices, in view of Theorem 1 of [4] and its Corollary 2, to demonstrate that $g = f + \text{const}$.

We consider first the case where f and g are everywhere finite and continuous. Then, for each $x \in E$, $\partial f(x)$ is a nonempty weak* compact set, and

$$f'(x; u) = \max \{\langle u, x^* \rangle \mid x^* \in \partial f(x)\},\ \forall u \in E\ , \tag{3.7}$$

where

$$f'(x; u) = \lim_{\lambda \downarrow 0} [f(x + \lambda u) - f(x)]/\lambda$$

[3, p. 65]. Similarly, $\partial g(x)$ is a nonempty weak* compact set, and

$$g'(x; u) = \max \{\langle u, x^* \rangle \mid x^* \in \partial g(x)\},\ \forall u \in E\ . \tag{3.8}$$

It follows from (3.6), (3.7) and (3.8) that

$$f'(x; u) \leqq g'(x; u),\ \forall x \in E,\ \forall u \in E\ . \tag{3.9}$$

On the other hand, for any $x \in E$ and $y \in E$, we have

$$f(y) - f(x) = \int_0^1 f'((1 - \lambda)x + \lambda y;\ y - x) d\lambda\ ,$$

$$g(y) - g(x) = \int_0^1 g'((1 - \lambda)x + \lambda y;\ y - x) d\lambda$$

(see [6, § 24]), so that by (3.9) we have

$$f(y) - f(x) \leqq g(y) - g(x),\ \forall x \in E,\ \forall y \in E\ .$$

Of course, the latter can hold only if $g = f + \text{const}$.

In the general case, we observe from (3.6) that

$$\partial g(x) + \partial j(x) \supset \partial f(x) + \partial j(x),\ \forall x \in E\ ,$$

and consequently

$$\partial(g + j)(x) \supset \partial(f + j)(x),\ \forall x \in E\ ,$$

by (3.1)(and its counterpart for g). This implies by Proposition 1 that

$$\partial(g + j)^*(x^*) \supset \partial(f + j)^*(x^*),\ \forall x^* \in E^*\ . \tag{3.10}$$

The functions $(f + j)^*$ and $(g + j)^*$ are finite and continuous on E^*, so we may conclude from (3.10) and the case already considered that

$$(g + j)^* = (f + j)^* + \alpha$$

for a certain real constant α. Taking conjugates, we then have

$$(g + j)^{**} = (f + j)^{**} - \alpha\ . \tag{3.11}$$

Since $(g + j)^{**}$ and $(f + j)^{**}$ agree on E with $g + j$ and $f + j$, respectively, (3.11) implies that

$$g + j = f + j - \alpha\ ,$$

and hence that $g = f + \text{const.}$

REMARK. The preceding proofs become much simpler if E is reflexive, since then ∂f^* and $\partial(f + j)^*$ are just the "inverses" of ∂f and $\partial(f + j)$, respectively, and Proposition 1 is superfluous. In this case, S may be replaced by the inverse of $T + \partial j$ in the proof of Theorem A.

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Received July 17, 1969. This work was supported in part by the Air Force Office of Scientific Research under grant AF-AFOSR-1202-67.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

CONVERGENCE OF A SEQUENCE OF TRANSFORMATIONS OF DISTRIBUTION FUNCTIONS-II

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A previous paper of the present author was devoted to the study of the convergence properties of the iterates of a certain transformation of distribution functions (d.f.'s) of a random variable (r.v.). In this paper the definitions and some of the results are extended to the case of bivariate d.f.'s.

1. Definition and preliminaries. Throughout this paper $F(x, y)$ will denote the bivariate $d.f.$ of a nonnegative random vector (X, Y). More precisely, (i) $F(x, y)$ is monotonic nondecreasing; i.e., for $a > c$, $b > d$ we have

$$[F(x, y)]_{c;d}^{a;b} = F(a, b) - F(a, d) - F(c, b) + F(c, d) \geqq 0 .$$

(ii) $F(x, 0) = F(0, y) = 0$ for all x and y. (iii) $F(+\infty, +\infty) = \lim_{x,y\to\infty} F(x, y) = 1$ and (iv) $F(x, y)$ is left continuous in each variable; i.e.,

$$\lim_{h\to 0-} F(x + h, y) = F(x, y)$$

for all x and y with a similar left continuity in y.

We shall let $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$ be the marginal $d.f.$'s of X and Y respectively and $\mu(i, j) = E(X^i Y^j)$ be a product moment of order $i + j$ when it exists finitely. Hence $\mu(i, 0)$ and $\mu(0, i)$ are the i-th moments of the marginal $d.f.$'s F_1 and F_2 respectively. For brevity we let $\mu = \mu(1, 1)$.

Let us remark at this point that (1) all of the results of this paper (and more) follow immediately from the univariate case if F is the $d.f.$ corresponding to a product measure; i.e., X and Y are independent and (2) although we are dealing explicitly with the bivariate case, the treatment and the results carry over in a direct way to distributions in the positive quadrant of R^n, $n \geqq 3$.

We develop now the requisite background material before introducing the bivariate transform in § 2.

The following two lemmas for integration by parts are basic. These formulas are known [11], but apparently not readily available, and so we give them in a form convenient for our use.

LEMMA 1.1. *Assuming the existence of the double Riemann-Stieltjes integral we have*

$$
\begin{aligned}
\int_0^a\int_0^b f(x,y)dg(x,y) &= \int_0^a\int_0^b [1-g(x,b)-g(a,y)+g(x,y)]df(x,y) \\
&\quad - \int_0^a [g(x,b)-g(x,0)]df(x,0) \\
&\quad - \int_0^b [g(a,y)-g(0,y)]df(0,y) \\
&\quad + [g(x,y)f(x,y)]_{0,0}^{a,b} - [f(x,y)]_{0,0}^{a,b}
\end{aligned} \tag{1.1}
$$

where

$$[h(x,y)]_{0,0}^{a,b} = h(a,b) - h(a,0) - h(0,b) + h(0,0)\ .$$

LEMMA 1.2.

$$\int_0^a\int_0^b f(x)dg(x,y) = \int_0^a f(x)d[g(x,b)-g(x,0)]\ . \tag{1.2}$$

It is well known that the double Riemann-Stieltjes integral exists when, for example, one of the functions f and g is continuous and the other is of bounded variation (cf. [3]).

LEMMA 1.3. *If $G(x, y)$ is continuous and the bivariate d.f. of a nonnegative random vector except that $G(\infty, \infty)$ is arbitrary, then*

$$\int_0^\infty\int_0^\infty G(x,y)dF(x,y) = \int_0^\infty\int_0^\infty [1-F_1(x)-F_2(y)+F(x,y)]dG(x,y)\ . \tag{1.3}$$

Proof. Let $a > 0, b > 0$ and $S = [0, a] \times [0, b]$. Using (1.1) and simplifying we get

$$
\begin{aligned}
\int_S G(x,y)dF(x,y) &= A + \int_0^a [F_1(x)-F(x,b)]dG(x,b) \\
&\quad + \int_0^b [F_2(y)-F(a,y)]dG(a,y) \\
&\quad - G(a,b)[1-F(a,b)] \\
&= A + B
\end{aligned} \tag{1.4}
$$

where

$$A = \int_S F^*(x,y)dG(x,y)$$

and

$$
\begin{aligned}
F^*(x,y) &= 1 - F_1(x) - F_2(y) + F(x,y) \\
&= Pr(X \geqq x, Y \geqq y) \geqq 0\ .
\end{aligned} \tag{1.5}
$$

Now $B \leqq 0$. In fact, since $F_1(x) - F(x, b)$ and $F_2(y) - F(a, y)$ are nondecreasing functions in x and y respectively we have

$$\begin{aligned} B &\leqq G(a, b)[F_1(a) - F(a, b)] + G(a, b)[F_2(b) - F(a, b)] \\ &\qquad\qquad - G(a, b)[1 - F(a, b)] \\ &= -G(a, b)F^*(a, b) \leqq 0 . \end{aligned}$$

Next, noting (1.2) and integrating by parts

$$\begin{aligned} \int_0^a\int_b^\infty G(x, y)dF(x, y) &\geqq \int_0^a G(x, b)d[F_1(x) - F(x, b)] \\ &= -\int_0^a [F_1(x) - F(x, b)]dG(x, b) \\ &\quad + G(a, b)[F_1(a) - F(a, b)] . \end{aligned}$$

Similar lower bounds can be computed for $\int_a^\infty\int_0^b$ and $\int_a^\infty\int_b^\infty$. Combining these results we obtain

$$\left\{\int_0^a\int_b^\infty + \int_a^\infty\int_0^b + \int_a^\infty\int_b^\infty\right\}G(x, y)dF(x, y) \geqq -B \geqq 0 . \tag{1.6}$$

If now

$$c = \int_0^\infty\int_0^\infty G(x, y)dF(x, y) < \infty$$

the left side in (1.6) is

$$c - \int_S G(x, y)dF(x, y) \geqq -B \geqq 0 ,$$

and letting $a \to \infty$, $b \to \infty$ we get $B \to 0$. Hence $A \to c$ as a and b approach ∞. If, however, $c = +\infty$, since $B \leqq 0$ it follows from (1.4) that $A \geqq \int_S G(x, y)dF(x, y)$ and letting $a, b \to \infty$ we get $A = +\infty$. The lemma is proved in any case.

COROLLARY. *For* $m \geqq 1, n \geqq 1$

$$\mu(m, n) = mn\int_0^\infty\int_0^\infty x^{m-1}y^{n-1}F^*(x, y)dydx \tag{1.7}$$

where F^* *is defined in* (1.5). *In particular,*

$$\mu = \int_0^\infty\int_0^\infty F^*(x, y)dydx . \tag{1.8}$$

We now recall that the characteristic function (*c.f.*) $f(t, t')$ of a d.f. $F(x, y)$ is called an analytic c.f. if there exists a function $A(z, z')$ of two complex variables which is defined and holomorphic in a neighborhood of the origin and which coincides with f for real values of z

and z'. The lemma below is an extension of the necessity part of Theorem 7.2.1 in [5].

LEMMA 1.4. *If $F(x, y)$ has an analytic* c.f. *then there exists a positive constant R such that*

$$F^*(x, y) = o[e^{-(rx+r'y)}],\ x, y \longrightarrow \infty$$

for all positive r, r' smaller than R.

Proof. If f is holomorphic in $\{(z, z'): |z| < p, |z'| < p'\}$ for some $p > 0, p' > 0$, then it is holomorphic at least in the "band" $\{(z, z'): |\mathrm{Im}\, z| < p, |\mathrm{Im}\, z'| < p'\}$ (cf. [2], [8]). Put $R = \min(p, p')$ and $\mathrm{Im}\, z = t$, $\mathrm{Im}\, z' = t'$. Let $x > 0$, $y > 0$. Then

$$\int_x^\infty \int_y^\infty \exp(tu + t'v) dF(u, v)$$

exists finitely for $\max(|t|, |t'|) < R$. Pick positive numbers r, r' such that $r < R$, $r' < R$ and then s, s' such that $r < s < R$ and $r' < s' < R$. Then there exists a positive constant C such that

$$\begin{aligned} C &\geqq \int_x^\infty \int_y^\infty \exp(su + s'v) dF(u, v) \\ &\geqq \exp(sx + s'y) F^*(x, y)\ . \end{aligned}$$

Thus for $0 < r < R, 0 < r' < R$

$$\begin{aligned} 0 &\leqq F^*(x, y) \exp(rx + r'y) \\ &= F^*(x, y) \exp(sx + s'y) \exp[-(s - r)x - (s' - r')y] \\ &\leqq C \exp[-(s - r)x - (s' - r')y] \longrightarrow 0 \text{ as } x, y \longrightarrow \infty\ . \end{aligned}$$

2. **The bivariate transform.** We now define the bivariate transform and its iterates. Let $F(x, y)$ have finite moments $\mu(i, j)$ of all orders ($i \geqq 0$, $j \geqq 0$). Define the sequence $\{G_n\}$ of absolutely continuous d.f.'s as follows. Put

$$G_1(x, y) = \mu^{-1} \int_0^x \int_0^y F^*(u, v) dv du$$

for $x > 0$, $y > 0$ and zero elsewhere. For $n \geqq 1$ let

$$G_{n+1}(x, y) = [\alpha(n, 1)]^{-1} \int_0^x \int_0^y G_n^*(u, v) dv du$$

for $x > 0$, $y > 0$ and zero elsewhere. Here

$$\alpha(n, 1) = \int_0^\infty \int_0^\infty G_n^*(u, v) dv du$$

and $G_n^*(u, v) = 1 - G_n^{(1)}(u) - G_n^{(2)}(v) + G_n(u, v)$ where

$$G_n^{(1)}(u) = G_n(u, +\infty) \text{ and } G_n^{(2)}(v) = G_n(+\infty, v) .$$

In view of (1.8) $G_n(x, y)$ is indeed an absolutely continuous d.f. for $n \geqq 1$. Furthermore, if X and Y are independent so that $F(x, y) = F_1(x)F_2(y)$ we see that the bivariate transform of F is the product of the univariate transforms introduced in [10] of the marginal d.f.'s F_1 and F_2. In the general case, however, no such simple relationship exists. This is important to the understanding of why a separate treatment of the two dimensional case is necessary and also helps explain the difficulty in strengthening part (v) of Theorem 4.1.

In this section we obtain the relation between the moments of F and of G_n for $n \geqq 1$.

THEOREM 2.1. *If the moment generating function* (m.g.f.) $M(t_1, t_2)$ *of* $F(x, y)$ *exists in a neighborhood* N *of the origin then the* m.g.f. $M^*(t_1, t_2)$ *of* $G_1(x, y)$ *exists in* N *and*

$$\begin{aligned} M^*(t_1, t_2) &= (\mu t_1 t_2)^{-1}[M(t_1, t_2) - M_1(t_1) - M_2(t_2) + 1],\ t_1 t_2 \neq 0 \\ M^*(t_1, 0) &= (\mu t_1)^{-1}[\partial M/\partial t_2|_{(t_1,0)} - \partial M/\partial t_2|_{(0,0)}],\ t_1 \neq 0 \\ M^*(0, t_2) &= (\mu t_2)^{-1}[\partial M/\partial t_1|_{(0,t_2)} - \partial M/\partial t_1|_{(0,0)}],\ t_2 \neq 0 \\ M^*(0, 0) &= 1 \end{aligned} \tag{2.1}$$

where the arguments of M^* *are in* N *and* M_1 *and* M_2 *are the* m.g.f.'s *of the marginal* d.f.'s F_1 *and* F_2, *respectively.*

Proof. Clearly $M^*(0, 0) = 1$. Further, the existence of the m.g.f. M in N implies the existence of $M_1(u)$ and $M_2(u)$ for $(u, 0) \in N$ and $(0, u) \in N$ respectively. Consider first the case $t_1 > 0$, $t_2 > 0$, $(t_1, t_2) \in N$. The first assertion in the theorem follows at once from Lemma 1.3 by using $G(x, y) = (e^{t_1 x} - 1)(e^{t_2 y} - 1)$ and noting that

$$M^*(t_1, t_2) = \int_0^\infty \int_0^\infty \exp(t_1 x + t_2 y) F^*(x, y) dy dx .$$

The result follows similarly when $t_1 t_2 \neq 0$, t_1 and/or t_2 negative. We now turn to the second equation in (2.1) and merely sketch the proof. Since the m.g.f. M defines a holomorphic function in a "band" containing N, the integral

$$\int_0^\infty \int_0^\infty \exp(t_1 x + t_2 y) dF(x, y)$$

converges uniformly in compact subsets of N. Hence, for $(t_1, t_2) \in N$,

$$
\begin{aligned}
(\partial/\partial t_2)\int_0^\infty\int_0^\infty &\exp(t_1x+t_2y)dF(x,y)\\
&=\int_0^\infty\int_0^\infty(\partial/\partial t_2)\exp(t_1x+t_2y)dF(x,y)\\
&=\int_0^\infty\int_0^\infty y\exp(t_1x+t_2y)dF(x,y)\ .
\end{aligned}
$$

Thus, the quantity in square brackets on the right side of the second equation in (2.1) reduces to

$$\int_0^\infty\int_0^\infty y(e^{t_1x}-1)dF(x,y)\ .$$

Use of Lemma 1.3 again gives us the result. The third equation in (2.1) is proved in the same way. The theorem is completely proved.

We shall write $\mu(i, j; n)$ to denote $E_{G_n}(X^iY^j)$, $i \geqq 0$, $j \geqq 0$, $n \geqq 1$. The following results are easily proved. If F has a m.g.f., these results are obtained as corollaries to Theorem 2.1.

THEOREM 2.2. *Let* $m \geqq 1$, $n \geqq 1$. *If* $\mu(i, j)$ *exists finitely for* $0 \leqq i \leqq m$, $0 \leqq j \leqq n$ *then* $\mu(i, j; 1)$ *exists finitely for* $0 \leqq i \leqq m-1$, $0 \leqq j \leqq n-1$. *In this case*

$$\mu(i,j;1)=\mu(i+1,j+1)/(i+1)(j+1)\mu\ . \tag{2.2}$$

THEOREM 2.3. *If* $\mu(i, j)$ *exists finitely for all nonnegative integers* i *and* j, *then for all such* i *and* j *and* $n \geqq 1$,

$$\mu(i,j;n)=\binom{n+i}{i}^{-1}\binom{n+j}{j}^{-1}\mu(n+i,n+j)/\mu(n,n)\ . \tag{2.3}$$

3. A convergence theorem for d.f.'s on a finite rectangle. In this section we prove the following theorem:

THEOREM 3.1. *If* $F(x, y)$ *is a finite distribution on the rectangle* $[0, a]\times[0, b]$, *i.e.,* $F(a, b) = 1$, *but* $F(x, y) < 1$ *for* $x < a$ *or* $y < b$. *Then*

$$\lim_{n\to\infty} G_n(x/n, y/n) = G(x,y) = \begin{cases}[1-\exp(-x/a)][1-\exp(-y/b)], & \min(x,y)\geqq 0\\ 0 & \text{elsewhere.}\end{cases}$$

To prove the theorem we need several inequalities concerning the growth rates of moments which we now obtain. For every nonnegative real number m, n, p, q and real number t, we have

$$\mu(2m, 2n) + 2t\mu(m + p, n + q) + t^2\mu(2p, 2q) = E(X^m Y^n + tX^p Y^q)^2 \geqq 0$$

so that, if the moments are positive and finite we get

$$\mu(2m, 2n)\mu(2p, 2q) \geqq \mu^2(m + p, n + q) . \tag{3.1}$$

Let r, s be positive integers. Letting $2m = r + 1, 2n = s + 1, 2p = r - 1, 2q = s - 1$ in (3.1) and then $s + 1 = r$ we obtain

$$\mu(r + 1, s + 1)/\mu(r, s) \geqq \mu(r, s)/\mu(r - 1, s - 1) \tag{3.2}$$

$$\mu(r + 1, r)/\mu(r, r - 1) \geqq \mu(r, r - 1)/\mu(r - 1, r - 2) . \tag{3.3}$$

Similarly,

$$\mu(s, s + 1)/\mu(s - 1, s) \geqq \mu(s - 1, s)/\mu(s - 2, s - 1) . \tag{3.4}$$

Setting $2m = 2p = r, 2n = 2q = s$ in (3.1) we get

$$\mu(r + 1, s)/\mu(r, s) \geqq \mu(r, s)/\mu(r - 1, s) \tag{3.5}$$

and its dual

$$\mu(r, s + 1)/\mu(r, s) \geqq \mu(r, s)/\mu(r, s - 1) . \tag{3.6}$$

Lemma 3.1 through 3.4 are proved under the hypothesis of Theorem 3.1.

LEMMA 3.1.

$$\lim_{n\to\infty} \mu^{1/n}(n, n) = ab . \tag{3.7}$$

$$\lim_{n\to\infty} \mu^{1/n}(n + i, n + j) = ab, i \geqq 0, j \geqq 0 . \tag{3.8}$$

Proof. Similar to Boas [1].

COROLLARY.

$$\lim_{n\to\infty} \mu(n + 1, n + 1)/\mu(n, n) = ab . \tag{3.9}$$

LEMMA 3.2.

$$\lim_{n\to\infty} \mu(n + i, n)/\mu(n + i - 1, n) = a, i \geqq 1 \tag{3.10}$$

$$\lim_{n\to\infty} \mu(n + i, n)/\mu(n, n) = a^i, i \geqq 0 . \tag{3.11}$$

Proof. It suffices to prove (3.10) since (3.11) follows from it. Let $i = 1$. Clearly $\limsup_{n\to\infty} \mu(n + 1, n)/\mu(n, n) \leqq a$. Since

$$\mu(n, n + 1)/\mu(n, n) \leqq b ,$$

we have from (3.5), for $n \geqq 2$,

$$\mu(n+1, n)/\mu(n, n) \geqq b^{-1}\mu(n, n)/\mu(n-1, n-1)$$

which implies that the lim inf of the left side is at least $b^{-1}ab = a$. For a general i we use (3.5) and induction on i to get

$$\begin{aligned} a &\geqq \mu(n+i+1, n)/\mu(n+i, n) \\ &\geqq \mu(n+i, n)/\mu(n+i-1, n) \to a, \text{ as } n \to \infty . \end{aligned}$$

Similarly we have the dual results

$$\lim_{n\to\infty} \mu(n, n+j)/\mu(n, n+j-1) = b, j \geqq 1 . \tag{3.12}$$

$$\lim_{n\to\infty} \mu(n, n+j)/\mu(n, n) = b^j, j \geqq 0 . \tag{3.13}$$

LEMMA 3.3.

$$\lim_{n\to\infty} \mu(n-i, n)/\mu(n-i-k, n) = a^k, i \geqq 0, k \geqq 0 . \tag{3.14}$$

Proof. It suffices to consider $k = 1$, $i \geqq 1$.

$$\begin{aligned} &\mu(n-i, n)/\mu(n-i-1, n) \\ &= \frac{\mu(n-i, n)}{\mu(n-i, n-i)} \frac{\mu(n-i-1, n-i-1)}{\mu(n-i-1, n)} \frac{\mu(n-i, n-i)}{\mu(n-i-1, n-i-1)} \\ &\sim \frac{\mu(n, n+i)}{\mu(n, n)} \frac{\mu(n, n)}{\mu(n, n+i+1)} \frac{\mu(n+1, n+1)}{\mu(n, n)} \\ &\to b^i(b^{-1})^{i+1}ab = a, \text{ as } n \to \infty \end{aligned}$$

in view of (3.10)-(3.13).

