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Let A be a fixed point in n-dimensional Euclidean space. Let $B_1, B_2, \cdots, B_{n+1}$ be the vertices of a simplex S_n of n-dimensions, that is, the $n+1$ vertices do not lie on a $(n-1)$ dimensional subspace. Let d_i, assumed to be positive, be the distance of B_i from A, and let l_{ij} be the cosine of the angle between the straight lines AB_i and AB_j for $i, j = 1, 2, \cdots, n+1$. Let π_i denote the $(n-1)$-dimensional hyperplane passing through all the vertices of S_n except B_i, let p_i, assumed positive, be the perpendicular distance of π_i from A, and let m_{ij} denote the cosine of the angle between the normals from A to π_i and π_j for $i, j = 1, 2, \cdots, n+1$. The present paper deals with the following problems.

(a) An expression for the content of S_n, $C(S_n)$ say, in terms of d_i and l_{ij} for $i, j = 1, 2, \cdots, n+1$ is first obtained. Then leaving $d_1, d_2, \cdots, d_{n+1}$ fixed, values of l_{ij}, say l^*_{ij}, are determined in such a manner that $C(S_n)$ is a maximum, and the maximum value of $C(S_n)$ is obtained for the two cases that arise: (i) when A is inside S_n, (ii) when A is outside S_n. The latter case does not arise when $d_1 = d_2 = \cdots = d_{n+1}$.

(b) An expression for $C(S_n)$ is obtained in terms of p_i and m_{ij}, $i, j = 1, 2, \cdots, n+1$. Then leaving $p_1, p_2, \cdots, p_{n+1}$ fixed, values for m_{ij}, say m^*_{ij}, are determined in such a manner that $C(S_n)$ is a minimum, and such $C(S_n)$ is computed for the two cases that arise depending on (i) whether A is inside S_n or (ii) A is outside S_n. The latter case does not arise when

$$p_1 = p_2 = \cdots = p_{n+1} .$$

The results are stated below.

(a) The content of S_n, max $C(S_n)$ and l^*_{ij} are given by

$$n!C(S_n) = |(l_{ij}d_id_j + 1)|^{1/2} \tag{1.1}$$

$$\max (n!C(S_n))^2 = -u^{-1}\prod_{i=1}^{n+1}(d_i^2 - u) \tag{1.2}$$

$$l^*_{ij} = u/(d_id_j) \text{ for } i, j = 1, 2, \cdots, n+1;\ i \neq j , \tag{1.3}$$

where u satisfies the equation

$$1 + u\sum_{i=1}^{n+2}(d_i^2 - u)^{-1} = 0 . \tag{1.4}$$

The unique negative root for u in (1.4) corresponds to the case when A is inside S_n. When the relation

$$d_1 = d_2 = \cdots = d_{n+1}$$

is not satisfied, the smallest positive root for u in (1.4) corresponds to the case when A is outside S_n. Other roots for u in (1.4), if any, are inadmissible.

(b) The content $C(S_n)$, min $(C(S_n))$ and m^*_{ij} are given by

$$(n!C(S_n))^2 = |(p_ip_j + m_{ij})|^n / \prod_{i=1}^{n+1} |M_{ii}| \tag{1.5}$$

where $|M_{ii}|$ is the cofactor of m_{ii} in $|(m_{ij})|$ and

$$\min (n!C(S_n))^2 = -v^{-1}n^{2n} \prod_{i=1}^{n+1} (p_i^2 - v) \tag{1.6}$$

and

$$m^*_{ij} = v/(p_ip_j) \text{ for } i \neq j \ ; \ i, j = 1, 2, \cdots, n+1 \ ; \tag{1.7}$$

where v satisfies the equation

$$1 + v \sum_{i=1}^{n+1} (p_i^2 - v)^{-1} = 0 \ . \tag{1.8}$$

The unique negative root for v in (1.8) corresponds to the case when A is inside S_n. When the relation

$$p_1 = p_2 = \cdots = p_{n+1}$$

is not satisfied, the smallest positive root for v in (1.8) corresponds to the case when A is outside S_n. All other roots, if any, are inadmissible.