In a similar fashion

$$\lim_{n\to\infty} \mu(n, n-i)/\mu(n, n-i-k) = b^k, i \geqq 0, k \geqq 0 . \tag{3.15}$$

LEMMA 3.4.

$$\lim_{n\to\infty} \mu(n+i, n+j)/\mu(n, n) = a^i b^j, i \geqq 0, j \geqq 0 . \tag{3.16}$$

Proof.

$$\begin{aligned} &\mu(n+i, n+j)\mu(n, n) \\ &\quad = [\mu(n+i, n+j)/\mu(n, n+j)][\mu(n, n+j)/\mu(n, n)] \\ &\quad\quad \to a^i b^j, \text{ as } n \to \infty \end{aligned}$$

by (3.14) and (3.15).

We are now ready to prove Theorem 3.1. The moment $E(X^iY^j)$, $i, j \geqq 0$, of $G_n(x/n, y/n)$ is

$$n^in^j\binom{n+i}{i}^{-1}\binom{n+j}{j}^{-1}\mu(n+i, n+j)/\mu(n, n) \quad \text{(Theorem 2.3)}$$

which converges to $a^ii!\, b^jj!$ (Lemma 3.4). This last quantity is the moment of order (i, j) of $G(x, y)$ given in the statement of the theorem. The result now follows by the bivariate moment convergence theorem. We observe that the limit distribution is the product of two univariate distributions; i.e., the limiting random variables are independent.

Examples 5.1 and 5.2 illustrate this theorem.

4. *D.F.*'s on an infinite range. In this section let F be distributed on the whole positive quadrant of the plane; i.e., $F(x, y) < 1$ for all real x and y.

Let $\{c_n\}$, $\{d_n\}$ be sequences of positive real numbers and use the following abbreviations. (Superscripts indicate the appropriate marginal d.f.'s) $H_n(x, y) = G_n(c_nx, d_ny)$,

$$H_n^*(x, y) = 1 - H_n^{(1)}(x) - H_n^{(2)}(y) + H_n(x, y),$$
$$G^*(x, y) = 1 - G^{(1)}(x) - G^{(2)}(y) + G(x, y)$$

$b_n = \int_0^\infty\int_0^\infty H_n^*(x, y)dydx$, and $b = \int_0^\infty\int_0^\infty G^*(x, y)dydx$. We note that $b_n = E_{H_n}(XY)$ and $b = E_G(XY)$. We further recall that a d.f. is proper if there is no straight line in the xy-plane which contains the whole mass of the distribution. The main result of this section is the following theorem.

THEOREM 4.1. *Let positive real numbers c_n and d_n exist such that $\lim_{n\to\infty} H_n(x, y) = G(x, y)$ and $\lim_{n\to\infty} H_n^*(x, y) = G^*(x, y)$ where $G(x, y)$ is a proper* d.f. *Let* $\limsup_{n\to\infty} c_n/c_{n-1} = l_1 < \infty$ *and* $\limsup_{n\to\infty} d_n/d_{n-1} = l_2 < \infty$. *Then*

(i) *$\{b_n\}$ is a bounded sequence.*

(ii) $\lim_{n\to\infty} b_n = b < \infty$.

(iii) *$\lim_{n\to\infty} c_n/c_{n-1} = l_1$ and $\lim_{n\to\infty} d_n/d_{n-1} = l_2$ exist.*

(iv) *$l_1l_2 \geqq 1$ and equality holds if F has an analytic* c.f.

(v) *For $i \geqq 0$, $\mu_{i,i}(H_n) \to \mu_{i,i}(G)$ as $n \to \infty$ where*

$$\mu_{i,j}(\varphi) = E_\varphi(X^iY^j)\,.$$

(vi) *If $a_n \to a$, $a'_n \to a'$ as $n \to \infty$ where a_n, a'_n, a, a', are all positive, then $\lim_{n\to\infty} H_n(a_nx, a'_ny) = G(ax, a'y)$.*

(vii) *$\lim_{n\to\infty} H_n^{(1)}(x) = G^{(1)}(x)$ and $\lim_{n\to\infty} H_n^{(2)}(y) = G^{(2)}(y)$ uniformly in x and y.*

(viii) *$G(x, y)$ is continuous and the convergence $H_n(x, y) \to G(x, y)$ is uniform in x and y.*

Proof. The first five parts of the theorem follow as in Theorem 4.1 in [10]. As for the remainder, we first prove that $G(x, y)$ is continuous. This involves several steps.

Step 1.

$$G(x, y) = b^{-1}\int_0^{l_1 x}\int_0^{l_2 y} G^*(u, v)dvdu,\ x > 0,\ y > 0\ . \tag{4.1}$$

This is easily proved.

Step 2. $\sup_{y>0} y\int_0^\infty H_n^*(x, y)dx$ is uniformly bounded for n sufficiently large.

Proof. Since $b_n \to b < \infty$, there exists N and $M > 0$ such that $n > N$ implies $\int_0^\infty\int_{y/2}^y H_n^*(u, v)dvdu \leqq M$ for all $y > 0$. Since $H_n^*(u, v)$ is monotonic decreasing in v, we have for $n > N$ and all $y > 0$

$$M \geqq \int_0^\infty\int_{y/2}^y H_n^*(u, v)dvdu \geqq \frac{y}{2}\int_0^\infty H_n^*(u, y)du$$

which proves our result.

Step 3. $\sup\limits_{x>0} x\int_0^\infty H_n^*(x, y)dy$ is uniformly bounded for n sufficiently large.

Step 4. Let

$$g_n(x, y) = \int_a^x\int_a^y H_n^*(u, v)dvdu,\ (x, y) \in [a, \infty)\times[a, \infty),\ a > 0\ .$$

Then there exists a subsequence $\{g_{n_k}(x, y)\}$ converging uniformly to

$$g(x, y) = \int_a^x\int_a^y G^*(u, v)dvdu\ .$$

Proof. It is clear by the bounded convergence theorem that $g_n \to g$ pointwise. To obtain a subsequence converging uniformly we shall show that $\{g_n\}$ is uniformly bounded and equicontinuous and then appeal to the Arzela-Ascoli theorem [6, *p.* 242].

First $\{g_n\}$ is uniformly bounded since $|g_n(x, y)| \leqq b_n \leqq M$. Now we prove that it is equicontinuous. Let ε be given. ($\varepsilon < 1$). Choose N and $M > 0$ such that for $n > N$

$$\sup_{x>0} x\int_0^\infty H_n^*(x, y)dy < M \text{ and } \sup_{y>0} y\int_0^\infty H_n^*(x, y)dx < M\ .$$

This is possible by Steps 2 and 3. Next, pick $\delta < \min(\varepsilon, \varepsilon a/M)$, $\delta > 0$.

Let $|x - x'| < \delta$, $|y - y'| < \delta$ and for definiteness let $x' < x$, $y' < y$. (Other cases are similarly handled.) Then, for $n > N$

$$|g_n(x, y) - g_n(x', y')| \leqq A + B + C$$

where

$$C = \left|\int_{x'}^{x}\int_{y'}^{y} H_n^*(u, v)dvdu\right| \leqq (x - x')(y - y') < \delta < \varepsilon .$$

$$A = \left|\int_{a}^{x'}\int_{y'}^{y} H_n^*(u, v)dvdu\right|$$

$$\leqq \frac{(y - y')}{y'} y'\int_{a}^{x'} H_n^*(u, y')du \leqq (y - y')M/a < \delta M/a < \varepsilon$$

using Step 2. In a similar fashion

$$B = \left|\int_{x'}^{x}\int_{a}^{y'} H_n^*(u, v)dvdu\right| < \varepsilon .$$

Step 4 is proved.

We now turn to the proof of the continuity of $G(x, y)$. Clearly, G is continuous at $(c, 0)$, $c > 0$ since by Step 1 $G(x, y) \leqq Mxy$ and $G(c, 0) = \lim_{n\to\infty} H_n(c, 0) = 0$. Similarly, G is continuous at $(0, 0)$ and at $(0, d)$, $d > 0$. Hence let $c > 0$, $d > 0$ and consider continuity at (c, d). Let $\varepsilon > 0$ be given, $\varepsilon < 1$. Choose $a > 0$, $4a < \min(c, d, \varepsilon, 1/l_1, 1/l_2, l_1c, l_2d)$ and let

$$g(x, y) = b^{-1}\int_{a}^{l_1x}\int_{a}^{l_2y} G^*(u, v)dvdu .$$

Note that $g(c, d)$ is defined. Since $H_n^*(u, v)$ is continuous for each $n \geqq 1$, it follows from Step 4 that $g(x, y)$ is continuous in $[a, \infty) \times [a, \infty)$. Let $\eta > 0$ be the delta needed for the given ε and (c, d) in the definition of continuity of g. Further by Step 1,

$$G(x, y) = g(x, y) + \left\{\int_{0}^{a}\int_{a}^{l_2y} + \int_{a}^{l_1x}\int_{0}^{a} + \int_{0}^{a}\int_{0}^{a}\right\} b^{-1}G^*(u, v)dvdu,\ l_1x > a,\ l_2y > a .$$

This equation is also true for $x = c$, $y = d$. Choose $\delta < \min(\eta, a)$. Then, for $|x - c| < \delta$, $|y - d| < \delta$ we have that (x, y) belongs to the domain of g and

$$A = |g(x, y) - g(c, d)| < \varepsilon$$

$$B = \left|\int_{0}^{a}\int_{l_2y}^{l_2d} G^*(u, v)dvdu\right| < al_2|y - d| < \varepsilon$$

$$C = \left|\int_{l_1x}^{l_1c}\int_{0}^{a} G^*(u, v)dvdu\right| < al_1|x - c| < \varepsilon .$$

Hence, $|G(x, y) - G(c, d)| \leqq A + B + C < 3\varepsilon$. The proof of the continuity of $G(x, y)$ is completed. Since $H_n(x, y)$ converges to $G(x, y)$ and these are all continuous d.f.'s, the bivariate version of a familiar result [9, p. 438] asserts that the convergence $H_n \to G$ is uniform. This uniform convergence now yields parts (vi) and (vii) of the theorem immediately. Theorem 4.1 is completely proved.

REMARK 1. A consequence of (ii) is the asymptotic equivalence: $c_n d_n \sim \mu(n+1, n+1)/b(n+1)^2\mu(n, n)$ where $b = E_G(XY)$. Thus, the theorem gives the asymptotic nature of only the product of the normalizing sequences in terms of the rate of growth of the moments of F. It might be natural to seek conditions under which the normalizers will be given by

$$\begin{aligned} c_n &\sim k\mu(n+1, n)/(n+1)\mu(n, n) \\ d_n &\sim \mu(n, n+1)/bk(n+1)\mu(n, n) \end{aligned} \tag{4.2}$$

for some constant $k > 0$. If (4.2) holds, it is natural to expect $1/k$ and bk to correspond to the first moments of the marginal d.f.'s G_1 and G_2 of the limiting d.f. G. This is true and is seen as follows. By a straightforward calculation

$$\mu_{1,0}(H_n) \sim \mu(n+1, n)/c_n(n+1)\mu(n, n) \to 1/k$$

under (4.2). Letting

$$f_n(x) = 1 - H_n^{(1)}(x), f(x) = 1 - G^{(1)}(x), g_n(x) = \int_0^x f_n(u)du$$

and $g(x) = \int_0^x f(u)du$ we have $g_n \to g$ by the bounded convergence theorem. In fact, applying the Arzela-Ascoli theorem to $\{g_n\}$, it is easy to conclude that $g_{n'} \to g$ uniformly in x, where n' is a suitable subsequence of the natural numbers. It now follows by the Moore-Osgood theorem [7, p. 285] that

$$\lim_{n'\to\infty}\lim_{x\to\infty} g_{n'}(x) = \lim_{x\to\infty}\lim_{n'\to\infty} g_{n'}(x);\ \text{i.e.,}$$

$$\lim_{n'\to\infty} \mu_{1,0}(H_{n'}) = \int_0^\infty [1 - G^{(1)}(u)]du = \mu_{1,0}(G)\ .$$

Since $\mu_{1,0}(H_n) \to 1/k$ it follows that $\mu_{1,0}(G) = 1/k$. Similarly, $\mu_{0,1}(G) = bk$. Incidentally, we have proved that $\mu_{1,0}(H_n) \to \mu_{1,0}(G)$, and $\mu_{0,1}(H_n) \to \mu_{0,1}(G)$ under the condition (4.2).

REMARK 2. Part (v) of the theorem asserts the convergence of $\mu_{i,j}(H_n)$ to $\mu_{i,j}(G)$ only for $i = j$. Remark 1 above extends this to the case $i = 0, j = 1$ and $i = 1, j = 0$ under the condition (4.2). It might be interesting to investigate if the general moment convergence is a consequence of (4.2) but we shall not pursue that in this paper.

REMARK 3. Under the conditions of the theorem and (4.2) the following relations for the growth rates of the moments of F are easily obtained:

(i) $\mu(n+1, n+1) \sim \mu(n+1, n)\mu(n, n+1)/\mu(n, n)$

(ii) $\mu(n+2, n+1)\mu(n, n)/\mu(n+1, n+1)\mu(n+1, n) \sim l_1$

(iii) $\mu(n+1, n+2)\mu(n, n)/\mu(\mu+1, n+1)\mu(n, n+1) \sim l_2$.

We observe that (4.2) is valid if, for example, X and Y are independent and the c_n and d_n are normalizers satisfying the conditions of Theorem 4.1 in [10] corresponding to the d.f.'s of X and Y respectively. Theorems 4.2 and 4.3, below, illustrate situations where X and Y are dependent and (4.2) holds.

THEOREM 4.2. *Let U and V be independent positive* r.v's *having analytic* c.f.'s. *Then the* n-*th iterated transform of the joint* d.f. *of $X = UV$ and $Y = V$, suitably normalized converges to the product of simple exponential* d.f.'s.

Proof. Uuder the stated conditions all the moments λ_n and σ_n respectively of U and V are finite and the moments of the d.f. of (X, Y) and of its nth iterated transform are given by

$$\mu(i, j) = E(X^i Y^j) = E(U^i)E(V^{i+j}) = \lambda_i \sigma_{i+j}$$

$$\mu(i, j; n) \sim i!\, n^{-i-j}(\lambda_{n+i}/\lambda_n)(\sigma_{2n+i+j}/\sigma_{2n}) .$$

Choosing the c_n and d_n as in (4.2) with $b = 1$, $k = 1$ we see after some simplification that $\mu_{i,j}(H_n) \to i!\, j!$ as $n \to \infty$. (Here we have used Lemmas 4.2 and 4.3 in [10]). Such a choice of c_n and d_n is valid since c_{n+1}/c_n and d_{n+1}/d_n are bounded. Indeed they approach 1. The theorem is proved.

REMARK. If U and V have independent exponential distributions then the joint probability density function (p.d.f.) of X and Y is the one considered in Example 5.3.

We close this section with the following result illustrating a situation where the normalizers are as in (4.2) but the limit d.f. is not necessarily a d.f. of independent r.v.'s. To prove the theorem we merely need to verify that the mements of $G(x, y)$ determine it uniquely. This follows readily from the following sufficient condition for the determinateness of a moment sequence $\{m_{ij}\}$, namely, that the series $\sum_{i,j=0}^{\infty} m_{ij}x^i y^j/i!\,j!$ have a nonvanishing radius of convergence (cf. [4, p. 217]).

In the present case

$$m_{ij}(G) = \frac{ai!\,j!}{k_1^i k_2^j} + \frac{bi!\,j!}{k_3^i k_4^j}$$

and this clearly satisfies the sufficiency condition.

THEOREM 4.3. *Let X and Y be independent positive* r.v.'s *with* d.d.f.'s *$f_1(x)$ and $f_2(y)$ respectively and having analytic* c.f.'s *(so that the moments λ_n and σ_n of X and Y respectively are all finite). Let further $\lambda_{n+1}\sigma_n/\lambda_n\sigma_{n+1} \sim \alpha$ where $0 < \alpha < \infty$. Define the* p.d.f.

$$f(x, y) = af_1(x)f_2(y) + bf_1(y)f_2(x)$$

where $a + b = 1$ and a, b are positive real numbers. Then the normalizers (4.2) *lead to the limiting* d.f.

$$G(x, y) = \int_0^x\int_0^y (Ae^{-uk_1-vk_2} + Be^{-uk_3-vk_4})dvdu$$

where $k_1 = (a + b/\alpha)$, $k_2 = (a + b\alpha)$, $k_3 = (b + a\alpha)$, and *$k_4 = (b + a/\alpha)$ $A = ak_1k_2$ and $B = bk_3k_4$.*

COROLLARY. *If $\alpha \neq 1$, $G(x, y)$ is not the* d.f. *of independent* r.v.'s. *If $\alpha = 1$, $G(x, y)$ is the product of exponential distributions.*

The hypothesis of the theorem are satisfied if, for example,

$$f_1(x) = \exp(-x),\ x > 0;\ f_2(y) = \alpha\exp(-\alpha y),\ y > 0$$

where $\alpha > 0$.

5. Examples. This section contains three examples. The first two examples illustrate Theorem 3.1; the third one illustrates Theorem 4.2.

EXAMPLE 5.1. Let a, b, c be positive real numbers such that $a + b + c < 1$. Then the d.f. of a bivariate Bernoullian random vector is:

$$F(x, y) = \begin{cases} a & 0 < x,\ y \leqq 1 \\ a + b & 0 < x \leqq 1,\ y > 1 \\ a + c & 0 < y \leqq 1,\ x > 1 \\ 1 & \max(x, y) > 1 \\ 0 & \min(x, y) \leqq 0 \end{cases}$$

with $\mu = E(XY) = \alpha$ where $\alpha = 1 - a - b - c$. It is easily verified that the n-th iterated transform of $F(x, y)$ is the joint d.f. of two independently distributed random variables with a common d.f. given by $[1 - (1 - x)^n]$ for $0 < x < 1$ and one for $x > 1$. Thus $G_n(x, y)$ converges to the degenerate distribution (degenerate at the origin). But $G_n(x/n, y/n)$ converges to the product of exponential d.f.'s.

EXAMPLE 5.2. Consider the bivariate distribution with p.d.f. $f(x, y) = x + y$ for $0 < x, y < 1$ and zero elsewhere. The computation of G_n is unwieldy but

$$\mu(i, j; n) = \frac{i!\,j!}{2}(n+1)^{-1}[(n+j+1)(n+2)_{(i)}^{-1}(n+3)_{(j)}^{-1} + (n+i+1)(n+3)_{(i)}^{-1}(n+2)_{(j)}^{-1}]$$

where $(a)_{(r)} = a(a+1)\cdots(a+r-1)$ for a positive integer r and $(a)_{(0)} = 1$. It follows that the moment of order (i, j) of $G_n(x/n, y/n)$ converges to $i!\,j!$ and hence the limiting d.f. is the product of simple exponential d.f.'s.

EXAMPLE 5.3. Let $f(x, y) = y^{-1}\exp(-y - x/y)$, $\min(x, y) > 0$ and zero elsewhere be a joint p.d.f. Here Theorem 4.2 applies and the limiting d.f. is again the product of simple exponential d.f.'s if we choose $c_n \sim 2n$, $d_n \sim 2$ as given by (4.2).

Acknowledgment. The author is grateful to Professor W. L. Harkness for his help in the preparation of this paper.

The author is also thankful to the referee for his many valuable comments.

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Received April 17, 1969.

THE STATE UNIVERSITY OF NEW YORK AT STONY BROOK

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

RINGS OF ANALYTIC FUNCTIONS

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If F is an open Riemann surface and $A(F)$ is the set of all analytic functions on F, then $A(F)$ is a ring under pointwise addition and multiplication. This paper is concerned with proper subrings R of $A(F)$ which are isomorphic images of $A(G)$, the ring of all analytic functions on an open Riemann surface G, under a homomorphism Φ which maps constant functions onto themselves. The ring R has the form $\{g \circ \phi: g \in A(G), \phi$ an analytic map from F into $G\}$, and will be denoted R_ϕ. Relations between ϕ, R_ϕ and the spectrum of R_ϕ are given as necessary and sufficient conditions for the existence of a Riemann surface G such that R is isomorphic to $A(G)$.