When $d_1 = d_2 = \cdots = d_{n+1}$, we obtain the special result that the largest simplex inscribed in a sphere of n-dimensions is a regular one, while when $p_1 = p_2 = \cdots = p_{n+1}$ the smallest simplex circumscribing a sphere is a regular one.

The coordinates of B_i referred to a n-dimensional Cartesian coordinate system with origin at A will be denoted by $(x_{i,1}, x_{i,2}, \cdots, x_{i,n})$. $(x_1, x_2, \cdots, x_n)$ will denote a general point in the n-space.

2. Extremal simplex determined by the distance of vertices. The content of S_n is given by (Sommerville, p. 124) $n!C(S_n) = |V|$ where

$$V = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} & 1 \\ x_{2,1} & \cdots & x_{2,n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} & 1 \end{bmatrix} \tag{2.1}$$

so that $(n!C(S_n))^2 = |VV'| = |(w_{ij})|$ say, where

$$w_{ij} = 1 + s_{ij} \text{ for } i, j = 1, 2, \cdots, n+1; \text{ and} \tag{2.2}$$

$$(s_{ij}) = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} \end{bmatrix} \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & & \vdots \\ x_{n+1,1} & \cdots & v_{n+1,n} \end{bmatrix}' \tag{2.3}$$

$$= (l_{ij}d_id_j) \ . \tag{2.4}$$

Hence we have proved (1.1).

We note that $s_{ii} = d_i^2$, for $i = 1, 2, \cdots, n+1$. From (2.3) we also note that the rank of (s_{ij}) is less than $n+1$ so that $|(s_{ij})| = 0$ and (s_{ij}) is semi-positive definite. Further we note that both (s_{ij}) and (w_{ij}) are symmetric matrices and since $B_1, \cdots, B_{n+1}$ do not lie on a $(n-1)$-dimensional subspace, we must have $|(w_{ij})| \neq 0$, in fact, $|(w_{ij})| > 0$ since (w_{ij}) is positive definite. Our problem of maximizing $C(S_n)$ with respect to the l_{ij}, $i \neq j$, for given values of d_i, $d_i > 0$, may be re-stated as follows.

We must maximize $|(w_{ij})|$ over the class of symmetric matrices (s_{ij}) or (w_{ij}) with respect to s_{ij}, $i, j = 1, \cdots, n+1$, subject to the conditions: $|(s_{ij})| = 0$ and $s_{ii} = d_i^2$ for $i = 1, \cdots, n+1$. Further (s_{ij}) should be semipositive definite and $|w_{ij}| \neq 0$.

Let θ and $\mu_1, \cdots, \mu_{n+1}$ be Lagrange multipliers. We seek the extreme values of the function L with respect to s_{ij}, $i, j = 1, \cdots, n+1$, where

$$L = |w_{ij}| - \theta |s_{ij}| + \sum_{i=1}^{n+1} \mu_i (s_{ii} - d_i^2) .$$

Hence s_{ij} must satisfy

$$\frac{1}{2} \frac{\partial L}{\partial s_{ij}} = |W_{ij}| - \theta |S_{ij}| = 0 \text{ for } i \neq j,\ i, j, = 1, \cdots, n+1$$

$$\text{and } \frac{\partial L}{\partial s_{ii}} = |W_{ii}| - \theta |S_{ii}| + \mu_i = 0 \text{ for } i = 1, \cdots, n+1 ;$$

where $|W_{kl}|$ and $|S_{kl}|$ denote co-factors of w_{kl} and s_{kl} in $|(w_{ij})|$ and $|(s_{ij})|$ respectively.