Open Riemann surfaces will be denoted by F and G, the rings of all analytic functions on F and G with pointwise addition and multiplication will be denoted by $A(F)$ and $A(G)$, and Φ will denote a homomorphism from $A(G)$ into $A(F)$ which maps constant functions onto themselves. Let Φ be such a homomorphism. In [5, pp. 272–273] H. L. Royden shows there is an analytic mapping ϕ of F into G such that $\Phi(g) = g \circ \phi$, and that if Φ is an isomorphism onto $A(F)$ then ϕ is a one-to-one, onto analytic mapping. If ϕ is an analytic mapping of F into G, then Φ defined by $\Phi(g) = g \circ \phi$, $g \in A(G)$, is a homomorphism from $A(G)$ into $A(F)$ which preserves constant functions. When ϕ is one-to-one and onto, Φ is an isomorphism.

The image of $A(G)$ under Φ is the set $\{g \circ \phi: g \in A(G), \phi$ is an analytic map of F into $G\}$ denoted by R_ϕ. R_ϕ is a subring of $A(F)$ and contains the constant functions, since $\Phi(\lambda) = \lambda$ for λ a constant function. The following conditions are equivalent: R_ϕ properly contains the constant functions, Φ is an isomorphism, ϕ is not a constant function. Theorems 1 and 2 give other relations between ϕ and R_ϕ.

THEOREM 1. *If R_ϕ properly contains the constant functions, then R_ϕ contains $1/f$ whenever $f \in R_\phi$, $f(z) \neq 0$ on F, if and only if ϕ maps F onto G.*

Proof. Let ϕ map F onto G, $f \in R_\phi$, $f(z) \neq 0$ on F. Then $f = \Phi h$ for some $h \in A(G)$ and $1/h \in A(G)$ if $h(y) \neq 0$ for $y \in G$. Suppose $h(a) = 0$. Since $a = \phi(z)$ for some $z \in F$, $0 = h(a) = h(\phi(z)) = \Phi h(z) = f(z)$. This contradicts $f(z) \neq 0$ on F. Thus $h(a) \neq 0$ for $a \in G$, $1/h \in A(G)$, and $1/f = \Phi(1/h) \in R_\phi$.

Suppose R_ϕ contains $1/f$ when $f \in R_\phi$, $f(z) \neq 0$ on F. Let $a \in G$.

There is $g \in A(G)$ such that $g(a) = 0$ and $g(w) \neq 0$ for $w \neq a$ [1, pp. 591–592]. The function $\Phi g \in R_\phi$. If $\Phi g(z) = g \circ \phi(z) \neq 0$ for $z \in F$, then there is $h \in R_\phi$ such that $(\Phi g)(h) = 1$. There is $k \in A(G)$ such that $h = \Phi k$. Then $(\Phi g)(\Phi k) = 1$ and $\Phi(gk) = 1$ but Φ is an isomorphism implies $gk = 1$ and $g(a)k(a) = 1$. This contradicts $g(a) = 0$. Therefore $g(\phi(z)) = 0$ and $\phi(z) = a$ for some $z \in F$.

A straightforward argument shows

THEOREM 2. *If R_ϕ properly contains the constant functions, then R_ϕ separates the points of F if and only if ϕ is one-to-one.*

Let R be a ring of analytic functions defined on F. The spectrum of R, ΣR, is the set of nonzero homomorphisms π from R into the complex numbers such that $\pi(\lambda) = \lambda$ for λ a constant function. For $x \in F$ the point evaluation mapping $\pi_x = \{(f, f(x)) \colon f \in R\}$ is a homomorphism from R into the complex numbers, and $\pi_x(\lambda) = \lambda$ for λ a constant function. Therefore ΣR always contains the point evaluation mappings defined on R. In [5, p. 272] H. L. Royden shows that the spectrum of $A(F)$ is the set of point evaluation mappings π_x defined on $A(F)$, $x \in F$. For $f \in R$ let $\hat{f} = \{(\pi, \pi f) \colon \pi \in \Sigma R\}$; $\hat{f}$ is a function from ΣR into the complex numbers. Let $\hat{R}$ denote $\{\hat{f} \colon f \in R\}$. With pointwise addition and multiplication $\hat{R}$ is a ring containing the constant functions and is isomorphic to R under $f \to \hat{f}$.

For $y \in G$, let ψ_y denote an element of $\Sigma A(G)$. The mapping $P = \{(y, \psi_y) \colon y \in G\}$ is a one-to-one function from G onto $\Sigma A(G)$. If $R = \Phi(A(G))$ and Φ is an isomorphism, $L = \{(\pi, \pi \circ \Phi) \colon \pi \in \Sigma R\}$ is a one-to-one function from ΣR onto $\Sigma A(G)$. The mapping $\pi \to \pi \circ \Phi = \psi_y \to y$ which is $P^{-1} \circ L$ defines a one-to-one correspondence between ΣR and G when Φ is an isomorphism.

THEOREM 3. *Let $R_\phi = \Phi(A(G))$, Φ be an isomorphism from $A(G)$ into $A(F)$ which preserves constant functions. Let M be the function from $\Sigma A(F)$ into ΣR_ϕ defined by $M(\pi_x) = \pi_x|_{R_\phi}$. Then M is onto if and only if ϕ is onto, and M is one-to-one if and only if ϕ is one-to-one.*

Proof. The proof that M is one-to-one if and only if ϕ is one-to-one follows from Theorem 2 and the fact that $A(F)$ separates the points of F.

Let $\pi \in \Sigma R_\phi$. Then $\pi \circ \Phi \in \Sigma A(G)$ implies there is $y \in G$ such that $\pi \circ \Phi = \psi_y$, where $\psi_y(g) = g(y)$ for $g \in A(G)$. There are two cases: $y \in \phi(F)$, $y \notin \phi(F)$. If $y \in \phi(F)$, then $y = \phi(x)$ for some $x \in F$ and $\pi(\Phi g) = g(y) = g(\phi(x)) = \Phi g(x)$ for every $g \in A(G)$, $\pi(\Phi g) = \Phi g(x)$ for every $f = \Phi g \in R_\phi$. This implies $\pi = M(\pi_x)$. If $y \notin \phi(F)$, then $y \neq \phi(x)$ for $x \in F$, and it may be shown that for every $x \in F$ there is $f \in R_\phi$ such that $\pi(f) \neq f(x)$. Let $x \in F$. Then $\phi(x) \in G$. $y \in G$, $y \neq \phi(x)$, and $A(G)$ separates the points of G implies there is a $g \in A(G)$ such that $g(y) \neq g(\phi(x))$. From $\Phi(g) \in R_\phi$ and $\pi(\Phi g) = g(y) \neq g(\phi(x)) = \Phi g(x)$ it follows that $\pi \neq M(\pi_x) = \pi_x|_{R_\phi}$.

For $\pi \in \Sigma R_\phi$, $\pi \circ \Phi = \psi_y \in \Sigma A(G)$, and it has been shown $\pi \in M(\Sigma A(F))$ if and only if $y \in \phi(F)$.

From Theorem 3 and since ΣR_ϕ and G are in one-to-one correspondence, it follows that the point evaluation maps in ΣR_ϕ are in one-to-one correspondence with the points $\phi(x) \in \phi(F)$, and the elements of ΣR_ϕ which are not point evaluation maps are in one-to-one correspondence with the points in $G - \phi(F)$.

Theorem 4 contains a necessary condition which a subring R of $A(F)$ must satisfy if R is to be $\Phi(A(G))$, the isomorphic image of $A(G)$ under Φ for some open Riemann surface G. The corollary to Theorem 5 gives a set of sufficient conditions on R in order that R be $\Phi(A(G))$ when $\Phi g = g \circ \phi$ and ϕ: $F \to G$ is an onto mapping.

Suppose F is an open Riemann surface, $p \in F$, f is analytic at p and τ is a local uniformizer which maps a neighborhood of p onto $\{z\colon |z| < g\}$ for some $\rho > 0$, $\tau(p) = 0$. There is a number $r > 0$ such that $f \circ \tau^{-1}(z) = \sum_{i=0}^{\infty} a_i z^i$ for $|z| < r$. The multiplicity of f at p is defined as $\inf\{k\colon k \neq 0$ and $a_k \neq 0\}$, denoted $n(p; f)$. The multiplicity $n(p; f)$ of f at p does not depend on τ. If R contains functions other than constants, $m = \inf\{n(p; f)\colon \mathrm{f} \in R\}$ is defined, and $n(p; f) = m$ for some $f \in R$.

THEOREM 4. *Let $p \in F$, R_ϕ contain functions other than constants and let $m = \{\inf n(p; f)\colon f \in R_\phi\}$. There is a local uniformizer τ at p with the properties: $\tau(0) = p$, for some $\rho > 0$, τ maps $\{z\colon |z| < \rho\}$ onto a neighborhood of p, and if $f \in R_\phi$, $f \circ \tau(z) = \sum_{i=0}^{\infty} a_i (z^m)^i$ for $|z| < \rho$.*

The proof of Theorem 4 is based on two lemmas:

LEMMA 1. *If $p \in F$, $m = \inf\{n(p; f)\colon f \in R_\phi\}$ and $f \in R_\phi$, then $n(p;f) = km$, where k is a positive integer.*

LEMMA 2. *Given $\sum_{i=m}^{\infty} c_i z^i$ convergent for $|z| < \rho$, $c_m \neq 0$, $m \neq 0$, there is $\sum_{i=1}^{\infty} b_i z^i$ convergent for $|z| < \rho$, $b_1 \neq 0$, such that $(\sum_{i=1}^{\infty} b_i z^i)^m = \sum_{i=m}^{\infty} c_i z^i$.*

Lemma 1 follows from the two relations: For $f \in R_\phi$, $f = g \circ \phi$ for some $g \in A(G)$, which implies $n(p; f) = (n(p; \phi))(n(\phi(p); g))$, and if $m = \inf\{n(p; f): f \in R_\phi\}$ then $n(p; \phi) = m$. Lemma 2 is proved by defining W a subset of the natural numbers N as $W = \{n \in N: b_1, b_2, \cdots, b_n$ can be defined in such a way that the coefficients of z^i for $1 \leqq m \leqq i \leqq m + n - 1$ of $(\sum_{i=1}^\infty b_i z^i)^m$ and $\sum_{i=m}^\infty c_i z^i$ are equal$\}$ and using induction to show $W = N$.

Proof of Theorem 4. Let τ_p be a local uniformizer about p such that $\tau_p(0) = p$. If $m = \inf\{n(p; f): f \in R_\phi\}$, there is $f_p \in R_\phi$ and $\rho > 0$ such that $f_p \circ \tau_p(z) = \sum_{i=m}^\infty c_i z^i$ for $|z| < \rho$, $c_m \neq 0$, and the range of $\sum_{i=m}^\infty c_i z^i$ contains $|z| < \rho^m$.

There is a power series $\sum_{i=1}^\infty b_i z^i$, $b_1^m = c_m$, such that $\sum_{i=m}^\infty c_i z^i = (\sum_{i=1}^\infty b_i z^i)^m$ for $|z| < \rho$ as stated in Lemma 2. $k(z) = \sum_{i=1}^\infty b_i z^i$ is defined for $|z| < \rho$, is one-to-one, and its range contains $|z| < \rho$. Thus $k^{-1}(y)$ is defined for $|y| < \rho$ and $f_p \circ \tau_p \circ k^{-1}(z) = (\sum_{i=1}^\infty b_i(k^{-1}(z))^i)^m = z^m$ for $|z| < \rho$, $\tau_p \circ k^{-1}(0) = p$. The function $\tau = \tau_r \circ k^{-1}$ is a local uniformizer about p and there is $f_p \in R_\phi$ such that $f_p \circ \tau(z) = z^m$ for $|z| < \rho$.

Let $f \in R_\phi$, f not a constant function. Then $f \circ \tau(z) = \sum_{i=0}^\infty a_i z^i$ for $|z| < \rho$. Let N denote the natural numbers and define $W = \{n \in N: f \circ \tau(z) = \sum_{i=0}^n a_{mj_i} z^{mj_i} + z^{mj_n} h_n(z)$, where $h_n(z) = \sum_{i=1}^\infty b_{n,i} z^i$ and j_i are nonnegative integers, $0 = j_0 < j_1 < \cdots < j_n\}$.

It follows from Lemma 1 that for $|z| < \rho, f \circ \tau(z) = \sum_{i=0}^\infty a_i z^i = a_0 + a_{mj_1} z^{mj_1} + z^{mj_1} h_1(z)$, where $h_1(0) = 0$. If $k \in W$, then $f \circ \tau(z) = \sum_{i=0}^k a_{mj_i} z^{mj_i} + z^{mj_k} h_k(z)$, $h_k(0) = 0$. Since $f \in R_\phi$, $z^m \in R_\phi$ and constants are contained in R_ϕ, $z^{mj_k} h_k(z) = f(z) - \sum_{i=0}^k a_{mj_i} z^{mj_i} \in R_\phi$. If $h_k \neq 0$, $n(p; z^{mj_k} h_k) = mj_{k+1}$ and $f \circ \tau(z) = \sum_{i=0}^{k+1} a_{mj_i} z^{mj_i} + z^{mj_{k+1}} h_{k+1}(z)$, where $h_{k+1}(z) = \sum_{i=1}^\infty b_{k+1,i} z^i$ on $|z| < \rho$ and $j_{k+1} > j_k$. If $h_k = 0$, then the above statement is true with $a_{mj_{k+1}} = 0$, $h_{k+1} = 0$. By induction $W = N$ and $f \circ \tau(z) = \sum_{i=0}^\infty a_{mi} z^{mi}$ on $|z| < \rho$.

If R, a subring of $A(F)$, has the property that for every $a \in F$, $f \in R$, for some local uniformizer τ about a, $f \circ \tau(z) = \sum_{i=0}^\infty a_i (z^{m(a)})^i$ for $m(a) = \inf\{n(a; f): f \in R\}$, then R has property (ξ). If R contains functions other than constants and has property (ξ), then for $a \in F$, $m(a) = \inf\{n(a; f): f \in R\} = 1$ if R separates the points of F.

THEOREM 5. *If R is a subring of $A(F)$ which contains functions other than constants and has property (ξ), then there is an open Riemann surface G, an analytic mapping ϕ of F onto G, and a separating subring S of $A(G)$ such that S is isomorphic to R under $\hat{f} \to \hat{f} \circ \phi$, $\hat{f} \in S$.*[1]

Proof. Let $G = \{\pi_p: p \in F\}$ where $\pi_p = \{(f, f(p)): f \in R\}$ and $\phi =$

$\{(p, \pi_p): p \in F\}$. The topology on G will be that which makes ϕ continuous and open. If N_p is an open neighborhood of $p \in F$, then $N_{\pi_p} = \{\pi_q: q \in N_p\}$ is an open neighborhood of π_p. The set G with this topology is a connected Hausdorff space.

Let $p \in F$, $\pi_p \in G$ and $m = \inf\{n(p; f): f \in R\}$. By the same argument used in the beginning of the proof of Theorem 4, there is a function $f_p \in R$ and a local uniformizer τ about p such that $\tau(0) = p$ and $f_p \circ \tau(z) = z^m$ for $|z| < \rho^{1/m}$ for some $\rho > 0$. Then for $f \in R$, $f \circ \tau(z) = \sum_{i=0}^{\infty} a_i(z^m)^i = g_f(z^m)$ for $|z| < \rho^{1/m}$, g_f analytic on $|z| < \rho$.

It will be shown that $\sigma_\tau = \{(z^m, \pi_{\tau(z)}): |z| < \rho^{1/m}\}$ is a local uniformizer about π_p. If $z_1^m = z_2^m$, then $f \circ \tau(z_1) = g_f(z_1^m) = g_f(z_2^m) = f \circ \tau(z_2)$, for $f \in R$ implies $\pi_{\tau(z_1)} = \pi_{\tau(z_2)}$, which implies that σ_τ is a function. If $\pi_{\tau(z_1)} = \pi_{\tau(z_2)}$ then in particular $f_p \circ \tau(z_1) = f_p \circ \tau(z_2)$, which implies $z_1^m = z_2^m$, and σ_τ is one-to-one. Since the relations $z^m \to z \to \tau(z) \to \phi(\tau(z)) = \pi_{\tau(z)}$ are open and continuous, σ_τ is open and continuous. Thus σ_τ is a homeomorphism from $\{w: |w| < \rho\}$ onto $\phi \circ \tau(\{z: |z| < \rho^{1/m}\}) = N_{\pi_p}$.

If $\pi \in W = \sigma_{\tau_2}(|z| < \rho_2) \cap \sigma_{\tau_1}(|z| < \rho_1)$, there are points z_1, z_2 such that $\pi_{\tau_1}(z_1) = \pi_{\tau_2}(z_2)$. Then $f \circ \tau_1(z_1) = f \circ \tau_2(z_2)$ for every $f \in R$, and $z_1^{m_1} = f_1(\tau_1(z_1)) = f_1(\tau_2(z_2)) = g_{f_1}(z_2^{m_2})$, so g_{f_1} is analytic on $\{w: |w| < \rho_2\}$, which contains $\sigma_{\tau_2}^{-1}(W)$. This shows that $z_1^{m_1} = \sigma_{\tau_1}^{-1} \circ \sigma_{\tau_2}(z_2^{m_2})$ is analytic on $\sigma_{\tau_2}^{-1}(W)$ to $\sigma_{\tau_1}^{-1}(W)$. The function σ_τ is a local uniformizer of a neighborhood of π_p, and G is a Riemann surface.

For $f \in R$, let $\hat{f} = \{(\pi_p, f(p)): p \in F\}$, $S = \{\hat{f}: f \in R\}$. Since f is continuous and ϕ is open, $\hat{f}$ is continuous. The function $\hat{f}$ is analytic at π_p, because if $|w| < \rho$, $w = z^m$, then $\hat{f} \circ \sigma_\tau(w) = \hat{f}(\pi_{\tau(z)}) = f(\tau(z)) = \sum_{i=0}^{\infty} a_i(z^m)^i = \sum_{i=0}^{\infty} a_i w^i$. The mapping ϕ is analytic at p, because $\sigma_\tau^{-1} \circ \phi \circ \tau(z) = \sigma_\tau^{-1}(\pi_{\tau(z)}) = z^m$ for $|z| < \rho^{1/m}$. With pointwise addition and multiplication, S is a ring and is isomorphic to R under the mapping $\hat{f} \to \hat{f} \circ \phi = f$. The ring S separates the points of G. Since S contains functions which are not constant and are analytic on G, G is an open Riemann surface.

If S is to be $A(G)$, then by Theorem 3 the mapping $M(\pi_p) = \pi_p|_R$ from $\Sigma A(F)$ to ΣR must be onto, since ϕ is an onto mapping of F to G. Thus ΣR may contain only point evaluation mappings and $\Sigma R = G$.

COROLLARY TO THEOREM 5. *If R is a subring of $A(F)$ which properly contains the constant functions and has property* (ξ), *if ΣR contains only point evaluation mappings, and R contains all $f \in A(F)$ such that $f \circ \tau_p(z) = \sum_{i=0}^{\infty} a_i(z^m)^i$ for $|z| < \rho^{1/m}$, $p \in F$, $m = \inf\{n(p: f): f \in R\}$, then $\Sigma R = G$ is an open Riemann surface, and R is isomorphic to $S = A(G)$.*

[1] This result and proof are similar to one given by M. Heins for a subfield of the field of all meromorphic functions on a Riemann surface [**2**, pp. 268-269].

Proof. Everything except $S = A(G)$ was shown in the proof of Theorem 5. The function $\hat{f} \in A(G)$ if and only if for every $\pi_p \in G$, $\hat{f} \circ \sigma_{\tau_p}(w) = \sum_{i=0}^{\infty} a_i w^i$ for $|w| < \rho$. Let $\hat{f} \in A(G)$, $p \in F$, $\pi_p \in G$, and $f = \hat{f} \circ \phi$. Then $f \in A(F)$ and $f \in R$, because for $|z| < \rho^{1/m}$, $f \circ \tau_p(z) = \hat{f} \circ \phi(\tau_p(z)) = \hat{f}(\pi_{\tau_p(z)}) = \hat{f} \circ \sigma_{\tau_p}(z^m) = \sum_{i=0}^{\infty} a_i (z^m)^i$.

If $R = \{\hat{f} \circ \phi : \hat{f} \in S\}$ and S separates the points of G, then R separates the points of F if and only if ϕ is a one-to-one function. If S separates the points of G, and $S = A(G)$, then R may not separate the points of F, because if it did ϕ would be a one-to-one, onto analytic function from F to G, and $R = A(F)$. If $S \neq A(G)$ there may be a surface H, a mapping ϕ_1 and a separating subring T of $A(H)$ such that ϕ_1 is analytic and one-to-one but not onto, and $T = A(H)$.

In this part of the paper it is noted that if $R = \Phi(A(G))$, then ΣR with the Gelfand topology is an open Riemann surface, and $\hat{R}$ which is isomorphic to R, is the ring of all analytic functions on ΣR. Theorem 8 gives sufficient conditions on a subring R of $A(F)$ and on $\hat{R}$ in order that ΣR be an open Riemann surface and $\hat{R}$ be a ring of analytic functions on ΣR. In conclusion sufficient conditions for $\hat{R}$ to be $A(\Sigma R)$ are given.