This implies that

$$\sum_{j=1}^{n+1} w_{kj} \cdot \frac{1}{2} \frac{\partial L}{\partial s_{ij}} + w_{ki} \frac{\partial L}{\partial s_{ii}} = 0$$

so that

$$\sum_{j=1}^{n+1} w_{kj} |W_{ij}| - \theta \sum_{j=1}^{n+1} w_{kj} |S_{ij}| + \mu_i w_{ki} = 0 .$$

Let $k \neq i$; then using (2.2), $w_{kj} = 1 + s_{kj}$ and by the well-known property that expansions in terms of alien co-factors vanish identically (Aitken, p. 51) we finally obtain

$$-\theta \sum_{j=1}^{n+1} |S_{ij}| + \mu_i w_{ki} = 0$$

so that $s_{ki} = w_{ki} - 1 = \theta/\mu_i \sum_{j=1}^{n+1} |S_{ij}| - 1$, for all $k \neq i$. Since the above expression for s_{ki} is constant for values of $k = 1, \cdots, n+1$, $k \neq i$, we conclude that the elements of the ith column of (s_{ij}), except

$s_{ii} = d_i^2$, must be equal. Since s_{ij} is a symmetric matrix, the above property extends to the rows of (s_{ij}) and it is easily seen that the extreme values of L correspond to values s_{ij}^* of s_{ij} where

$$s_{ij}^* = u \text{ for } i \neq j,\ i, j = 1, \cdots, n+1 \tag{2.5}$$

while

$$s_{ii}^* = d_i^2,\ i = 1, \cdots, n+1 .$$

Now u can be determined from the relation $|s_{ij}| = 0$ so that we must have

$$\begin{vmatrix} d_1^2 & u & \cdot & \cdot & u \\ u & d_2^2 & \cdot & \cdot & u \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u & u & \cdot & \cdot & d_{n+1}^2 \end{vmatrix} = 0 . \tag{2.6}$$

Let us define the determinant

$$D_k(x; a_1, \cdots, a_k) = \begin{vmatrix} a_1 & x & \cdot & \cdot & x \\ x & a_2 & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & a_k \end{vmatrix} . \tag{2.7}$$

From the relation due to Grabeiri (1874) (see Muir, vol. 3, 4, p. 110), or by subtracting the first row of the above determinant from the remaining rows and by the use of Cauchy expansion in terms of the first row and first column, we have

$$D_k(x; a_1, \cdots, a_k) = \left(1 + x \sum_{i=1}^{k} (a_i - x)^{-1}\right) \prod_{i=1}^{k} (a_i - x) . \tag{2.8}$$

Hence from (2.6) u must satisfy the equation

$$\left(1 + u \sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right) \prod_{i=1}^{n+1} (d_i^2 - u) = 0 . \tag{2.9}$$

From (2.2) and (2.5) the extreme value of $(n!C(S_n))^2$ for any u satisfying (2.9) is equal to

$$\begin{aligned} &D_{n+1}(1 + u; 1 + d_1^2, \cdots, 1 + d_{n+1}^2) \\ &= \left(1 + (1 + u) \sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right)\left(\prod_{i=1}^{n+1} (d_i^2 - u)\right) \\ &= \left(\sum_{i=1}^{n+1} (d_i^2 - u)^{-1}\right)\left(\prod_{i=1}^{n+1} d_i^2 - u)\right) \end{aligned} \tag{2.10}$$

by the use of (2.9).

Since $u = 0$ does not satisfy (2.6), we immediately obtain from

(2.9) that the expression (2.10) is equal to

$$-u^{-1}\prod_{i=1}^{n+1}(d_i^2-u) \tag{2.11}$$

which is the extreme value of $(n!C(s_n))^2$ in terms of u. In order that the content is nonzero we must have $u \neq d_i^2$ for $i = 1, \cdots, n+1$. This statement along with (2.9) implies that u must satisfy the equation

$$1 + u\sum_{i=1}^{n+1}(d_i^2-u)^{-1} = 0 \ . \tag{2.12}$$

The roots for u, temporarily assuming that $d_1, \cdots, d_{n+1}$ are distinct, can be located by Decartes rule of signs by checking the signs of the left-handside of (2.12) for values of u, equal to $-\infty$, 0, $+\infty$ and in the neighborhood of d_i^2, $i = 1, \cdots, n+1$. Relabelling d_i such that $d_1 < d_2 < \cdots < d_{n+1}$, it is easily verified that all the roots for u are real, say $u_1, \cdots, u_{n+1}$ and may be labelled in such a manner that

$$u_1 < 0 < d_1^2 < u_2 < d_2^2 < \cdots < u_{n+1} < d_{n+1}^2 \ . \tag{2.13}$$

Consider the characteristic roots of (s_{ij}^*) given by $|s_{ij}^* - \lambda I| = 0$. By (2.5) and (2.7) λ must satisfy $D_{n+1}(u; d_1^2-\lambda, \cdots, d_{n+1}^2-\lambda) = 0$. Hence from (2.9)

$$\left(1 + u\sum_{i=1}^{n+1}(d_i^2-\lambda-u)^{-1}\right)\prod_{i=1}^{n+1}(d_i^2-\lambda-u) = 0 \ .$$

By similar method as used to obtain (2.13) we find that the roots for λ may be so labelled that $\lambda_1 = 0$ and

$$d_i^2 < \lambda_{i+1} + u < d_{i+1}^2 \qquad i = 1, \cdots, n \ .$$

In order that all the roots for λ are nonnegative it is easily seen that the relation

$$d_2^2 - u > \lambda_2 \geqq 0 \tag{2.14}$$

must be satisfied so that we must have $u < d_2^2$. From (2.13) we find that the only admissible roots for u are u_1 and u_2.

To establish (1.4) it only remains to show that u_1 corresponds to the case when A is inside the extremal simplex whereas u_2 corresponds to the case when A is outside the extremal simplex.

Consider the equation of π_i, passing through all the vertices of S_n except B_i having the coordinates $(x_{i,1}, \cdots, x_{i,n})$, given by

$$L_i(x_1, \cdots, x_n) = 0 \ ,$$

where

$$L_i(x_1, \cdots, x_n) = \begin{vmatrix} x_{1,1} & \cdots & x_{1,n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{i-1,1} & \cdots & x_{i-1,n} & 1 \\ x_1 & \cdots & x_n & 1 \\ x_{i+1,1} & \cdots & x_{1+1,n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,n} & 1 \end{vmatrix} .$$

Now A and B_i lie on the same side of π_i if and only if $L_i(x_{i,1}, \cdots, x_{i,n})$. $L_i(0, \cdots, 0) > 0$ while A and B_i lie on opposite sides of π_i if and only if $L_i(x_{i,1}, \cdots, x_{i,n})$. $L_i(0, \cdots, 0) < 0$.

Now by direct multiplication of the determinant $L_i(x_{i,1}, \cdots, x_{i,n})$ with the transpose of the determinant $L_i(0, 0, \cdots, 0)$ we obtain

$$L_i(x_{i,1}, \cdots, x_{i,n}) \cdot L_i(0, 0, \cdots, 0)$$
$$= \begin{vmatrix} 1 + s_{11} & 1 + s_{12} & \cdots 1 \cdots & 1 + s_{1\, n+1} \\ 1 + s_{21} & 1 + s_{22} & \cdots 1 \cdots & 1 + s_{2\, n+1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 + s_{n+11} & 1 + s_{n+12} & \cdots 1 & 1 + s_{n+1\, n+1} \end{vmatrix} .$$

We now assume that S_n is an extremal simplex so that from (2.5) $s_{\nu\nu} = d_\nu^2$, $\nu = 1, \cdots, n+1$ and $s_{\nu k} = u$, $\nu \neq k$, $\nu, k = 1, \cdots, n+1$. Then in the last determinant each entry in the i-th column is 1, the jth diagonal entry is $d_j^2 + 1$ for $j \neq i, j = 1, \cdots, n+1$ while the remaining entries are $1 + u$. Subtracting $(1 + u)$ times the i-th column from the remaining columns we immediately obtain

$$L_i(x_{i,1}, \cdots, x_{i,n}) \cdot L_i(0, \cdots, 0) = (d_i^2 - u)^{-1} \prod_{j=1}^{n+1} (d_j^2 - u)$$
$$= \frac{-u^{-1} \prod_{1}^{n+1} (d_j^2 - u)}{(-u^{-1}(d_i^2 - u))} .$$

Since from (2.11) the numerator of the last expression is positive, we find that A and B_i lie on the same side of π_i if and only if

$$-u^{-1}(d_i^2 - u) > 0 ,$$

while they lie on opposite sides of π_i if and only if $-u^{-1}(d_i^2 - u) < 0$.