If R is a ring of complex valued functions on F, then the Gelfand topology on ΣR is the weakest topology on ΣR which makes each element of $\hat{R}$ continuous, where $\hat{R} = \{\hat{f} : f \in R\}$, $\hat{f} = \{(\pi, \pi f) : \pi \in \Sigma R\}$. Let $\pi_0 \in \Sigma R$, K be a finite subset of $\hat{R}$, $\varepsilon > 0$. An open neighborhood of π_0 will be $\{\pi \in \Sigma R : |\hat{f}(\pi) - \hat{f}(\pi_0)| < \varepsilon$ for $\hat{f} \in K\}$. If $R = \Phi(A(G))$ and Φ is an isomorphism, then ΣR and $\Sigma A(G)$ with the Gelfand topology are homeomorphic under the mapping $L(\pi) = \pi \circ \Phi$ from ΣR onto $\Sigma A(G)$. The mapping $P(y) = \psi_y$ from G onto $\Sigma A(G)$ with the Gelfand topology is one-to-one, onto and continuous. The mapping P is also open. As Royden observes [4, pp. 287–288], this is a consequence of a theorem of Remmert that an open Riemann surface can be mapped one-to-one and holomorphically into C^3 [3, p. 118]. Thus $P^{-1} \circ L$ is a homeomorphism from ΣR with the Gelfand topology onto G.

THEOREM 6. *If R is a subring of $A(F)$ such that $R = \Phi(A(G))$, and if Φ is an isomorphism which preserves constant functions, then ΣR with the Gelfand topology is an open Riemann surface, and $\hat{R}$ is the ring of all analytic functions on ΣR. Moreover $\hat{R}$ is isomorphic to R.*

Proof. The spectrum of R with the Gelfand topology is a Hausdorff space. It is homeomorphic to G under the mapping $L^{-1} \circ P$, and is connected. Let $\pi_q \in \Sigma R$ where $q \in G$, $\psi_q \in \Sigma A(G)$, and $L^{-1} \circ P$ maps $q \rightarrow \psi_q \rightarrow \pi_q$. If N_q is a neighborhood of q then $N_{\pi_q} = L^{-1} \circ P(N_q)$ is a neighborhood of π_q. There exists $h_q \in A(G)$ which has a simple zero at q [1, pp. 591–592]. h_q is a local uniformizer on a neighborhood of q, $N_q = h_q^{-1}(|z| < \rho)$ for some $\rho > 0$. If $\sigma_q = h_q|_{N_q}$, then $h_q \circ \sigma_q^{-1}(z) = z$ for $|z| < \rho$. For $h \in A(G)$, $y \in N_q$, $h(y) = \sum_{i=0}^{\infty} a_i(h_q(y))^i$.

If $f_q = \Phi h_q$ then $\hat{f}_q$ is a local uniformizer on $N_{\pi_q} = L^{-1} \circ P(N_q)$. From $\hat{f}_q(\pi_y) = h_q(y)$ follows $\hat{f}_q(\pi_y) = h_q \circ P^{-1} \circ L(\pi_y)$, $\pi_y \in N_{\pi_q}$, which implies $\hat{f}_q$ is a homeomorphism of N_{π_q} onto $|z| < \rho$. If $\pi_y \in N_{\pi_{q_1}} \cap N_{\pi_{q_2}}$, then $\hat{f}_{q_1}(\pi_y) = h_{q_1}(y) = \sum_{i=0}^{\infty} a_i(h_{q_2}(y))^i = \sum_{i=0}^{\infty} a_i(\hat{f}_{q_2}(\pi_y))^i$ since $\pi_y \in N_{\pi_{q_2}}$ or $y \in N_{q_2}$. The function $\hat{f}_q$ is a local uniformizer on N_{π_q} and ΣR is a Riemann surface.

The ring $\hat{R}$ is contained in $A(\Sigma R)$, because if $\hat{f} \in \hat{R}$, $\pi_y \in N_{\pi_q}$, $z = \hat{f}_q(\pi_y)$, then $\hat{f} \circ \hat{f}_q^{-1}(z) = \hat{f}(\pi_y) = h(y) = \sum_{i=0}^{\infty} a_i(h_q(y))^i = \sum_{i=0}^{\infty} a_i(\hat{f}_q(\pi_y))^i = \sum_{i=0}^{\infty} a_i z^i$. The function $T(q) = \pi_q$ is an analytic map of G onto ΣR. If θ is analytic on ΣR, then $\theta \circ T \in A(G)$ and $\theta \in \hat{R}$ because $\theta(\pi_q) = \theta \circ T(q) = \psi_q(\theta \circ T) = \pi_q(f)$ for $f = \Phi(\theta \circ T)$. This implies $\theta = \hat{f}$. Thus $\hat{R} = A(\Sigma R)$. Since $\hat{R}$ contains functions which are analytic and are not constant on ΣR, ΣR is an open Riemann surface.

THEOREM 7. *Let $R = \Phi(A(G))$. If $\hat{\pi} \in \Sigma R$, then $\hat{\pi}^{-1}(0)$ is a principal maximal ideal of R, and every principal maximal ideal of R is the kernel of $\pi \in \Sigma R$. If $\hat{\pi}^{-1}(0)$ is generated by f, then $\hat{f}$ is a local homeomorphism on a neighborhood $N_{\hat{\pi}}$ of $\hat{\pi}$ and if $\pi \in N_{\hat{\pi}}$, $\hat{k} \in \hat{R}$, then $\hat{k}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$.*

Proof. If $\hat{\pi} \in \Sigma R$, then $\hat{\pi} \circ \Phi = \psi_q \in \Sigma A(G)$ and $\hat{\pi}^{-1}(0) = \Phi(\psi_q^{-1}(0))$. The kernel of ψ_q, $M_q = \psi_q^{-1}(0)$, is a principal maximal ideal of $A(G)$, and every principal maximal ideal of $A(G)$ is a kernel of $\psi \in \Sigma A(G)$ [5, pp. 271–272]. If h generates M_q, then h has a single zero and it is a simple zero at q [5]. Thus h is a homeomorphism on a neighborhood of q, N_q. If $f = \Phi h$, then $\hat{\pi}^{-1}(0)$ is the ideal generated by f. Also $\hat{f}$ is a uniformizer on $N_{\hat{\pi}} = L^{-1} \circ P(N_q)$, and if $\pi \in N_{\hat{\pi}}$, $\hat{k} \in \hat{R}$, then $\hat{k}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$ as shown in the proof of Theorem 6.

LEMMA. *Let S be a ring of continuous functions on X with identity. Then X is not connected if and only if S is contained in a ring Q of continuous functions on X, where $Q = I_1 + I_2$, I_1, I_2 proper ideals of Q, $I_1 \cap I_2 = \{0\}$.*

THEOREM 8. *Let R be a subring of $A(F)$ which properly contains the constant functions, and suppose $\hat{R}$ is not contained in a ring Q of continuous functions on ΣR where $Q = I_1 + I_2$, I_1, I_2 proper ideals of Q, $I_1 \cap I_2 = \{0\}$. If for $\hat{\pi} \in \Sigma R$, $\hat{\pi}^{-1}(0)$ is a principal ideal of R*

generated by f and $\hat{f}$, the function in $\hat{R}$ which corresponds to f in R, is a homeomorphism on a neighborhood of $\hat{\pi}$, and for π in this neighborhood, $g \in R$, $\pi g = \sum_{i=0}^{\infty} a_i(\pi f)^i$, then ΣR is an open Riemann surface and $\hat{R}$ is a ring of analytic functions on ΣR.

Proof. The spectrum of R with the Gelfand topology is a Hausdorff space. By the lemma ΣR is connected. Let $\hat{\pi} \in \Sigma R$. There is $\hat{f}$ a homeomorphism of $N_{\hat{\pi}}$ onto $|z| < \rho$ for some $\rho > 0$. If $\pi \in N_{\hat{\pi}}$, $g \in R$, then $\hat{g}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$. If $\pi \in N_{\pi_1} \cap N_{\pi_2} = W$ then $\hat{f}_1 \circ \hat{f}_2^{-1}(\hat{f}_2(\pi)) = \hat{f}_1(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}_2(\pi))^i$ implies $\hat{f}_1 \circ \hat{f}_2^{-1}$ is analytic on $\hat{f}_2(W)$. $\{(N_\pi, \hat{f}_\pi)$: $\pi \in \Sigma R\}$ defines an analytic structure on ΣR. It is immediate that $\hat{R} \subset A(\Sigma R)$. Since $\hat{R}$ contains functions which are not constant and are analytic on ΣR, ΣR is an open Riemann surface.

If $\{R_n\}$ is a sequence of subrings of $A(F)$ such that R_n satisfies the conditions of Theorem 8, $\Sigma R_n|_{R_1} = \Sigma R_1$, $R_{n-1} \subset R_n$, then the chain has a maximal element, $\{\hat{f} \circ \phi$: $\hat{f} \in A(\Sigma R_1)$ and $\phi(x) = \pi_x$, $x \in F\}$. Let $\hat{\pi} \in \Sigma R_1$ and $\hat{f}$ be a local homeomorphism at $\hat{\pi}$. If R_1 satisfies the conditions of Theorem 8 and contains all functions g in $A(F)$ such that $\hat{g}(\pi) = \sum_{i=0}^{\infty} a_i(\hat{f}(\pi))^i$ for $\pi \in N_{\hat{\pi}}$, π and $\hat{\pi}$ elements of ΣR_1, then $\hat{R}_1 = A(\Sigma R_1)$, because if $\hat{g} \notin \hat{R}_1$, then there is $\hat{\pi} \in \Sigma R_1$ such that $\hat{g} \circ \hat{f}^{-1}$ is not analytic on $\{z$: $|z| < \rho\}$ which implies $\hat{g} \notin A(\Sigma R_1)$.

The author wishes to thank Professors H. B. Curtis and H. E. Lacey for their suggestions and encouragement.

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Received July 24, 1968, and in revised form October 14, 1969. This paper is part of the author's Ph. D. thesis submitted to the University of Texas at Austin.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

THE PRINCIPLE OF SUBORDINATION APPLIED TO FUNCTIONS OF SEVERAL VARIABLES

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In this paper we consider univalent maps of domains in $C^n (n \geqq 2)$. Let P be a polydisk in C^n. We find necessary and sufficient conditions that a function $f: P \to C^n$ be univalent and map the polydisk P onto a starlike or a convex domain. We also consider maps from

$$(1)\qquad \begin{aligned} &D_p = \{z: |z|_p < 1\} \subset C^n \\ &|z|_p = |(z_1, z_2, \cdots, z_n)|_p = \left[\sum_{j=1}^{n} |z_j|^p\right]^{1/p}, \quad p \geqq 1 \end{aligned}$$

into C^n and give necessary and sufficient conditions that such a map have starlike or convex image.

In [4] Matsuno has considered a similar problem for the hypersphere $D_2 \subset C^n$. His definition of starlikeness is different from that used in this paper, but the results show that the two definitions are equivalent. However, his definition of convex-like is not equivalent to geometrically convex.

1. Preliminary lemmas.

For $(z_1, z_2, \cdots, z_n) = z \in C^n$, define $|z| = \max_{1 \leqq j \leqq n} |z_j|$. Let $E_r = \{z \in C^n: |z| < r\}$ and $E = E_1$. Let $\mathscr{P}$ be the class of mappings $w: E \to C^n$ which are holomorphic and which satisfy $w(0) = 0$, $\operatorname{Re}[w_j(z)/z_j] \geqq 0$ when $|z| = |z_j| > 0$, $(1 \leqq j \leqq n)$ where $w = (w_1, w_2, \cdots, w_n)$. The following lemmas are generalizations of Theorems A and B of Robertson [5, p. 315–317].

LEMMA 1. *Let $v(z; t): E \times I \to C^n$ be holomorphic for each $t \in I = [0, 1]$, $v(z; 0) = z$, $v(0, t) = 0$ and $|v(z; t)| < 1$ when $z \in E$. If*

$$(2)\qquad \lim_{t \to 0^+} [(z - v(z; t))/t^\rho] = w(z)$$

exists and is holomorphic in E for some $\rho > 0$, then $w \in \mathscr{P}$.

Proof. The hypothesis (2) implies that $\lim_{t \to 0^+} v_j(z; t) = z_j$ (here $v(z; t) = (v_1(z; t), v_2(z; t), \cdots, v_n(z; t))$ so

$$\frac{2z_j(z_j - v_j(z; t))}{z_j + v_j(z; t)} \equiv \psi_j(z; t)$$

is holomorphic for $z \in E$, $z_j \neq 0$ $(1 \leqq j \leqq n)$. By Schwarz lemma, $|v(z; t)| \leqq |z|$ and hence $\operatorname{Re}[\psi_j(z; t)/z_j] \geqq 0$ when $|z| = |z_j| > 0$. Setting $\psi(z; t) = (\psi_1, \psi_2, \cdots, \psi_n)$, $(z \in E, z_1 z_2 \cdots z_n \neq 0)$ we observe that

$$\lim_{t\to 0^+} \psi(z; t)/t^\rho = w(z)$$

for these values of z and using continuity of w we conclude $w \in \mathscr{P}$.

LEMMA 2. *Let $f: E \to C^n$ be holomorphic and univalent and satisfy $f(0) = 0$. Let $F(z; t): E \times I \to C^n$ be a holomorphic function of z for each $t \in I = [0, 1]$, $F(z; 0) = f(z)$, $F(0, t) = 0$ and suppose $F(z; t) \prec f$ for each $t \in I$ (i.e., $F(E; t) \subset f(E)$ for each $t \in I$). Let $\rho > 0$ be such that $\lim_{t\to 0^+} F(z; 0) - F(z; t)/t^\rho = F(z)$ exists and is holomorphic. Then $F(z) = Jw$ where $w \in \mathscr{P}$. Here F and w are written as column vectors and J is the complex Jacobian matrix for the mapping f.*

Proof. Since $F(z; t) \prec f$ for each $t \in I$, there exists $v: E \times I \to E$ such that $f(v(z; t)) = F(z; t)$ where $|v(z; t)| \leqq |z|$. Writing f as a column vector we have $f(v(z; t)) = f(z) + J(v(z; t) - z) + R(v(z; t), z)$ where $|R(\zeta, z)|/|\zeta - z| \to 0$ as $|\zeta - z| \to 0$. Hence

$$\frac{F(z; 0) - F(z; t)}{t^\rho} = J\left(\frac{z - v(z; t)}{t^\rho}\right) - \frac{R(v(z; t), z)}{t^\rho}$$

and the lemma follows from Lemma 1.

2. Starlike and convex mappings of the polydisk.

DEFINITION. A holomorphic mapping $f: E \to C^n$ is starlike if f is univalent, $f(0) = 0$ and $(1 - t)f \prec f$ for all $t \in I$.

THEOREM 1. *Suppose $f: E \to C^n$ is starlike and that J is the complex Jacobian matrix of f. There exists $w \in \mathscr{P}$ such that $f = Jw$ where f and w are written as column vectors.*

Proof. Apply Lemma 2 with $F(z; t) = (1 - t)f(z)$. Then

$$f(z) = \lim_{t\to 0^+} \frac{f(z) - (1 - t)f(z)}{t} = \lim_{t\to 0^+} \frac{F(z; 0) - F(z; t)}{t}$$

and the theorem follows from Lemma 2.

We now consider the conclusion of Theorem 1 in component form. Let J_j be the matrix obtained by replacing the jth column in J by the column vector f, $1 \leqq j \leqq n$. Then the jth component w_j of w is $\det(J_j)/\det J$. Theorem 1 therefore says that if f is starlike then $\operatorname{Re}[\det(J_j)/z_j \det J] \geqq 0$ when $|z| = |z_j| > 0$. Also,

$$(3) \qquad f_j = \frac{\partial f_j}{\partial z_1} w_1 + \frac{\partial f_j}{\partial z_2} w_2 + \cdots + \frac{\partial f_j}{\partial z_n} w_n\,, \qquad 1 \leqq j \leqq n$$

and equating coefficients in the power series using (3) we find

$$w_j(z) = z_j + \text{terms of total degree 2 or greater} .$$

Now suppose $|z^{(0)}| = |z_j^{(0)}| > 0$ and let α_k, $(1 \leqq k \leqq n)$ be such that $z_k^{(0)} = \alpha_k z_j^{(0)}$. Then $|\alpha_k| \leqq 1$, $(1 \leqq k \leqq n)$. Consider $w_j(z)/z_j = u(z_j)$ where z is restricted to the set,

$$z = (\alpha_1, \alpha_2, \cdots, \alpha_n)z_j , \qquad |z_j| < 1 .$$

Then $\operatorname{Re} u(z_j) \geqq 0$, $0 < |z_j| < 1$ and $u(z_j) \to 1$ as $z_j \to 0$. Since $\operatorname{Re} u(z_j)$ is a harmonic function of z_j, we conclude $\operatorname{Re} u(z_j) > 0$, $|z_j| < 1$ and

$$\operatorname{Re}[w_j(z)/z_j] > 0 \quad \text{when} \quad |z| = |z_j| > 0 . \tag{4}$$

We now prove the converse of Theorem 1.

THEOREM 2. *Suppose* $f: E \to C^n$ *is holomorphic,* $f(0) = 0$, J *is nonsingular and that*

$$f(z) = Jw, \ w \in \mathscr{P} . \tag{5}$$

Then f *is starlike.*

Proof. Since $\det J \neq 0$ when $z = 0$, f is univalent in a neighborhood of 0. It is clear that $\{r: 0 \leqq r \leqq 1$ and f is univalent in $E_r\} = A$ is a closed subset of $[0, 1]$. We will show that A is also open and that if f is univalent in E_r then $f(E_r)$ is starlike with respect to 0.

Let $r > 0$ be such that f is univalent in E_r, $(0 < r < 1)$. Let z be fixed, $|z| \leqq r$ and let $v(z; t)$ be such that $f(v(z; t)) = (1 - t)f(z)$, $-\varepsilon < t < t_0$ where ε is small and positive and $t_0 > 0$. This is possible since $\det J \neq 0$.

Then

$$\begin{aligned} v(z; t) &= v(z; 0) + J^{-1}\cdot(-f(z))\cdot t + g(t) \\ &= z - J^{-1}\cdot J\cdot w\cdot t + g(t) \\ v(z; t) &= z - tw + g(t) \end{aligned} \tag{6}$$

by (5). Here $|g(t)|/t \to 0$ as $t \to 0$. Using (4), we conclude $|v(z; t)|$ is a strictly decreasing function of t. Hence each point of the ray $(1 - t)f(z)$, $0 < t \leqq 1$ is the image of a point $v(z; t) \in E_r$ for each z such that $|z| \leqq r$. We conclude that $f(E_r)$ is starlike with respect to 0. We now show A is open. Observe that f is one-to-one in the closed polydisk $\bar{E}_r$ for if $|z| \leqq |\zeta| = r$, $z \neq \zeta$ and $f(z) = f(\zeta)$ then by (6) and (4) we can conclude that for t positive and sufficiently small there are functions $v(\zeta; t)$, $v(z; t)$ such that $v(\zeta; t), v(z, t) \in E_r$, $v(\zeta; t) \neq v(z; t)$ and

$f(v(z; t)) = (1 - t)f(z) = (1 - t)f(\zeta) = f(v(\zeta, t))$ which is a contradiction.

We now define a continuous nonnegative function $\phi: E \times E \to R$ (R is the real numbers) such that $\phi(z, \zeta) = 0$ if and only if $f(z) = f(\zeta)$, $z \neq \zeta$. We show that ϕ is positive on the closed set $\bar{E}_r \times \bar{E}_r$ and hence has a positive minimum on this set. This will imply f is univalent in $E_{r+\varepsilon}$ for some $\varepsilon > 0$ and hence A is open. For $z, \zeta \in E$, define $G(z, \zeta) = \det(a_{ij})$ where

$$a_{ij} = \begin{cases} \dfrac{f_i(z_1, z_2, \cdots, z_j, \zeta_{j+1} \cdots, \zeta_n) - f_i(z_1, z_2, \cdots, z_{j-1}, \zeta_j, \cdots, \zeta_n)}{z_j - \zeta_j}, & (z_j \neq \zeta_j) \\ \dfrac{\partial f_i}{\partial z_j}(z_1, z_2, \cdots, z_j, \zeta_{j+1}, \cdots, \zeta_n), & (z_j = \zeta_j) \end{cases}$$

and $f = (f_1, f_2, \cdots, f_n)$.

Now set $\phi(z, \zeta) = |G(z, \zeta)| + \sum_{j=1}^{n} |f_j(z) - f_j(\zeta)|$. Then $\phi(z, z) = |\det(J(z)| > 0$ while

$$\phi(z, \zeta) > 0 \quad \text{when} \quad f(z) \neq f(\zeta) .$$

If $f(z) = f(\zeta)$ for some $z, \zeta \in E, z \neq \zeta$ then the columns of $G(z, \zeta)$ are not linearly independent so $G(z, \zeta) = 0$ and $\phi(z, \zeta) = 0$. The proof is now complete.

THEOREM 3. *Suppose $f: E \to C^n$ is holomorphic, $f(0) = 0$ and that J is nonsingular for all $z \in E$. Then f is a univalent map of E onto a convex domain if and only if there exist univalent mappings f_j $(1 \leqq j \leqq n)$ from the unit disk in the plane onto convex domains in the plane such that $f(z) = T(f_1(z_1), f_2(z_2) \cdots, f_n(z_n))$ where T is a nonsingular linear transformation.*

Proof. It is clear that if f satisfies the conditions given in the theorem, then f is univalent and $f(E)$ is convex. We will prove the converse.