Since $-u_2^{-1}(d_2^2 - u_2) < 0$ and $-u_1^{-1}(d_1^2 - u_1) > 0$, it is readily checked that we have proved (1.2), (1.3) and (1.4) in the case when $d_1, \cdots, d_{n+1}$ are distinct.

Necessary modifications are easily made when some or all of the d_i are not distinct.

Finally we remark that the simplex corresponding to u_1 has larger content than that for u_2. This is because

$$d_i^2 - u_1 > d_i^2 - u_2 > 0 \text{ for } i = 2, \cdots, n-1$$

and

$$-u_1^{-1}(d_1^2 - u_1) = 1 - \frac{d_1^2}{u_1} > 1 - d_1^2/u_2 = -u_2^{-1}(d_2^2 - u_2) ,$$

so that

$$-u_1^{-1} \prod_1^{n+1} (d_j^2 - u_1) > -u_2^{-1} \prod_1^{n+1} (d_j^2 - u_2) . \tag{2.15}$$

We also note that when $d_1 = d_2 = \cdots = d_{n+1}$ (1.4) has a unique negative root for u and the point A corresponding to this value of u must lie inside the extremal simplex.

3. Simplex determined by distances of faces. We recall that the $(n-1)$-dimensional hyperplane π_i passes through all the vertices of S_n except B_i. The distance of π_i from A is p_i. The point B_i does not lie on π_i but does lie on all the remaining n hyperplanes

$$\pi_j, j \neq i, j = 1, \cdots, n+1 .$$

Let π_i be given by (in normal form)

$$\pi_i\colon e_{i,1}x_1 + e_{i,2}x_2 + \cdots + e_{i,n}x_n = e_{i,n+1} \tag{3.1}$$

where for notational convenience we have written

$$p_i = e_{i,n+1} , \tag{3.2}$$

and $e_{i,1}, \cdots, e_{i,}n$ are the direction cosines of the normal to π_i, so that we have

$$\sum_{j=1}^{k} e_{i,j}e_{k,j} = m_{ik};\ i, k = 1, 2, \cdots, n+1;\ m_{ii} = 1 . \tag{3.3}$$

The notations used in this section will be listed first and some relations needed later will be established in order to avoid future digression.

We define the $(n+1) \times (n+1)$ matrix E in double suffix notation as

$$E = (e_{i,j}) \tag{3.4}$$

and $E_{i,j}$ will denote the co-factor of $e_{i,j}$ in E. We also define the $(n+1) \times (n+1)$ matrix M as

$$M = (m_{ij}) \tag{3.5}$$

and M_{ij} as co-factor of m_{ij} in M.
Let σ_i denote the signature of $|E_{i,n+1}|$ so that

$$\sigma_i = \begin{cases} 1 \text{ if } |E_{i,n+1}| > 0 \\ -1 \text{ if } |E_{i,n+1}| < 0 \end{cases} \quad \text{for } i = 1, \cdots, n+1 . \tag{3.6}$$

We remark here that $E_{i,n+1}$ is nonsingular. This is because

$$\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_{n+1}$$

have one and only one point in common, namely $(x_{i,1}, \cdots, x_{i,n})$. Since π_i does not pass through the above common point, it is easily seen that the matrix E is also nonsingular, so that

$$|E| \neq 0 \text{ and } |E_{i,n+1}| \neq 0, i = 1, \cdots, n+1 . \tag{3.7}$$

Furthermore it is easily seen that

$$|E_{i,n+1}| = \sigma_i |E_{i,n+1}E'_{i,n+1}|^{1/2} = \sigma_i |M_{ii}|^{1/2} \quad \text{for } i = 1, \cdots, n+1 \tag{3.8}$$

where the radical above as well as all radicals appearing in this paper will be always taken as positive. Hence from (3.2) and (3.4) we have

$$|E| = \sum_{i=1}^{n+1} p_i |E_{i,n+1}| = \sum_{i=1}^{n+1} \sigma_i p_i |M_{ii}|^{1/2} = \rho \text{ (say)} . \tag{3.9}$$