Suppose f is a univalent map of E onto a convex domain. Let $A = (A_1, A_2, \cdots, A_n)$ where $A_j \geqq 0$ $(1 \leqq j \leqq n)$ and let

$$A_t(z) = (z_1 e^{iA_1 t}, z_2 e^{iA_2 t}, \cdots, z_n e^{iA_n t})$$

where $-1 \leqq t \leqq 1$. Then

$$F(z; t) = 1/2[f(A_t(z)) + f(A_{-t}(z))] \prec f \qquad 0 \leqq t \leqq 1$$

and $F(z; t)$ satisfies the hypotheses of Lemma 2 with $\rho = 2$. Using the same notation as in Lemma 2, we have

$$F(z) = (F_1, F_2, \cdots, F_n)$$

$$(7)\qquad \begin{aligned} 2F_j &= \sum_{k=1}^{n} A_k^2\Big(z_k^2\frac{\partial^2 f_j}{\partial z_k^2} + z_k\frac{\partial f_j}{\partial z_k}\Big) \\ &\quad + 2\sum_{k=2}^{n}\sum_{l=1}^{k-1} A_k A_l z_k z_l \frac{\partial^2 f_j}{\partial z_l \partial z_k} \end{aligned}$$

and also $F = Jw$, $w \in \mathscr{P}$. Hence we find that $w_j = \det J^{(j)}/\det J$ where $J^{(j)}$ is obtained from J by replacing the jth column by F written as a column vector. Fix k, $1 \leqq k \leqq n$ and choose $A_k = 1$, $A_l = 0$, $l \neq k$, $1 \leqq l \leqq n$. Suppose $|z| = |z_j| > 0$, $j \neq k$ and $z_k = 0$. Then $w_j/z_j = 0$ and since $\operatorname{Re}(w_j/z_j) \geqq 0$ when $|z| = |z_j| > 0$ we must have $w_j \equiv 0$. We have therefore shown that for $1 \leqq j \leqq n$ and $1 \leqq k \leqq n$ we have

$$(8)\qquad z_k^2\frac{\partial^2 f_j}{\partial z_k^2} + z_k\frac{\partial f_j}{\partial z_k} = \frac{\partial f_j}{\partial z_k}\psi_k$$

where $\operatorname{Re}[\psi_k(z)/z_k] \geqq 0$ when $|z| = |z_k| > 0$. With k as before, fix l, $1 \leqq l \leqq n$, $l \neq k$ and choose $A_k = 1$, $A_l = \varepsilon > 0$ and $A_m = 0$, $1 \leqq m \leqq n$, $m \neq k, l$.

Using (8) we conclude

$$w_j = \varepsilon\frac{z_k z_l G_j}{\det J} + O(\varepsilon^2) \qquad (j \neq k)$$

where G_j is obtained from $\det J$ by replacing the jth column by the column $\partial^2 f_m/\partial z_l \partial z_k (1 \leqq m \leqq n)$. Hence $\operatorname{Re}[z_k z_l/z_j \cdot G_j/\det J] \geqq 0$ when $|z| = |z_j| > 0$. Since $\operatorname{Re}[z_k z_l/z_j \cdot G_j/\det J] = 0$ when $z_k z_l = 0$ we see that $G_j \equiv 0$ for each j, $1 \leqq j \leqq n$.

Since the system of equations

$$\sum_{j=1}^{n}\frac{\partial f_m}{\partial z_j}\phi_j = \frac{\partial^2 f_m}{\partial z_l \partial z_k} \qquad 1 \leqq m \leqq n$$

has solution

$$\phi_j = \frac{G_j}{\det J} = 0 \qquad 1 \leqq j \leqq n$$

we conclude

$$\frac{\partial^2 f_m}{\partial z_l \partial z_k} = 0 \qquad 1 \leqq m \leqq n\,.$$

This implies

$$(9)\qquad f_m(z) = \sum_{j=1}^{n} a_{j,m}\phi_{j,m}(z_j) \qquad 1 \leqq m \leqq n$$

where $\phi_{j,m}$ is analytic on the unit disk in the complex plane. Using

(8) we conclude $\phi_{j,m} = \phi_{j,k}$ $(1 \leqq m, k \leqq n)$ provided the constants $a_{j,m}$ in (9) are appropriately chosen. The theorem now follows readily from (8).

EXAMPLE 1. Let $f: E \to C^2$ be given by $f(z) = (z_1 + az_2^2, z_2)$ where a is a complex number, $a \neq 0$. Clearly f is univalent. Letting $f = Jw$, we find $w_1 = z_1 - az_2^2$, $w_2 = z_2$ so f is starlike provided $|a| < 1$. Note that Theorem 3 implies the suprising result that none of the sets $f(E_r)$ is convex $(1 > r > 0)$.

EXAMPLE 2. Let $f: E \to C^2$ be given by $f(z) = (z_1 g(z), z_2 g(z))$, $g: E \to C$ where g is holomorphic, $0 \notin g(E)$. Then $f = Jw$ implies

$$w_1/z_1 = w_2/z_2 = 1 + \left[z_1\frac{\partial g}{\partial z_1} + z_2\frac{\partial g}{\partial z_2}\right]\Big/ g \tag{10}$$

and f is starlike if and only if $\operatorname{Re}(w_1(z)/z_1) \geqq 0$, $z \in E$. Conversely, one can show that if $f: E \to C^2$ is holomorphic, $f = Jw$ where $w \in \mathscr{P}$ and $w_1/z_1 = w_2/z_2$ then there exists $g: E \to C$, g holomorphic, $0 \notin g(E)$ such that (10) holds and $f = ((a_1 z_1 + a_2 z_2)g, (b_1 z_1 + b_2 z_2)g)$, $(a_1 b_2 \neq a_2 b_1)$. In these cases the intersection of the polydisk E with an analytic plane $\alpha z_1 + \beta z_2 = 0$ maps into an analytic plane $\delta f_1 + \gamma f_2 = 0$. Interesting choices of g are $g(z) = (1 - z_1 z_2)^{-1}$ and $g(z) = [(1 - z_1)(1 - z_2)]^{-1}$.

3. Extension to convex and starlike maps of D_p. Since the details of the proofs for the results in this section are similar to those in §'s 2 and 3, we omit the details. We wish to find lemmas which apply to D_p (D_p is defined in equation (1)) in the same way that Lemmas 1 and 2 apply to the polydisk. The crucial point is that given equation (6) with $0 \neq z \in D_p$ we wish to conclude

$$|v(z; t)|_p \leqq |z|_p \quad \text{when} \quad 0 < t < \varepsilon$$

for some $\varepsilon > 0$. This will be true provided $\sum_{j=1}^n |z_j - tw_j|^p < \sum_{j=1}^n |z_j|^p$ for t sufficiently small. That is

$$\sum_{\substack{j=1\\ z_j \neq 0}}^n |z_j|^p(1 - 2t \operatorname{Re} w_j/z_j + t^2 |w_j/z_j|^2)^{p/2} + \sum_{z_j=0} t^p |w_j|^p < \sum_{j=1}^n |z_j|^p$$

or

$$t\Big(\sum_{\substack{j=1\\ z_j \neq 0}}^n -p \operatorname{Re} |z_j|^p \operatorname{Re}(w_j/z_j) + \sum_{z_j=0} t^{p-1} |w_j|\Big) < 0$$

when t is sufficiently small, $t > 0$. Hence we define $\mathscr{P}_p$ for $p \geqq 1$ by $w \in \mathscr{P}_p$ if $w: D_p \subset C^n \to C^n$, $w(0) = 0$, w holomorphic and

$$
\begin{aligned}
&\operatorname{Re}\sum_{j=1}^{n} w_j \cdot |z_j|^p/z_j \geqq 0 \qquad \text{if } p > 1 \\
(11) \qquad &\operatorname{Re}\sum_{\substack{j=1\\ z_j\neq 0}}^{n} w_j \cdot |z_j|/z_j - \sum_{z_j=0} |w_j| \geqq 0 \qquad \text{if } p = 1\,,
\end{aligned}
$$

$z \in D_p$, $w = (w_1, w_2, \cdots, w_n)$.

We now have the following lemmas and theorems which correspond to the lemmas and theorems of §§ 2 and 3.

LEMMA 3. *Let $v(z; t)\colon D_p \times I \to C^n$ be holomorphic for each $t \in I$, $v(z, 0) = z$, $v(0, t) = 0$ and $|v(z; t)|_p < 1$ when $z \in D_p$. If*

$$\lim_{t\to 0^+} [(z - v(z; t))/t^\rho] = w(z)$$

exists and is holomorphic in D_p for some $\rho > 0$, then $w \in \mathscr{P}_p$.

LEMMA 4. *Let $f\colon D_p \to C^n$ be holomorphic and univalent and satisfy $f(0) = 0$. Let $F(z; t)\colon D_p \times I \to C^n$ be a holomorphic function of z for each $t \in I$, $F(z, 0) = f(z)$, $F(0; t) = 0$ and suppose $F(z; t) \prec f$ for each $t \in I$. Let $\rho > 0$ be such that $\lim_{t\to 0^+}(F(z; 0) - F(z; t))/t^\rho = F(z)$ exists and is holomorphic. Then $F(z) = Jw$ where $w \in \mathscr{P}_p$.*

THEOREM 4. *If $f\colon D_p \to C^n$ is starlike then there exists $w \in \mathscr{P}_p$ such that $f = Jw$. Conversely, if $f\colon D_p \to C^n$, $f(0) = 0$, J is nonsingular and $f = Jw$, $w \in \mathscr{P}_p$ then f is starlike.*

THEOREM 5. *Let $f\colon D_p \to C^n$, $f(0) = 0$ and suppose J is nonsingular. Then $f(D_p)$ is convex if and only if $F = Jw$ where $w \in \mathscr{P}_p$ for each choice of $A = (A_1, A_2, \cdots, A_n)$, $A_j \geqq 0$ $(1 \leqq j \leqq n)$ and F is given by (7) with $z \in D_p$.*

Now set $p = 2$. It is easy to see that Theorem 4 above is equivalent to Matsuno's Theorem 1 [4, p. 91]. Consider $f\colon D_2 \to C^2$ given by $f(z) = (z_1 + az_2^2, z_2)$. Theorem 5 shows that $f(D_2)$ is convex if and only if $|a| \leqq 1/2$ while Matsuno's Lemma 3 [4, p. 94] implies f is convex-like if and only if $|a| \leqq 3\sqrt{3}/4$. This shows that convex-like is not equivalent to geometrically convex.

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Received May 19, 1969. This research was supported by the National Science Foundation, Grant GP 8225.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON SECONDARY CHARACTERISTIC CLASSES IN COBORDISM THEORY

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This paper introduces into cobordism theory a new notion borrowed from ordinary cohomology theory. Specifically, let ξ be a $U(n)$-bundle over the CW-complex X. Let E and E_0 be the total spaces of the associated bundles whose fibers are respectively the unit disc $E^{2n} \subset C^n$ and the unit sphere $S^{2n-1} \subset C^n$. The classifying map for ξ gives rise to an element $U_\xi \in \Omega_U^{2n}(E, E_0)$. One defines the Thom isomorphism φ: $\Omega_U^q(X) \rightarrow \Omega_U^{q+2n}(E, E_0)$ by $\varphi(x) = (p^*x)U_\xi$ and Euler class, $e(\xi)$ of ξ, by $e(\xi) = p^{*-1}j^*(U_\xi)$. For each $\alpha = (\alpha_1, \alpha_2, \cdots)$, let $cf_\alpha(\xi) \in \Omega_U^{2|\alpha|}(X)$ be the Conner-Floyd Chern class of ξ, and S_α: $\Omega_U^q(X, Y) \rightarrow \Omega_U^{q+2|\alpha|}(X, Y)$ be the operation defined by Novikov. Then one has the relation, $S_\alpha(e(\xi)) = cf_\alpha(\xi) \cdot e(\xi)$. Now if ξ is a bundle such that $e(\xi) = 0$, then one can define a secondary characteristic class

$$\Sigma_\alpha(\xi) \in \Omega_U^*(X) \text{ mod } (S_\alpha - cf_\alpha(\xi))\Omega_U^*(X)$$

by using the above relation. The object of this paper is to study some of the properties of such secondary characteristic classes.

Secondary characteristic classes adapt particularly to the study of embedding and immersion problems. Massey and Peterson and Stein developed secondary characteristic classes in ordinary cohomology theory [4][7][8], and Lazarov has studied secondary characteristic classes in K-theory [3]. We hope the secondary characteristic classes given here, and the operations on cobordism, defined by Novikov, will have some applications on embedding and immersion problems.

The organization of the papers is as follows. In §1 we collect some results on cobordism theory and give the definition of secondary characteristic classes of cobordism theory. In §2 we give an example and carry out some computations of these characteristic classes.

1. Definition of secondary characteristic classes.

Let ξ be a $U(n)$-bundle over the CW-complex X. Let E and E_0 be the total spaces of the associated bundles whose fibres are respectively the unit disc $E^{2n} \subset C^n$ and the unit sphere $S^{2n-1} \subset C^n$. Then the Thom complex is the quotient space E/E_0. In particular, if we take ξ to be the universal $U(n)$-bundle over $BU(n)$, then the resulting Thom complex $M(\xi)$ is written $MU(n)$. The sequence of spaces

$$(MU(0), MU(1), \cdots, MU(n), \cdots)$$

is a spectrum. Associated with this spectrum we have a cohomology functor, the groups of this cohomology functor are written $\Omega_U(X, Y)$ and called complex cobordism groups. We know that $\Omega_U^*(.)$ is a multiplicative cohomology theory and $\Omega_U(P)$, where P is a point, is a polynomial ring $Z[x_1, x_2, \cdots, x_i, \cdots]$ where $x_i \in \Omega_U^{-2i}(P)$.

Next for each $U(n)$-bundle ξ over X the classifying map for ξ induces a map

$$\gamma\colon M(\xi) \longrightarrow MU(n) .$$

The map γ represents an element $U_\xi \in \Omega_U^{2n}(E, E_0)$. We define the Thom isomorphism

$$\varphi\colon \Omega_U^q(X) \longrightarrow \Omega_U^{q+2n}(E, E_0)$$

by $\varphi(x) = (p^*x)U_\xi$.

Now we need the following known theorems:

THEOREM 1 (*Conner-Floyd*) [1]. *To each ξ over X and each $\alpha = (\alpha_1, \alpha_2, \cdots)$ we can assign classes $cf_\alpha(\xi) \in \Omega_U^{2|\alpha|}(X)$, called the Conner-Floyd classes, with the following properties*:

(i) $cf_0(\xi) = 1$;

(ii) $cf_\alpha(g^*\xi) = g^*cf_\alpha(\xi)$;

(iii) *Whitney sum formula* $cf_\alpha(\xi \oplus \eta) = \sum_{\beta+\gamma=\alpha} (cf_\beta\xi)(cf_\gamma\eta)$;

(iv) *Let ξ be a $U(1)$-bundle over X, classified by a map $X \longrightarrow BU(1)$, and let the composite $X \xrightarrow{f} BU(1) \longrightarrow MU(1)$ represent the element $w \in \Omega_U^2(X)$. Then $cf_1(\xi) = w$.*

THEOREM 2 (*Novikov*) [1]. *For each $\alpha = (\alpha_1, \alpha_2, \cdots)$ there exists an operation*

$$S_\alpha\colon \Omega_U^q(X, Y) \longrightarrow \Omega_U^{q+2|\alpha|}(X, Y)$$

with the following properties:

(i) $S_0 = 1$;

(ii) $S_\alpha f^* = f^* S_\alpha$;

(iii) *S_α is stable*: $S_\alpha\delta = \delta S_\alpha$;

(iv) *Cartan formula*

$$S_\alpha(xy) = \sum_{\beta+\gamma=\alpha} (S_\beta x)(S_\gamma y);$$

(vi) *If $w \in \mathrm{Map}\,(X, MU(1)) \subset \Omega_U^2(X)$ then $S_{(k)}(w) = w^{k+1}$, and*

$$S_\alpha(w) = 0 \ \textit{if} \ \alpha \neq (k) ;$$

(vii) *Suppose that ξ is an $U(n)$-bundle over X then we have*

$$cf_\alpha(\xi) = \varphi^{-1}S_\alpha\varphi(1);$$

where φ is the Thom isomorphism for Ω_U^*.

DEFINITION 3. The Euler class of a $U(n)$-bundle ξ over X, denoted $e(\xi)$, is $p^{*-1}j^*(U_\xi)$, where $j^*\colon \Omega_U^i(E_1, E_0) \longrightarrow \Omega_U^i(E)$ is induced by the inclusion $j\colon E \longrightarrow (E, E_0)$, and the isomorphism $p^*\colon \Omega_U^i(X) —— \Omega_U^i(E)$ is induced by the projection $p\colon E \longrightarrow X$.

The following propositions are not difficult to prove:

PROPOSITION 4. *If ξ is a trivial, then $e(\xi) = 0$.*

PROPOSITION 5. *For the Euler class, the relation*

$$e(\xi \oplus \eta) = e(\xi)e(\eta)$$

holds.

PROPOSITION 6. *If a $U(n)$-bundle has an nonzero cross section, then $e(\xi) = 0$.*

From Theorem 2 we have $cf_\alpha(\xi) = \varphi^{-1}S_\alpha\varphi(1)$ so that

$$S_\alpha U_\xi = \varphi cf_\alpha(\xi) = p^*(cf_\alpha(\xi))U_\xi\ .$$

Therefore we have $S_\alpha e(\xi) = cf_\alpha(\xi)e(\xi)$.

Now let ξ be a bundle such that $e(\xi) = 0$, then the long exact sequence for (E, E_0) breaks up into short exact sequences.

$0 \longrightarrow \Omega_U^i(X) \longrightarrow \Omega_U^i(E_0) \xrightarrow{\delta} \Omega_U^{i+1}(E, E_0) \longrightarrow 0$. Let $a_\xi \in \Omega_U^{2n-1}(E_0)$ such that $\delta(a_\xi) = U_\xi$. Then every element in $\Omega_U^i(E_0)$ can be written uniquely as $xa_\xi + y$ where $x \in \Omega_U^{i-(2n-1)}(X)$ and $y \in \Omega_U^i(X)$. In particular, write $S_\alpha(a) = xa_\xi + y$. Then we apply δ and find that $x = cf_\alpha(\xi)$. If a^1 is another element with $(a^1) = U_\xi$, then $S_\alpha(a^1) = cf_\alpha(\xi)a^1 + y^1$. Then $y - y^1 \in (S_\alpha = cf_\alpha(\xi))\Omega_U^{2n-1}(X)$. Thus we can define a natural transformation Σ_α, from $U(n)$-bundle whose $e(\xi)$-class vanishes, to a natural quotient of Ω_U^*. If ξ is a such bundle $\Sigma_\alpha(\xi)$ takes values in $\Omega_U^*(X)$ mod $(S_\alpha - cf_\alpha(\xi))\Omega_U^*(X)$ and is the coset of y.

The following property can be easily proved:

PROPOSITION 7. *If ξ has a nonzero cross section then $\Sigma_\alpha(\xi) = 0$.*

2. Example. Consider $U(n+1)$ as a principal $U(n)$-bundle over S^{2n+1} for $n > 1$. Let ξ be the associated complex vector bundle. Then the sphere bundle is the complex Stiefel manifold $U(n+1)/U(n-1)$. Since $\Omega_U^{2n}(S^{2n+1}) = 0$, then $\Sigma_\alpha(\xi)$ is defined.

Let t_n be the Thom space of S^{2n+1} with respect to ξ, we have the short exact sequence

$$0 \longrightarrow \Omega_U^{2n-1}(S^{2n+1}) \longrightarrow \Omega_U^{2n-1}(U(n+1)/U(n-1)) \longrightarrow \Omega_U^{2n}(t_n) \longrightarrow 0 .$$

Since $H^*(U(n+1)/U(n-1)) = \Lambda[\gamma_{2n-1}, \gamma_{2n+1}]$ be the exterior algebra generated by γ_{2n-1} and γ_{2n+1} of dimensions $2n-1$, $2n+1$ respectively. Therefore by [2] we have $\Omega_U^*(U(n+1)/U(n-1))\Lambda[\gamma_{2n-1}, \gamma_{2n+1}] \otimes \Omega_U^*(P)$. Let $\tilde{\gamma}_{2n-1} \in \Omega_U^*(U(n+1)/U(n-1))$, $\tilde{\gamma}_{2n+1} \in \Omega_U^*(U(n+1)/U(n-1))$ such that $\mu_z(\tilde{\gamma}_{2n-1}) = \gamma_{2n-1}$, $\mu_z(\tilde{\gamma}_{2n+1}) = \gamma_{2n+1}$, where μ_z: $\Omega_U^* \longrightarrow H^*(\ , Z)$ is the map defined by the Thom class (see [2] for definition), the group $\Omega_U^{2n-1}(U(n+1)/U(n-1))$ is $Z+Z$ with generators $\tilde{\gamma}_{2n-1}$ and $\tilde{\gamma}_{2n+1}[CP^1]$ where $[CP^1] \in \Omega_U^{-2}(P)$ is a generator of $\Omega_U^*(P)$. The group $\Omega_U^{2n-1}(S^{2n+1})$ is infinite cyclic with generator $\tilde{\gamma}_{2n+1}[CP^1]$, and so $\delta(\tilde{\gamma}_{2n-1}) = \pm U_\xi$. We know that $S_{(1)}\tilde{\gamma}_{2n-1} = cf_{(1)}\tilde{\gamma}_{2n-1} + b\tilde{\gamma}_{2n+1}$ where $\pm b\tilde{\gamma}_{2n+1}$ represents $\Sigma_{(1)}(\xi)$. Since $\Omega_U^*(U(n+1)/U(n-1))$ injects into $\Omega_U^*(U(n+1))$, we can compute it in $\Omega_U^*(U(n+1))$. Now by using the notation of [9, p. 40] we have the monomorphism

$$\mu^*\colon \Omega_U^*(U(n+1)) \longrightarrow \Omega_U^*(Q_{n+1} \times U(n)) .$$

By induction, we can determine $S_{(1)}$ if we know $S_{(1)}$ in Q_{n+1} and its behavior under cross products. By [9] we have $Q_{n+1} = SCP^n VS^1$ and since $S_{(1)}$ commutes with the suspension map

$$s\colon \Omega_U^i(CP^n) \longrightarrow \Omega_U^{i+1}(SCP^n) ,$$

so we need only know $S_{(1)}$ in $\Omega_U^*(CP^n)$. By [2, p. 52] we know that $\Omega_U^*(CP^n)$ is a free $\Omega_U^*(P)$-module with basis $1, w_n, \cdots, (w_n)^n$ where $w_n \in \mathrm{Map}\,[CP^n, MU(1)] \subset \Omega_U^2(CP^n)$. Moreover, the inclusion

$$i\colon CP^{n-1} \subset CP^n$$

has $i^*w_n = w_{n-1}$. By Theorem 2 we have $S_{(1)}(w_n)^j = j(w_n)^{j+1}$, hence $S_{(1)}s(w_n)^j = sS_{(1)}(w_n)^j = sj(w_n)^{j+1} = js(w_n)^{j+1}$, here $s(w_n)^j$, $s(w_n)^{j+1}$ in $\Omega_U^*(SCP^n)$ are the images of $(w_n)^i$, $(w_n)^{i+1}$ under the suspension map s respectively. From above data and an argument, similar to [9, p. 53], we obtain $S_{(1)}\tilde{\gamma}_{2n-1} = (n-1)\tilde{\gamma}_{2n+1}$, hence $cf_{(1)} = 0$ and $b = n-1$. Now we compute $(S_{(1)} - cf_{(1)})\,\Omega_U^{2n-1}(S^{2n+1}) = S_{(1)}\Omega_U^{2n-1}(S^{2n+1})$, which is generated by $S_{(1)}(\tilde{\gamma}_{2n+1}[CP^1])$. By [5] we have $S_{(1)}(\tilde{\gamma}_{2n+1}[CP^1]) = 2\tilde{\gamma}_{2n+1}$. Therefore $\Sigma_{(1)}(\xi) \neq 0$ if $n - 1 \neq 0 \bmod (2)$.

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Received August 29, 1969. Supported by the Research Foundation, State University of New York.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

CONTINUOUS COMPLEMENTORS ON B^*-ALGEBRAS

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This paper is concerned with continuous and uniformly continuous complementors on a B^*-algebra. Let A be a B^*-algebra with a complementor p and E_p the set of all p-projections of A. We show that if A has no minimal left ideals of dimension less than three, then p is uniformly continuous if and only if E_p is a closed and bounded subset of A. We also give a characterization of the boundedness of E_p.

Let A be a complex Banach algebra and let L_r be the set of all closed right ideals of A. Following [4], we shall say that A is a right complemented Banach algebra if there exists a mapping $p: R \to R^p$ of L_r into itself having the following properties:

$$(C_1) \qquad R \cap R^p = (0) \qquad (R \in L_r) ;$$

$$(C_2) \qquad R + R^p = A \qquad (R \in L_r) ;$$

$$(C_3) \qquad (R^p)^p = R \qquad (R \in L_r) ;$$

$$(C_4) \qquad \text{if } R_1 \subset R_2, \text{ then } R_2^p \subset R_1^p \ (R_1, R_2 \in L_r) .$$

The mapping p is called a right complementor on A. In this paper a complemented Banach algebra will always mean a right complemented Banach algebra. We also use $p(R)$ for R^p.

For any set S in a Banach algebra A, let S_l and S_r denote the left and right annihilators of S in A, respectively. Then A is called an annihilator algebra if, for every closed left ideal I and for every closed right ideal R, we have $I_r = (0)$ if and only if $I = A$ and $R_l = (0)$ if and only if $R = A$. If $I_{rl} = I$ and $R_{lr} = R$, then A is called a dual algebra.

We say that a Banach algebra A has an approximate identity if there exists a net $\{e_\alpha\}$ in A such that $\|e_\alpha\| \leqq 1$, for all α, and $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$, for all $x \in A$. Every B^*-algebra has an approximate identity.

A minimal idempotent f in a complemented Banach algebra A is called a p-projection if $(fA)^p = (1 - f)A$. If A is a semi-simple annihilator complemented Banach algebra, then every nonzero right ideal, no matter whether closed or not (see [4; p. 653]), contains a p-projection. Let A be a complemented B^*-algebra with a complementor p. Since, by [4; p. 655, Lemma 5], the socle of A is dense in A, A is dual (see [3; p. 222, Th. 2.1]). Let E (resp. E_p) be the set of all self-adjoint minimal idempotents (resp. p-projections) in A. Then, for each $e \in E$, there exists a unique $P(e) \in E_p$ such that $P(e)A = eA$. It can be shown that P is a one-to-one mapping of E onto E_p. We call P the p-derived mapping of p. The complementor p is said to be continuous if P is continuous in the relative topologies of E and E_p induced by the given norm on A (see [1; p. 463, Definition 3.7]).

Let A be a dual B^*-algebra. It has been shown in [1; p. 463, Th. 3.6] that the mapping $p: R \rightarrow (R_l)^*$ is a complementor on $A(R \in L_r)$. In this case $E_p' = E$, P is the identity map, and therefore p is uniformly continuous.

The concept "p is continuous" can be defined for any semi-simple annihilator complemented Banach $*$-algebra in which $xx^* = 0$ implies $x = 0$. In fact, let A be such an algebra and p a given complementor on A. By [2; p. 155, Th. 1], every maximal closed right ideal of A is modular. Therefore [1, p. 462, Corollary 3.4] holds for A. Hence the mapping P exists as in the case of B^*-algebra and so the concept of continuity of p can be defined.

In this paper, all algebras and spaces under consideration are over the complex field C.

2. Lemmas. In this section, unless otherwise stated, H will denote a complex Hilbert space and $A = LC(H)$, the set of all compact operators on H. There exist many complementors on A. If H is infinite dimensional, then all complementors on A are continuous ([1; p. 471, Th. 6.8]). However if $\dim H$ is finite, this is not true in general as is shown in [1; p. 475]. If $\dim H \geqq 3$, then every continuous complementor on A is uniformly continuous (see [1; p. 471, Corollary 6.6]).

If u and v are elements of H, $u \otimes v$ will denote the operator on H defined by the relation $(u \otimes v)(h) = (h, v)u$, for all $h \in H$.

LEMMA 1. *Let A be any C^*-subalgebra of bounded operators on H and $E \subset A$ the set of all self-adjoint minimal idempotents. The E is a closed subset of $L(H)$, all bounded operators on H.*

Proof. Let $\{e_n\} \subset E$ be a sequence converging to some $e \in A$. Clearly $e^2 = e$ and $e^* = e$. In order that $e \in E$, it suffices to show that $e(H)$ is one dimensional. Since $(u \otimes v)^* = v \otimes u$ and since each e_n is a self-adjoint minimal idempotent, we can write $e_n = u_n \otimes u_n$, where $u_n \in H$ and $||u_n|| = 1$ $(n = 1, 2, \cdots)$. Let $v, w \in H$ be such that $e(v) \neq 0$, $e(w) \neq 0$. Since $\{(v, u_n)\}$ is bounded, there exists a subsequence $\{v, u_k)\}$ of $\{(v, u_n)\}$ and a nonzero constant $a \in C$ such that $(v, u_k) \rightarrow a$. Since

$$||au_k - e(v)|| \leqq |a - (v, u_k)|\, ||u_k|| + ||e_k - e||\, ||v|| ,$$

we have $au_k \rightarrow e(v)$. Similarly we can show that there exist a subsequence $\{u_t\}$ of $\{u_k\}$ and a nonzero constant $b \in C$ such that $bu_t \rightarrow e(w)$.

It follows now that $be(v) = ae(w)$, which shows that $e(H)$ is one dimensional. This completes the proof.

LEMMA 2. *Let H be finite dimensional, p a complementor on A and E_p the set of all p-projections in A. If E_p is a closed and bounded subset of A, then p is continuous.*

Proof. Let $e \in E$ and let $\{e_n\}$ be a sequence in E such that $e_n \to e$. Write $e_n = u_n \otimes u_n$, $e = u \otimes u$, where $u_n, u \in H$ and $||u_n|| = ||u|| = 1$ $(n = 1, 2, \cdots)$. Since H is finite dimensional, there exists a subsequence $\{u_k\}$ of $\{u_n\}$ such that $u_k \to u'$ for some $u' \in H$; clearly $||u'|| = 1$ and $u' \otimes u' = u \otimes u$. Thus $u = au'$, where $a = (u, u')$ and $|a| = 1$. Let $u'_k = au_k$. Then $e_k = u'_k \otimes u'_k$. Let P be the p-derived mapping of p. Since $P(e_k)$ is a minimal idempotent and since $P(e_k)A = e_kA$, we can write $P(e_k) = u'_k \otimes v'_k$, where $v'_k \in H$ $(k = 1, 2, \cdots)$. Similarly $P(e) = u \otimes v$ with $v \in H$. Since E_p is bounded and since $||u'_k|| = 1$, $\{v'_k\}$ is bounded. Since H is finite dimensional, there exists a subsequence $\{v'_t\}$ of $\{v'_k\}$ such that $v'_t \to v'$ for some $v' \in H$. As $||P(e_t)|| \geqq 1$, $v' \neq 0$. Since $P(e_t) = u'_t \otimes v'_t \to u \otimes v'$ and since E_p is closed, it follows that also $u \otimes v' \in E_p$. Then both $u \otimes v', u \otimes v \in E_p$. However, by [1, p. 466, Lemma 5.1] for any $u \in H$, there exists a unique such v. Thus $v = v'$. Hence $P(e_t) \to P(e)$. Therefore P is continuous and so is p. This completes the proof.

3. Main theorem. Throughout this section A will be a B^*-algebra with a complementor p. Then A is dual (see § 1). Let $\{I_t : t \in T\}$ be the family of all minimal closed two-sided ideals of A. Then, by [3; p. 221, Lemma 2.3], $A = (\sum_t I_t)_0$, the $B^*(\infty)$-sum of I_t. Since each I_t is a simple dual B^*-algebra, $I_t = LC(H_t)$ for some Hilbert space $H_t (t \in T)$. It has been shown in [4; p. 652, Lemma 1] that p induces a complementor p_t on I_t, which is given by $p_t(R) = p(R) \cap I_t$ for all closed right ideals R of $I_t (t \in T)$.

Let E (resp. E_t) be the set of all self-adjoint minimal idempotents in A (resp. in I_t) and let E_p (resp. E_p^t) be the set of all p-projections in A (resp. in I_t). Clearly $E_t = E \cap I_t$ and $E_p^t = E_p \cap I_t (t \in T)$. It can be shown that, if $u \neq v (u, v \in T)$, then $||e_u - e_v|| = 1$, for all $e_u \in E_u$, and $e_v \in E_v$. Since each $e \in E$ belongs to some I_t, $E = \bigcup_t E_t$. Similarly, if $u \neq v (u, v \in T)$, then $||f_u - f_v|| = \text{maximum}\,(||f_u||, ||f_v||) \geqq 1$, for all $f_u \in E_p^u$ and $f_v \in E_p^v$; $E_p = \bigcup_t E_p^t$. Thus p is continuous if and only if p_t is continuous for all $t \in T$ (see [1; p. 464]).

THEOREM 3. *Let A be a B^*-algebra which has no minimal left ideals of dimension less than three and p a complementor on A. Then the following statements are equivalent:*

(i) *p is uniformly continuous.*

(ii) *There exists an involution $*'$ on A for which $R^p = (R_l)^{*'}$, for every closed right ideal R of A (and hence there exists an equivalent norm $\|\cdot\|'$ on A which satisfies the B^*-condition for $*'$).*

(iii) *The set E_p of all p-projections in A is a closed and bounded subset of A.*

Proof. (i) $\rightarrow$ (ii). This is [**1**; p. 477, Th. 7.4].

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let E_p^t be the set of all p-projections in $I_t (t \in T)$. By [**1**; p. 465, Corollary 4.4], each $f_t \in E_p^t$ is self-adjoint in $*'$. Hence $\|f_t\|' = 1$. Since each E_p^t is the set of all self-adjoint (in $*'$) minimal idempotents in I_t, by Lemma 1, E_p^t is closed in $\|\cdot\|'$. It is now easy to show that E_p is closed and bounded in $\|\cdot\|$. This proves (iii).

(iii) $\Rightarrow$ (i). Suppose (iii) holds. If H_t is finite dimensional, then since $I_t = LC(H_t)$, it follows from Lemma 2 that p_t is continuous. If H_t is infinite dimensional, then by [**1**; p. 471, Th. 6.8], p_t is continuous. Therefore each p_t is continuous and so p is continuous. We now show that p is uniformly continuous. For each $t \in T$, let Q_t be a p_t-representing operator of H_t onto itself (see [**1**; p. 467, Definition 5.4]). By [**1**; p. 470, Th. 6.4], Q_t is a continuous positive linear operator with continuous inverse Q_t^{-1}. We may assume that $\|Q_t^{-1}\| = 1$, where $\|Q_t^{-1}\|$ denotes the operator bound of Q_t^{-1} on $H_t (t \in T)$ (see [**1**; p. 472, Corollary 6.10]). We claim that $\{\|Q_t\|\}$ is bounded above. On the contrary, we assume that there exists a sequence $\{Q_n\} \subset \{Q_t\}$ such that $\|Q_n^{1/2}\| \geqq 5n$, where $Q_n^{1/2}$ denotes the square root of Q_n $(n = 1, 2, \cdots)$. Since $\|Q_n^{-1}\| = 1$, we can choose $u_n \in H_n$ such that $\|u_n\| = 1$ and $\|Q_n u_n\| \leqq 2$. Since $\|Q_n^{1/2}\| \geqq 5n$, we can choose $v_n \in H_n$ such that $\|v_n\| = 1$, $(u_n, v_n) = 0$ and $\|Q_n^{1/2} v_n\| \geqq 5n$. Let $a_n = \|Q_n^{1/2} v_n\|^{-1}$ and $h_n = a_n v_n + u_n$. Then

$$\begin{aligned}(h_n, Q_n h_n) - (u_n, Q_n u_n) &= a_n^2 (v_n, Q_n v_n) + a_n (Q_n u_n, v_n) \\ &\quad + a_n (v_n, Q_n u_n) \\ &\geqq 1 - 2a_n \|Q_n u_n\| \\ &\geqq 1 - 4a_n \,.\end{aligned}$$

Since $a_n \leqq 1/5n$, we have

$$(h_n, Q_n h_n) - (u_n, Q_n u_n) \geqq 1 - \frac{4}{5n} \geqq \frac{1}{5} \,.$$

Therefore

$$\begin{aligned}\frac{1}{5} \leqq (h_n, Q_n h_n) - (u_n, Q_n u_n) &= a_n (v_n, Q_n h_n) + a_n (u_n, Q_n v_n) \\ &\leqq a_n |(v_n, Q_n h_n)| + 2a_n \,.\end{aligned}$$

Hence we get

$$(\sharp) \qquad |(v_n, Q_nh_n)| \geqq \frac{1}{5a_n} - 2 \geqq n - 2 .$$

Now let

$$f_n = \frac{h_n \otimes Q_nh_n}{(h_n, Q_nh_n)} .$$

By the definition of Q_n, $f_n \in E_p$. Since $||h_n|| \geqq ||u_n|| = 1$ and since

$$\begin{aligned}(h_n, Q_nh_n) &= a_n^2(v_n, Q_nv_n) + a_n(Q_nu_n, v_n) \\ &\quad + a_n(v_n, Q_nu_n) + (u_n, Q_nu_n) \\ &< 1 + 1 + 1 + 2 = 5 ,\end{aligned}$$

it follows from ($\sharp$) that

$$||f_n(v_n)|| = \frac{|(v_n, Q_nh_n)|\,||h_n||}{(h_n, Q_nh_n)} > \frac{n-2}{5} .$$

Since $||v_n|| = 1$, $||f_n|| > (n-2)/5$, contradicting the boundedness of E_p. Therefore $\{||Q_t||\}$ and $\{||Q_t^{-1}||\}$ are bounded. By using the argument in [1; p. 479], it is easy to show that p is uniformly continuous. This completes the proof of the theorem.

Finally we give a characterization of the boundedness of E_p.

Let R be a closed right ideal of A and let P_R be the projection on R along R^p, i.e., $P_R(x + y) = x$ for all $x \in R$, $y \in R^p$. Since $R^p = \{x \in A: P_R(x) = 0\}$, P_R is continuous. Now let $\{J_\lambda: \lambda \in \Lambda\}$ be the set of all minimal right ideals of A. Since A is dual, each J_λ is automatically closed. For every $\lambda \in \Lambda$, let P_λ be the projection on J_λ along $p(J_\lambda)$.

THEOREM 4. *Let A be a B^*-algebra with a complementor p. Then the following statements are equivalent*:

(i) *The set E_p of all p-projections in A is a bounded subset of A.*

(ii) *$\{|P_\lambda|: \lambda \in \Lambda\}$ is bounded, where $|P_\lambda|$ denotes the operator bounded of P_λ.*

(iii) *There exists a constant k such that*

$$k\,||x_1 + x_2|| \geqq ||x_i|| \qquad (i = 1, 2) ,$$

for all $x_1 \in J_\lambda$, $x_2 \in p(J_\lambda)$ $(\lambda \in \Lambda)$.

Proof. (i) $\Rightarrow$ (ii). Suppose $\sup\{||f||: f \in E_p\} \leqq c$, where c is a constant. Let J be a minimal right ideal of A. Then there exists an $f \in E_p$ such that $J = fA$ and $J^p = (1 - f)A$. Let $x \in A$. Since

$$||P_\lambda(x)|| = ||fx|| \leqq c\,||x|| ,$$

$|P_\lambda| \leqq c$. *This proves* (ii).

(ii) $\Rightarrow$ (iii). Suppose that $\sup\{|P_\lambda|: \lambda \in \Lambda\} \leqq k - 1$, where k is a constant. Then, for all $x_1 \in J_\lambda$, $x_2 \in p(J_\lambda)$ $(\lambda \in \Lambda)$, we have

$$||x_1|| \leqq (k-1)\,||x_1 + x_2|| \leqq k\,||x_1 + x_2||\,.$$

It now follows from $||x_2|| - ||x_1|| \leqq ||x_1 + x_2||$ that $||x_2|| \leqq k\,||x_1 + x_2||$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $f \in E_p$ and $x \in A$. Since $x = (1-f)x + fx$, by (iii), $k\,||x|| \geqq ||fx||$. As a B^*-algebra, A has an approximate identity $\{e_\alpha\}$. Since $||e_\alpha|| \leqq 1$, $||fe_\alpha|| \leqq k\,||e_\alpha|| \leqq k$. It now follows from $||fe_\alpha|| \to ||f||$ that $||f|| \leqq k$. This completes the proof of the theorem.

It is Professor B. J. Tomiuk who aroused my interest in this topic. I wish to express my hearty thanks to him. I also wish to thank the referee for discovering an error in my previous demonstration of Theorem 3.

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Received September 5, 1969.

UNIVERSITY OF OTTAWA
OTTAWA, CANADA

PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

ON A REGULAR SEMIGROUP IN WHICH THE IDEMPOTENTS FORM A BAND

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This paper is a continuation of a previous paper, in which the structure of certain regular semigroups, called generalized inverse semigroups, has been studied. A semigroup is called strictly regular if it is regular and the set of all its idempotents is a subsemigroup. A generalized inverse semigroup is strictly regular, but the converse is not true. Hence, the class of generalized inverse semigroups is properly contained in the class of strictly regular semigroups. The main purpose of this paper is to establish some results which clarify the structure of strictly regular semigroups. The concept of a quasi-direct product of a band (that is, an idempotent semigroup) and an inverse semigroup is introduced, and in particular it is proved that any semigroup is strictly regular if and only if it is a quasi-direct product of a band and an inverse semigroup.

A regular semigroup S (for the definition, see [1]) is called *strictly regular* if the set E of idempotents of S is a subsemigroup of S. If the set E of a regular semigroup S satisfies a (nontrivial) permutation identity $x_1x_2 \cdots x_n = x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n)}$, where π is a (nontrivial) permutation of $1, 2, \cdots, n$, then it can be proved (see [6]) that E is a subsemigroup of S (in fact, E is a normal band[1]) and hence S is strictly regular. In this case, S is particularly called a *generalized inverse semigroup*. Thus any generalized inverse semigroup is strictly regular, but the converse is not true. In the previous paper [6] the author studied the structure of generalized inverse semigroups and established the following theorem:

THEOREM. *A semigroup is a generalized inverse semigroup if and only if it is isomorphic to the quasi-direct product of a left normal band, an inverse semigroup and a right normal band.*

The main purpose of this paper is to establish a similar result for the class of strictly regular semigroups. Any notation and terminology should be referred to [1], [6], unless otherwise stated.

2. Greatest inverse semigroup decompositions. In this section, we shall determine the greatest inverse semigroup decomposition of a given strictly regular semigroup.

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[1] An idempotent semigroup T is called a *band*. If $abcd = acbd$ is satisfied for any elements a, b, c, d of T, then T is said to be *normal*.

Let R be a regular semigroup. Then for any $a \in R$, there exists $x \in R$ such that $axa = a$ and $xax = x$. Such an element x is called an *inverse* of a. An inverse of a is not necessarily unique.

Reilly and Scheiblich [4] has proved the following lemma:

LEMMA 1. ([4], *Lemma* 1.3.) *Let e be an idempotent of a strictly regular semigroup S. Then, every inverse of e is an idempotent.*

According to a recent information, the following two lemmas have been also obtained by a paper of T. E. Hall submitted to the Bull. Australian Math. Soc., though the author did not see yet the paper.

LEMMA 2. *Let S be a strictly regular semigroup, and E the band (i.e., the idempotent semigroup) consisting of all idempotents of S. Let e, f be elements of E such that $efe = e$ and $fef = f$. Then, for any $a, c \in S^1$[2], any inverse x of aec is also an inverse of afc.*

Proof. By the assumption, we have $(aec)x(aec) = aec$, $x(aec)x = x$, $efe = e$ and $fef = f$. Let a^*, c^* be any inverses of a, c respectively. (If $a = 1$ or $c = 1$, then we take 1 as 1^*.) Since $aecc^*(cxa)a^*aec = aec$, we have $a^*aecc^*(cxa)a^*aecc^* = a^*aecc^*$. Moreover, $cxa(a^*aecc^*)cxa = cxaecxa = cxa$. Since a^*a, cc^* and e are all idempotents and since S is strictly regular, the element a^*aecc^* is an idempotent. Since a^*aecc^* is an inverse of cxa and is an idempotent, it follows from Lemma 1 that cxa is also an idempotent. This means that cxa is an inverse of a^*afcc^*. (In general, let $E \sim \sum\{E_\gamma: \gamma \in \Gamma\}$($\Gamma$ semilattice; E_γ rectangular band) be the structure decomposition (for the definition, see [5] or [6]) of E. Since $efe = e, fef = f$, there exists E_γ such that $e, f \in E_\gamma$. Hence for any $\xi \in E_\alpha$ and $\eta \in E_\beta$, we have $\xi e\eta, \xi f\eta \in E_{\alpha\gamma\beta}$. Therefore any idempotent τ which is an inverse of $\xi e\eta$ is also an inverse of $\xi f\eta$.)

Hence we have

$$a^*afcc^*(cxa)a^*afcc^* = a^*afcc^*,\ a^*afcxafcc^* = a^*afcc^*$$

and accordingly

$$(afc)x(afc) = afc\ . \tag{2.1}$$

Next, we shall consider about the element $x(afc)x$.

$$afc(x(afc)x)afc = afc \text{ (by (2.1))}$$

and

[2] S^1 means the adjunction of an identity 1 to S if S has no identity. If S has an identity, then S^1 means S itself.

$$(x(afc)x)afc(x(afc)x) = x(afc)x \text{ (by (2.1))} .$$

Therefore, $x(afc)x$ is an inverse of afc. Accordingly, by using the same method used to get the relation (2.1), we have

$$(2.2) \qquad (aec)(x(afc)x)(aec) = aec .$$

Hence, $x(aec)x(afc)x(aec)x = x(aec)x$. Since x is an inverse of aec, we have

$$(2.3) \qquad x(afc)x = x .$$

Therefore, it follows from (2.1), (2.3) that x is an inverse of afc.

Let R be a regular semigroup. If a mapping $\varphi: R \rightarrow R$ satisfies the condition

$$(2.4) \qquad \text{for any } x \in R,\ x\varphi(x)x = x \text{ and } \varphi(x)x\varphi(x) = \varphi(x) ,$$

then φ is called an *inverse operator* in R. It is obvious that R has at least one inverse operator. It is also easy to see that an inverse operator in a regular semigroup R is unique if and only if R is an inverse semigroup.

Now, let S be a regular semigroup. Let Ω be the set of all inverse operators in S. We define a relation σ on S as follows:

$$(2.5) \qquad a\sigma b \text{ if and only if } \{\varphi(cad): \varphi \in \Omega\} = \{\varphi(cbd): \varphi \in \Omega\}$$

for any elements c, d of S^1. Then, σ is clearly an equivalence relation on S.

Further, we have

LEMMA 3. *If S is strictly regular, then σ is a congruence relation on S.*

Proof. Let a, b be elements of S such that $a\sigma b$. Let h be any element of S, and c, d any elements of S^1. Suppose that

$$x \in \{\varphi(c(ah)d): \varphi \in \Omega\} .$$

Then, $x \in \{\varphi(ca(hd)): \varphi \in \Omega\}$. Since $a\sigma b$,

$$x \in \{\varphi(cb(hd)): \varphi \in \Omega\} = \{\varphi(c(bh)d): \varphi \in \Omega\} .$$

Hence $\{\varphi(c(ah)d): \varphi \in \Omega\} \subset \{\varphi(c(bh)d): \varphi \in \Omega\}$. We can also easily prove the converse relation. Therefore, we have $ah\sigma bh$. By a similar method, we can prove that $ha\sigma hb$. Hence, σ is a congruence relation on S.

LEMMA 4. *If S is a strictly regular semigroup, then the factor* *semigroup* S/σ *of* S mod σ *is an inverse semigroup. Let* E *be the band consisting of all idempotents of* S, *and* $E \sim \sum \{E_\gamma : \gamma \in \Gamma\}$($\Gamma$ *semilattice;* E_γ *rectangular band) the structure decomposition of* E.

Then,

(1) *for any* $e \in E_\gamma$, *the congruence class* ($\in S/\sigma$) *containing* e *is* E_γ, *and*

(2) *the basic semilattice (i.e., the semilattice of idempotents) of* S/σ *is* $E/\sigma_E = \{E_\gamma : \gamma \in \Gamma\}$, *where* σ_E *is the restriction of* σ *to* E.

Proof. It is obvious that S/σ is regular. Let $\bar{x}$ denote the congruence class ($\in S/\sigma$) containing x. If $\bar{s} \in S/\sigma$ is an idempotent, then $s\sigma s^2$. Hence an inverse s^* of s is also an inverse of s^2, and hence we have $s^2 = (ss^*s)(ss^*s) = s(s^*s^2s^*)s = ss^*s = s$. Thus, s is an idempotent. It is clear that $\bar{x}$ is an idempotent if x itself is an idempotent. Therefore, it follows that $\bar{x} \in S/\sigma$ is an idempotent if and only if x itself is an idempotent. Next for any $e, f \in E$, we shall show that $e\sigma f$ if and only if $efe = e$ and $fef = f$. Suppose at first that $e\sigma f$. Then, $\{\varphi(e): \varphi \in \Omega\} = \{\varphi(f): \varphi \in \Omega\}$. Since $e \in \{\varphi(e): \varphi \in \Omega\}$, we have $e \in \{\varphi(f): \varphi \in \Omega\}$. Hence $efe = e$ and $fef = f$. Conversely, let $efe = e$ and $fef = f$. Then, $e\sigma f$ follows from Lemma 2. Thus, $e\sigma f$ if and only if $efe = e$ and $fef = f$. This means that σ gives the structure decomposition of E and accordingly that E/σ_E is isomorphic to Γ. Since the set E/σ_E of idempotents of S/σ is commutative, S/σ is an inverse semigroup having E/σ_E as its basic semilattice.

Let G be an inverse semigroup, and L the basic semilattice of G. Let S be a strictly regular semigroup, and E the band consisting of all idempotents of S. If there exists a homomorphism ξ of S onto G such that $\bigcup \{\xi^{-1}(t): t \in L\} = E$ and the structure decomposition of E is $E \sim \sum \{\xi^{-1}(t): t \in L\}$, then we say that S is a *regular extension* of E by G.

REMARK. According to Clifford and Preston [2], the above mentioned ξ is unique if it exists. Further, we have the following result: Let G_1, G_2 be inverse semigroups having L_1, L_2 as their basic semilattices respectively. Let S be a strictly regular semigroup, and E the band consisting of all idempotents of S. Let ξ_1, ξ_2 be homomorphisms of S onto G_1, G_2 respectively such that $\bigcup \{\xi_1^{-1}(t): t \in L_1\} = \bigcup \{\xi_2^{-1}(u): u \in L_2\} = E$ and the structure decomposition of E is given as each of $\{\xi_1^{-1}(t): t \in L_1\}$ and $\{\xi_2^{-1}(u): u \in L_2\}$(that is, $E \sim \sum \{\xi_1^{-1}(t): t \in L_1\}$ and $E \sim \sum \{\xi_2^{-1}(u): u \in L_2\}$). Then $G_1 \cong G_2$, $L_1 \cong L_2$, and ξ_1, ξ_2 induce the same congruence relation on S.

THEOREM 1. *Let* S *be a strictly regular semigroup, and* E *the* *band consisting of all idempotents of S. Then, S is a regular extension of E by an inverse semigroup.*

Proof. Let σ be a congruence relation on S defined by (2.5). Then, it is easy to see from Lemma 4 that S is a regular extension of E by S/σ.

Now for σ defined by (2.5), we have the following theorem:

THEOREM 2. *If S is a strictly regular semigroup, then σ defined by* (2.5) *gives the greatest inverse semigroup decomposition of S.*

Proof. Let δ be any congruence relation on S such that S/δ is an inverse semigroup. Let $\tilde{a}$, $a \in S$, denote the congruence class containing a mod δ. Now, let x, y be elements of S such that $x\delta y$. Since $x\delta y$, any inverse x^* of x is also an inverse of y. Hence, $\tilde{x}\tilde{x}^*\tilde{x} = \tilde{x}$, $\tilde{x}^*\tilde{x}\tilde{x}^* = \tilde{x}^*$, $\tilde{y}\tilde{x}^*\tilde{y} = \tilde{y}$ and $\tilde{x}^*\tilde{y}\tilde{x}^* = \tilde{x}^*$. Therefore, each of $\tilde{x}, \tilde{y}$ is an inverse of $\tilde{x}^*$. By the assumption, S/δ is an inverse semigroup and hence an inverse of $\tilde{x}^*$ must be unique. Thus we have $\tilde{x} = \tilde{y}$, that is, $x\delta y$.

3. Quasi-direct products. In the previous paper [6], the author introduced the concept of quasi-direct products. We shall generalize that concept in this section.

Let R be an inverse semigroup, and L the basic semilattice of R. Let E be a band whose structure decomposition is $E \sim \sum \{E_\alpha : \alpha \in L\}$. Define equivalence relations π_1, π_2 on E as follows:

$$(3.1) \qquad e\pi_1 f \text{ if and only if } ef = f \text{ and } fe = e\,.$$

$$(3.2) \qquad e\pi_2 f \text{ if and only if } ef = e \text{ and } fe = f\,.$$

For an element $e \in E$, let $\tilde{e}, \tilde{\tilde{e}}$ be the equivalence classes containing e mod π_1, π_2 respectively. Put $\tilde{E} = \{\tilde{e} : e \in E\}$, $\tilde{\tilde{E}} = \{\tilde{\tilde{e}} : e \in E\}$, $\tilde{E}_\alpha = \{\tilde{e} : e \in E_\alpha\}$ and $\tilde{\tilde{E}}_\alpha = \{\tilde{\tilde{e}} : e \in E_\alpha\}$, $\alpha \in L$. Then, clearly $\tilde{E} = \sum \{\tilde{E}_\alpha : \alpha \in L\}$ and $\tilde{\tilde{E}} = \sum \{\tilde{\tilde{E}}_\alpha : \alpha \in L\}$(where Σ means disjoint sum). Further, for any $e \in E_\alpha$, $(\tilde{e}, \tilde{\tilde{e}})$ is contained in the product set $\tilde{E}_\alpha \times \tilde{\tilde{E}}_\alpha$ of $\tilde{E}_\alpha$ and $\tilde{\tilde{E}}_\alpha$. Conversely for any $(\tilde{e}, \tilde{\tilde{f}}) \in \tilde{E}_\alpha \times \tilde{\tilde{E}}_\alpha$, there exists a unique element h of E_α such that $(\tilde{h}, \tilde{\tilde{h}}) = (\tilde{e}, \tilde{\tilde{f}})$. Since R is an inverse semigroup, every element ξ of R has a unique inverse. We shall denote it by ξ^{-1}.

To each ordered pair (ξ, η) of elements ξ, η of R, let correspond a mapping $\rho_{(\xi,\eta)} : (\tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi^{-1}\xi}) \times (\tilde{E}_{\eta\eta^{-1}} \times \tilde{\tilde{E}}_{\eta^{-1}\eta}) \rightarrow \tilde{E}_{\xi\eta(\xi\eta)^{-1}} \times \tilde{\tilde{E}}_{(\xi\eta)^{-1}\xi\eta}$. If the system $\{\rho_{(\xi,\eta)} : \xi, \eta \in R\}$ of these mappings $\rho_{(\xi,\eta)}$ satisfies the following condition (3.3), then this system $\{\rho_{(\xi,\eta)}: \xi, \eta \in R\}$ is called *a set of quasi-direct factors of E with respect to R*:

Hereafter, we shall use the following notations.

$$\rho_{(\xi\eta,\nu)} \circ_L \rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2), (\tilde{e}_3, \tilde{\tilde{f}}_3))$$

means

$$\rho_{(\xi\eta,\nu)}(\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)), (\tilde{e}_3, \tilde{\tilde{f}}_3)) ,$$

and

$$\rho_{(\xi,\eta\nu)} \circ_R \rho_{(\eta,\nu)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2), (\tilde{e}_3, \tilde{\tilde{f}}_3))$$

means

$$\rho_{(\xi,\eta\nu)}((\tilde{e}_1, \tilde{\tilde{f}}_1), \rho_{(\eta,\nu)}(\tilde{e}_2, \tilde{\tilde{f}}_2), (\tilde{e}_3, \tilde{\tilde{f}}_3)))$$

for elements $e_1, f_1, e_2, f_2, e_3, f_3$ such that $e_1 \in E_{\xi\xi^{-1}}, f_1 \in E_{\xi^{-1}\xi}, e_2 \in E_{\eta\eta^{-1}}$, $f_2 \in E_{\eta^{-1}\eta}$, $e_3 \in E_{\nu\nu^{-1}}$ and $f_3 \in E_{\nu^{-1}\nu}$.

(3.3)
(1) If $\xi, \eta \in L$, then $\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)) = (\widetilde{ef}, \approx{ef})$, where e, f are elements of E_ξ, E_η respectively such that $\tilde{e} = \tilde{e}_1$, $\tilde{\tilde{e}} = \tilde{\tilde{f}}_1, \tilde{f} = \tilde{e}_2$ and $\tilde{\tilde{f}} = \tilde{\tilde{f}}_2$.

(2) $\rho_{(\xi\eta,\nu)} \circ_L \rho_{(\xi,\eta)} = \rho_{(\xi,\eta\nu)} \circ_R \rho_{(\eta,\nu)}$ for all $\xi, \eta, \nu \in R$.

(3) For any $\xi \in R, e \in E_{\xi\xi^{-1}}$ and $f \in E_{\xi^{-1}\xi}$, there exist $h \in E_{\xi^{-1}\xi}$ and $k \in E_{\xi\xi^{-1}}$ such that

$$\rho_{(\xi,\xi^{-1}\xi)} \circ_R \rho_{(\xi^{-1},\xi)}((\tilde{e}, \tilde{\tilde{f}}), (\tilde{h}, \tilde{\tilde{k}}), (\tilde{e}, \tilde{\tilde{f}})) = (\tilde{e}, \tilde{\tilde{f}}) .$$

The author does not know whether such a system $\{\rho_{(\xi,\eta)}: \xi, \eta \in R\}$ always exists or not for given R and E. However, we shall show later that a set of quasi-direct factors of E with respect to R always exists if E, R have some special types.

Now, suppose that $\{\rho_{(\xi,\eta)}: \xi, \eta \in R\}$ is a set of quasi-direct factors of E with respect to R. Let $E \times R = \{((\tilde{e}, \tilde{\tilde{f}}), \xi): e \in E_{\xi\xi^{-1}}, f \in E_{\xi^{-1}\xi}, \xi \in R\}$, and define multiplication in $E \times R$ as follows:

$$((\tilde{e}_1, \tilde{\tilde{f}}_1), \xi)((\tilde{e}_2, \tilde{\tilde{f}}_2), \eta) = (\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)), \xi\eta) . \tag{3.4}$$

Then, $E \times R$ becomes a strictly regular semigroup which has R as its homomorphic image and embeds E as the band of its idempotents. It is easy to see from the definition of the multiplication in $E \times R$ and (1) of (3.3) that E is embedded in $E \times R$ as the band of idempotents of $E \times R$ and R is a homomorphic image of $E \times R$, while it follows from (2), (3) of (3.3) that $E \times R$ is a strictly regular semigroup. Hereafter, we shall call $E \times R$ *the quasi-direct product of E and R determined by* $\{\rho_{(\xi,\eta)}: \xi, \eta \in R\}$.

EXAMPLES. I. Let R be a weakly C-inversive semigroup (see [6]; Ljapin [3] has called R a completely regular inverse semigroup), that is a semigroup such that

(1) the idempotents of R form a semilattice L,

(2) there exists a subgroup $R(\alpha)$ of R containing α for every $\alpha \in L$, and the collection $\{R(\alpha): \alpha \in L\}$ of all $R(\alpha)$ satisfies (a) $R = \sum \{R(\alpha): \alpha \in L\}$, and (b) $R(\beta)R(\gamma) \subset R(\beta\gamma)$ for all $\beta, \gamma \in L$.

Of course, R is an inverse semigroup and satisfies $\xi\xi^{-1} = \xi^{-1}\xi$ and $(\xi\eta)(\xi\eta)^{-1} = (\xi\eta)^{-1}(\xi\eta) = \xi\xi^{-1}\eta\eta^{-1}$ for all $\xi, \eta \in R$. Let E be a band having $E \sim \sum \{E_\alpha: \alpha \in L\}$ as its structure decomposition. Now, define a mapping $\rho_{(\xi,\eta)}: (\tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi\xi^{-1}}) \times (\tilde{E}_{\eta\eta^{-1}} \times \tilde{\tilde{E}}_{\eta\eta^{-1}}) \rightarrow \tilde{E}_{\xi\eta(\xi\eta)^{-1}} \times \tilde{\tilde{E}}_{(\xi\eta)^{-1}\xi\eta}$ for every ordered pair (ξ, η) of elements of R as follows:

$$\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2))(= \rho_{(\xi,\eta)}((\tilde{e}, \tilde{\tilde{e}}), (\tilde{f}, \tilde{\tilde{f}}))) = (\widetilde{ef}, \widetilde{\widetilde{ef}}) , \tag{3.5}$$

where e, f are elements of $E_{\xi\xi^{-1}}$ and $E_{\eta\eta^{-1}}$ respectively such that $\tilde{e} = \tilde{e}_1, \tilde{\tilde{e}} = \tilde{\tilde{f}}_1, \tilde{f} = \tilde{e}_2$ and $\tilde{\tilde{f}} = \tilde{\tilde{f}}_2$. The existence of such elements e, f and their uniqueness are easily verified.

Then the system $\{\rho_{(\xi,\eta)}: \xi, \eta \in R\}$ satisfies the conditions (1), (2), (3) of (3.3) and becomes a set of quasi-direct factors of E respect to R. Hence, there exists the quasi-direct product $E \times R$ of E and R determined by $\{\rho_{(\xi,\eta)}: \xi, \eta \in R\}$. That is,

$$\begin{cases} E \times R = \{((\tilde{e}, \tilde{\tilde{f}}), \xi): e \in E_{\xi\xi^{-1}}, f \in E_{\xi^{-1}\xi}, \xi \in R\}, \text{ and multi-} \\ \text{plication in } E \times R \text{ is given by} \\ ((\tilde{e}_1, \tilde{\tilde{f}}_1), \eta)((\tilde{e}_2, \tilde{\tilde{f}}_2), \nu) = (\rho_{(\eta,\nu)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)), \eta\nu) \\ (= (\rho_{(\eta,\nu)}((\tilde{e}, \tilde{\tilde{e}}), (\tilde{f}, \tilde{\tilde{f}})), \eta\nu)) = ((\widetilde{ef}, \widetilde{\widetilde{ef}}), \eta\nu) , \\ \text{where } e, f \text{ are elements of } E_{\eta\eta^{-1}} \text{ and } E_{\nu\nu^{-1}} \text{ respectively} \\ \text{such that } \tilde{e} = \tilde{e}_1, \tilde{\tilde{e}} = \tilde{\tilde{f}}_1, \tilde{f} = \tilde{e}_2 \text{ and } \tilde{\tilde{f}} = \tilde{\tilde{f}}_2 . \end{cases}$$

On the other hand, let $E \bowtie R\,(L)$ be the spined product (for the definition of spined products, see [5] or [6]) of E and R with respect to L. Then $E \bowtie R\,(L) = \sum \{E_\alpha \times R(\alpha): \alpha \in L\}$ by the definition of spined products. Define a mapping $\varphi: E \bowtie R\,(L) \rightarrow E \times R$ as follows: $\varphi(e, \xi) = ((\tilde{e}, \tilde{\tilde{e}}), \xi)$, $(e, \xi) \in E \bowtie R\,(L)$. Then it is easy to see that φ is an isomorphism of $E \bowtie R\,(L)$ onto $E \times R$. Hence, in this case the quasi-direct product $E \times R$ means the spined product $E \bowtie R\,(L)$.

II. Let R be an inverse semigroup, and L the basic semilattice of R. Let E be a normal band having the structure decomposition $E \sim \sum \{E_\alpha: \alpha \in L\}$. Since E is a normal band, $\tilde{E}$ and $\tilde{\tilde{E}}$ are a left normal band and a right normal band respectively (see [6], [7]); hence $\tilde{e}\tilde{f} = \widetilde{ef}$ for $\tilde{e}, \tilde{f} \in \tilde{E}$ and $\tilde{\tilde{e}}\tilde{\tilde{f}} = \widetilde{\widetilde{ef}}$ for $\tilde{\tilde{e}}, \tilde{\tilde{f}} \in \tilde{\tilde{E}}$.

Now, define a mapping $\rho_{(\xi,\eta)}\colon (\tilde{E}_{\xi\xi^{-1}} \times \tilde{\tilde{E}}_{\xi^{-1}\xi}) \times (\tilde{E}_{\eta\eta^{-1}} \times \tilde{\tilde{E}}_{\eta^{-1}\eta}) \rightarrow \tilde{E}_{\xi\eta(\xi\eta)^{-1}} \times \tilde{\tilde{E}}_{(\xi\eta)^{-1}\xi\eta}$ for every ordered pair (ξ, η) of elements ξ, η of R as follows:

$$\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)) = (\widetilde{e_1h}, \widetilde{\widetilde{gf_2}}) , \tag{3.6}$$

where h, g are any elements of $E_{(\xi\eta)(\xi\eta)^{-1}}$ and $E_{(\xi\eta)^{-1}(\xi\eta)}$ respectively.

It was proved by [6] that $\widetilde{e_1h}$ and $\widetilde{\widetilde{gf_2}}$ do not depend on the selection of h, g and hence $\rho_{(\xi,\eta)}$ is well-defined. It is also seen from [6] that the system $\{\rho_{(\xi,\eta)}\colon \xi, \eta \in R\}$ satisfies (1), (2), (3) of (3.3) and becomes a set of quasi-direct factors of E with respect to R. Hence, we can consider the quasi-direct product $E \times R$ of E and R determined by $\{\rho_{(\xi,\eta)}\colon \xi, \eta \in R\}$.

That is,

$$\begin{cases} E \times R = \{((\tilde{e}, \tilde{\tilde{f}}), \nu)\colon e \in E_{\nu\nu^{-1}}, f \in E_{\nu^{-1}\nu}, \nu \in R\}, \text{ and multi-} \\ \text{plication in } E \times R \text{ is defined by} \\ ((\tilde{e}_1, \tilde{\tilde{f}}_1), \xi)((\tilde{e}_2, \tilde{\tilde{f}}_2), \eta) = (\rho_{(\xi,\eta)}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)), \xi\eta) \\ \quad = ((\widetilde{e_1h}, \widetilde{\widetilde{gf_2}}), \xi\eta) = ((\tilde{e}_1\tilde{h}, \tilde{\tilde{g}}\tilde{\tilde{f}}_2), \xi\eta) , \\ \text{where } h, g \text{ are any elements of } E_{\xi\eta(\xi\eta)^{-1}} \text{ and } E_{(\xi\eta)^{-1}\xi\eta} \text{ re-} \\ \text{spectively.} \end{cases}$$

On the other hand, we can also consider the quasi-direct product $Q(\tilde{E} \otimes R \otimes \tilde{\tilde{E}}; L)$ of $\tilde{E}$, $\tilde{\tilde{E}}$ and R in the sense of [6]. Define a mapping $\varphi\colon Q(\tilde{E} \otimes R \otimes \tilde{\tilde{E}}; L) \rightarrow E \times R$ by $\varphi((\tilde{e}, \xi, \tilde{\tilde{f}})) = ((\tilde{e}, \tilde{\tilde{f}}), \xi), e \in E_{\xi\xi^{-1}}, f \in E_{\xi^{-1}\xi}$, $\xi \in R$. Then, it is easy to verify that this φ is an isomorphism of $Q(\tilde{E} \otimes R \otimes \tilde{\tilde{E}}; L)$ onto $E \times R$.[3] Hence, the concept of quasi-direct products just introduced above is a generalization of the old concept of quasi-direct products introduced by [6].

Now, let R be an inverse semigroup whose basic semilattice is L. Let E be a band having L as its structure semilattice (for the definition of structure semilattices, see [6]). Examples I and II show that there exists a quasi-direct product of E and R if, in particular, R is a union of groups or E is a normal band. However, in case that R and E have no restriction we do not know whether there exists a quasi-direct product of E and R or not. Therefore, we state it as an open problem:

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[3] Moreover, we have the following result: If R, E are the inverse semigroup and the normal band given in II, then a quasi-direct product of E and R is uniquely determined up to isomorphisms and is isomorphic to $Q(\tilde{E} \otimes R \otimes \tilde{\tilde{E}}; L)$(hence of course to the above-mentioned $E \times R$). A proof of this result will be given later elsewhere.

PROBLEM. Let R be an inverse semigroup whose basic semilattice is L. Let E be a band having L as its structure semilattice. Is there a quasi-direct product of E and L? In case that a quasi-direct product of E and L exists, is it unique?

4. A structure theorem. In this section, we shall show that any strictly regular semigroup is isomorphic to a quasi-direct product of a band and an inverse semigroup. More precisely, let S be a strictly regular semigroup and E the band consisting of all idempotents of S. Let σ be the congruence relation on S which gives the greatest inverse semigroup decomposition of S. Then as was shown in Theorem 1, S is a regular extension of E by S/σ. Further it will be shown in this section that such a regular extension of E by S/σ which is isomorphic to S can be obtained as a quasi-direct product of E and S/σ.

Let S be a strictly regular semigroup, and E the band consisting of all idempotents of S. Let $E \sim \sum \{E_\alpha : \alpha \in L\}$($L$ semilattice) be the structure decomposition of E. Let σ be the congruence relation on S which gives the greatest inverse semigroup decomposition of S. Put $S/\sigma = R$. Let $\bar{x}$ denote the congruence class containing $x \in S \bmod \sigma$. As was shown in the § 2, E/σ_E (where σ_E is the restriction of σ to E) is the basic semilattice of S/σ. Hence we can assume that $E/\sigma_E \equiv L$. Of course, in this case $E/\sigma_E = \{E_\alpha : \alpha \in L\} = \{E_{\bar{e}} : \bar{e} \in E/\sigma_E\}$.[4]

Now, we construct a set of quasi-direct factors $\rho_{(\bar{x},\bar{y})}$ of E with respect to R as follows: Let $\tilde{E} = E/\pi_1$ and $\tilde{\tilde{E}} = E/\pi_2$, where π_1, π_2 are the equivalence relations on E defined by (3.1) and (3.2) respectively. Let $\tilde{E}_{\bar{e}} = E_{\bar{e}}/\pi_1$ and $\tilde{\tilde{E}}_{\bar{e}}/\pi_2$. For every ordered pair $(\bar{x}, \bar{y})$ of elements $\bar{x}, \bar{y}$ of R, define a mapping

$$\rho_{(\bar{x},\bar{y})} : (\tilde{E}_{\bar{x}\bar{x}^{-1}} \times \tilde{\tilde{E}}_{\bar{x}^{-1}\bar{x}}) \times (\tilde{E}_{\bar{y}\bar{y}^{-1}} \times \tilde{\tilde{E}}_{\bar{y}^{-1}\bar{y}}) \longrightarrow \tilde{E}_{\overline{xy}(\overline{xy})^{-1}} \times \tilde{\tilde{E}}_{(\overline{xy})^{-1}\overline{xy}}$$

by

$$\rho_{(\bar{x},\bar{y})}((\tilde{e}_1, \tilde{\tilde{f}}_1), (\tilde{e}_2, \tilde{\tilde{f}}_2)) = (\widetilde{uv(uv)^*}, \widetilde{\widetilde{(uv)^*uv}}) , \tag{4.1}$$

where u, v are elements of S such that $\bar{u} = \bar{x}$, $\bar{v} = \bar{y}$, $\widetilde{uu^*} = \tilde{e}_1$, $\widetilde{\widetilde{u^*u}} = \tilde{\tilde{f}}_1$, $\widetilde{vv^*} = \tilde{e}_2$ and $\widetilde{\widetilde{v^*v}} = \tilde{\tilde{f}}_2$ ($u^*, v^*, (uv)^*$ are inverses of u, v, uv respectively[5]). For an element x of a regular semigroup, hereafter we shall use the notation x^* to denote an inverse of x. Hence, for example, a^* means any inverse of a.

The existence of u, v in (4.1) and their uniqueness are obvious

[4] When we regard $\bar{e}$ as a subset of E, we denote it by $E_{\bar{e}}$. Hence, $E_{\bar{e}} = E_\alpha$ if and only if $\bar{e} \equiv \alpha$, i.e., $E_\alpha \ni e$.

[5] For any two inverses u_1, u_2 of u, $\widetilde{uu_1} = \widetilde{uu_2}$ and $\widetilde{\widetilde{u_1u}} = \widetilde{\widetilde{u_2u}}$. Hence, $\widetilde{uu^*}$ and $\widetilde{\widetilde{u^*u}}$ do not depend on the selection of an inverse u^* of u.

from the following result:

LEMMA 5. *For any elements $\bar{x}$ of R, e of $E_{\bar{x}\bar{x}^{-1}}$ and f of $E_{\bar{x}^{-1}\bar{x}}$, there exists a unique element u of S such that $\bar{u} = \bar{x}$, $\widetilde{uu^*} = \tilde{e}$ and $\widetilde{\widetilde{u^*u}} = \tilde{\tilde{f}}$. In fact, $u = exf$ has these properties.*

Proof. Let $u = exf$. Since $\bar{x} = \overline{xx^*}\,\overline{x}\,\overline{x^*x} = (\bar{x}\bar{x}^{-1})\bar{x}(\bar{x}^{-1}\bar{x}) = \bar{e}\,\bar{x}\bar{f} = \overline{efx}$, we have $\bar{x} = \overline{exf} = \bar{u}$. Now, we can take fx^*e as an inverse of u (see [4]). Hence, let $u^* = fx^*e$. Since $\bar{e} = \bar{x}\bar{x}^{-1} = \bar{x}\overline{x^*} = \bar{u}\overline{u^*} = \overline{uu^*}$, both e and uu^* are contained in $E_{\bar{e}}$. Hence, $e = euu^*e = e(exf)(fx^*e)e = (exf)(fx^*e) = uu^*$. That is, $e = uu^*$. Similarly, we obtain $u^*u = f$. Therefore, of course $\tilde{e} = \widetilde{uu^*}$ and $\tilde{\tilde{f}} = \widetilde{\widetilde{u^*u}}$. Next, we shall prove that such an element u is unique. Let v be any element of S such that $\bar{v} = \bar{x}$, $\widetilde{vv^*} = \tilde{e}$ and $\widetilde{\widetilde{v^*v}} = \tilde{\tilde{f}}$. Since $\widetilde{uu^*} = \widetilde{vv^*}$, $\widetilde{\widetilde{u^*u}} = \widetilde{\widetilde{v^*v}}$ and $\bar{u} = \bar{v}$, we have $vv^*uu^* = uu^*$, $u^*uv^*v = u^*u$ and $u\sigma v$. Since $u\sigma v$, $v^*uv^* = v^*$. Hence,

$$\begin{aligned} u = uu^*u = (vv^*uu^*)u &= vv^*u(u^*u) = vv^*u(u^*uv^*v) \\ &= vv^*(uu^*u)v^*v = v(v^*uv^*)v = vv^*v = v\ . \end{aligned}$$

Consequently, $u = v$.

When we consider an element $\bar{x}$ of R as a subset of S, we shall denote it by $S_{\bar{x}}$. Of course $S_{\bar{x}} = S_{\bar{y}}$ if and only if $\bar{x} = \bar{y}$, i.e., $x\sigma y$.

LEMMA 6. *For $\bar{x} \in R$,*

(1) $S_{\bar{x}} = \{exf\colon e \in S_{\bar{x}\bar{x}^{-1}}(= E_{\bar{x}\bar{x}^{-1}}), f \in S_{\bar{x}^{-1}\bar{x}}(= E_{\bar{x}^{-1}\bar{x}})\}$,

(2) $|S_{\bar{x}}| = |\tilde{E}_{\bar{x}\bar{x}^{-1}}|\,|\tilde{\tilde{E}}_{\bar{x}^{-1}\bar{x}}|$[6], *and*

(3) *for $e, e' \in E_{\bar{x}\bar{x}^{-1}}$ and for $f, f' \in E_{\bar{x}^{-1}\bar{x}}$, $exf = e'xf'$ if and only if $\tilde{e} = \tilde{e}'$ and $\tilde{\tilde{f}} = \tilde{\tilde{f'}}$.*

Proof. Let exf be an element of $\{exf; e \in S_{\bar{x}\bar{x}^{-1}}, f \in S_{\bar{x}^{-1}\bar{x}}\}$. Then since $\overline{exf} = \bar{x}\bar{x}^{-1}\bar{x}\bar{x}^{-1}\bar{x} = \bar{x}$, exf is an element of $S_{\bar{x}}$. Conversely let $y \in S_{\bar{x}}$, and put $yy^* = e'$ and $y^*y = f'$. $\bar{y} = \bar{x}$ implies y^* is an inverse of x. Hence $y = yy^*y = yy^*xy^*y = e'xf'$. Therefore, y is contained in the set $\{exf\colon e \in E_{\bar{x}\bar{x}^{-1}}, f \in E_{\bar{x}^{-1}\bar{x}}\}$. Thus (1) is satisfied. Since (2) is obvious from (1) and (3), we next prove only the part (3). Suppose that $exf = e'xf'$, $e, e' \in E_{\bar{x}\bar{x}^{-1}}$ and $f, f' \in E_{\bar{x}^{-1}\bar{x}}$. Then $\overline{exf} = \overline{e'xf'} = \bar{x}$. As is seen from Lemma 5, these elements satisfy $\widetilde{(exf)(exf)^*} = \tilde{e}$, $\widetilde{(e'xf')(e'xf')^*} = \tilde{\tilde{e}}'$, $\widetilde{\widetilde{(exf)^*(exf)}} = \tilde{\tilde{f}}$ and $\widetilde{\widetilde{(e'xf')^*(e'xf')}} = \tilde{\tilde{f'}}$. Since $exf = e'xf'$, it follows from the above that $\tilde{e} = \tilde{\tilde{e}}'$ and $\tilde{\tilde{f}} = \tilde{\tilde{f'}}$. Conversely,

[6] If A is a set, the notation $|A|$ means the cardinality of A.

suppose that $\widetilde{e} = \widetilde{e'}, \widetilde{\widetilde{f}} = \widetilde{\widetilde{f'}}$, $e, e' \in E_{\bar{x}\bar{x}^{-1}}$ and $f, f' \in E_{\bar{x}^{-1}\bar{x}}$. Then, we have $\overline{exf} = \bar{x} = \overline{e'xf'}$, $\widetilde{(exf)(exf)^*} = \widetilde{e} = \widetilde{\widetilde{e'}} = \widetilde{(e'xf')(e'xf')^*}$ and

$$\widetilde{\widetilde{(exf)^*(exf)}} = \widetilde{\widetilde{f}} = \widetilde{\widetilde{f'}} = \widetilde{\widetilde{(e'xf')^*(e'xf')}} .$$

Hence by Lemma 5, two elements $exf, e'xf'$ must be the same.

COROLLARY. *If R is finite, then* $|S| = \sum_{\bar{x} \in R} |\widetilde{E}_{\bar{x}\bar{x}^{-1}}|\,|\widetilde{\widetilde{E}}_{\bar{x}^{-1}\bar{x}}|$.

Proof. Obvious.

For every ordered pair $(\bar{x}, \bar{y})$ of elements $\bar{x}, \bar{y}$ of R, anyway $\rho_{(\bar{x},\bar{y})}$ is well-defined. Let $\Omega = \{\rho_{(\bar{x},\bar{y})} : \bar{x}, \bar{y} \in R\}$ be the collection of all these $\rho_{(\bar{x},\bar{y})}$. Then, it is easy to see that Ω becomes a set of quasi-direct factors of E with respect to R, that is, Ω satisfies the conditions (1), (2), (3) of (3.3). We shall give a proof only for the condition (2) which is the most complicated condition among the three.

We should prove

$$\begin{aligned} (2) \text{ of } (3.3)\colon & \rho_{(\bar{x}\bar{y},\bar{z})}(\rho_{(\bar{x},\bar{y})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), (\widetilde{e}_2, \widetilde{\widetilde{f}}_2)), (\widetilde{e}_3, \widetilde{\widetilde{f}}_3)) \\ &= \rho_{(\bar{x},\bar{y}\bar{z})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), \rho_{(\bar{y},\bar{z})}((\widetilde{e}_2, \widetilde{\widetilde{f}}_2), (\widetilde{e}_3, \widetilde{\widetilde{f}}_3))) . \end{aligned}$$

By Lemma 5, there exist unique u, v, w such that $\bar{u} = \bar{x}$, $\widetilde{uu^*} = \widetilde{e}_1$, $\widetilde{\widetilde{u^*u}} = \widetilde{\widetilde{f}}_1$, $\bar{v} = \bar{y}$, $\widetilde{vv^*} = \widetilde{e}_2$, $\widetilde{\widetilde{v^*v}} = \widetilde{\widetilde{f}}_2$, $\bar{w} = \bar{z}$, $\widetilde{ww^*} = \widetilde{e}_3$ and $\widetilde{\widetilde{w^*w}} = \widetilde{\widetilde{f}}_3$. Hence $\rho_{(\bar{x},\bar{y})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), (\widetilde{e}_2, \widetilde{\widetilde{f}}_2)) = (\widetilde{uv(uv)^*}, \widetilde{\widetilde{(uv)^*uv}})$, and hence

$$\begin{aligned} \rho_{(\bar{x}\bar{y},\bar{z})}(\rho_{(\bar{x},\bar{y})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), (\widetilde{e}_2, \widetilde{\widetilde{f}}_2)), (\widetilde{e}_3, \widetilde{\widetilde{f}}_3)) &= \rho_{(\bar{x}\bar{y},\bar{z})}((\widetilde{uv(uv)^*}, \widetilde{\widetilde{(uv)^*uv}}), (\widetilde{e}_3, \widetilde{\widetilde{f}}_3)) \\ &= (\widetilde{uvw(uvw)^*}, \widetilde{\widetilde{(uvw)^*uvw}}) . \end{aligned}$$

On the other hand, $\rho_{(\bar{y},\bar{z})}((\widetilde{e}_2, \widetilde{\widetilde{f}}_2), (\widetilde{e}_3, \widetilde{\widetilde{f}}_3)) = (\widetilde{vw(vw)^*}, \widetilde{\widetilde{(vw)^*vw}})$. Hence $\rho_{(\bar{x},\bar{y}\bar{z})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), \rho_{(\bar{y},\bar{z})}((\widetilde{e}_2, \widetilde{\widetilde{f}}_2), (\widetilde{e}_3, \widetilde{\widetilde{f}}_3))) = \rho_{(\bar{x},\bar{y}\bar{z})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), (\widetilde{vw(vw)^*}, \widetilde{\widetilde{(vw)^*vw}})) = (\widetilde{uvw(uvw)^*}, \widetilde{\widetilde{(uvw)^*uvw}})$. Accordingly, (2) of (3.3) is satisfied. Since Ω is a set of quasi-direct factors of E with respect to R, we can consider the quasi-direct product $E \times R$ of E and R determined by Ω.

Now,

$$(4.2) \qquad E \times R = \{((\widetilde{e}, \widetilde{\widetilde{f}}), \bar{x}) : \bar{x} \in R,\ e \in E_{\bar{x}\bar{x}^{-1}},\ f \in E_{\bar{x}^{-1}\bar{x}}\}$$

and multiplication in $E \times R$ is of course given by

$$((\widetilde{e}_1, \widetilde{\widetilde{f}}_1), \bar{x})((\widetilde{e}_2, \widetilde{\widetilde{f}}_2), \bar{y}) = (\rho_{(\bar{x},\bar{y})}((\widetilde{e}_1, \widetilde{\widetilde{f}}_1)), (\widetilde{e}_2, \widetilde{\widetilde{f}}_2)), \bar{x}\bar{y}) .$$

As to the connection between these S and $E \times R$, we have the following theorem which is the main result of this paper:

THEOREM 3. *Let S be a strictly regular semigroup, and E the band consisting of all idempotents of S. Let R be the greatest inverse* *semigroup homomorphic image of* S. *Then,* S *is isomorphic to a quasi-direct product of* E *and* R.

Proof. Take the quasi-direct product $E \times R$ obtained by (4.2), and consider the mapping $\varphi: S \rightarrow E \times R$ defined by $\varphi(x) = ((\widetilde{xx^*}, \widetilde{\widetilde{x^*x}}), \bar{x}), x \in S$. It is obvious from Lemmas 5 and 6 that φ is one-to-one and onto. Further, we have

$$\begin{aligned} \varphi(x)\varphi(y) &= ((\widetilde{xx^*}, \widetilde{\widetilde{x^*x}}), \bar{x})((\widetilde{yy^*}, \widetilde{\widetilde{y^*y}}), \bar{y}) \\ &= (\rho_{(\bar{x},\bar{y})}((\widetilde{xx^*}, \widetilde{\widetilde{x^*x}}), (\widetilde{yy^*}, \widetilde{\widetilde{y^*y}})), \bar{x}\bar{y})(((\widetilde{(xy)(xy)^*}, \widetilde{\widetilde{(xy)^*(xy)}}), \overline{xy}) \\ &= \varphi(xy) . \end{aligned}$$

Hence, φ is an isomorphism.

Finally the author is indebted to Professor Naoki Kimura for valuable discussions and for substantial help in the development of the material of this paper.

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Received May 12, 1969. This research has been partially supported by the National Science Foundation under Grant GP-7429.

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Pacific Journal of Mathematics
Vol. 33, No. 1 March, 1970