D will denote the diagonal matrix

$$D = \text{Diag.} (p_1, \cdots, p_{n+1}) \tag{3.10}$$

and let

$$R = (r_{ij}) = D^{-1}MD^{-1} \tag{3.11}$$

so that $r_{ii} = p_i^{-2}$ for $i = 1, \cdots, n+1$. Since

$$M = \begin{bmatrix} e_{1,1} & \cdots & e_{1,n} \\ \vdots & & \vdots \\ e_{n+1,1} & \cdots & e_{n+1,n} \end{bmatrix} \begin{bmatrix} e_{1,1} & \cdots & e_{1,n} \\ \vdots & & \vdots \\ e_{n+1,1} & \cdots & e_{n+1,n} \end{bmatrix}'$$

we also remark that M and consequently R are symmetric positive semi-definite matrices, so that $|M| = 0$ and $|R| = 0$.

Finally, it follows that

$$|M_{ii}| = |R_{ii}| \left(\prod_{j=1}^{n+1} p_j^2 \right) \Big/ p_i . \tag{3.12}$$

To obtain the content $C(S_n)$, we will use the formula (2.1). Since $(x_{i,1}, \cdots x_{i,n})$ lies on π_j; $j \neq i, j = 1, \cdots, n+1$, we may directly solve for $x_{i,j}$ from the following n linear equations:

$$\begin{bmatrix} e_{1,1}, & \cdots, & e_{1,n} \\ \vdots & & \vdots \\ e_{i-1,1}, & \cdots, & e_{i-1,n} \\ e_{i+1,1}, & \cdots, & e_{i+1,n} \\ \vdots & & \vdots \\ e_{n+1,1}, & \cdots, & e_{n+1,n} \end{bmatrix} \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \cdot \\ \cdot \\ \cdot \\ x_{i,n} \end{bmatrix} = \begin{bmatrix} e_{1,n+1} \\ \vdots \\ e_{i-1,n+1} \\ e_{i+1,n+1} \\ \vdots \\ e_{n+1,n+1} \end{bmatrix} .$$

A simple calculation shows that (see (3.4))

$$x_{i,j} = (-1)^{n-j}(-1)^{i+j} \mid E_{i,j} \mid /((-1)^{n+1+i} \mid E_{i,n+1} \mid) .$$

Hence we obtain

$$x_{i,j} = - \mid E_{i,j} \mid / \mid E_{i,n+1} \mid ;\ i, j = 1, \cdots, n + 1 .$$

Substituting these values in $| V |$ of (2.1) and factoring out -1 from each of the first n columns of V and also factoring out $| E_{i,n+1} |^{-1}$ from the ith row of V for $i = 1, \cdots, n + 1$, we readily obtain

$$(3.13)\qquad \begin{aligned} n!C(S_n) &= (-1)^n \mid \operatorname{Adj} E \mid \Big/ \prod_{i=1}^{n+1} \mid E_{i,n+1} \mid \\ &= (-1)^n \mid E \mid^n \Big/ \prod_{i=1}^{n+1} \mid E_{i,n+1} \mid \end{aligned}$$

where $| \operatorname{Adj} E |$ is the adjoint determinant of $| E |$. In order to avoid the ambiguity of sign in $C(S_n)$ we consider $(n!C(S_n))^2$ instead and from (3.9) and (3.12) we obtain

$$\begin{aligned} (n!C(S_n))^2 &= \mid E \mid^{2n} \Big/ \prod_{i=1}^{n+1} \mid E_{i,n+1} \mid^2 \\ &= \left(\sum_{i=1}^{n+1} \sigma_i p_i \mid M_{ii} \mid^{1/2} \right)^{2n} \Big/ \prod_{i=1}^{n+1} \mid M_{ii} \mid \\ &= \left(\sum_{i=1}^{n+1} \sigma_i \mid R_{ii} \mid^{1/2} \right)^{2n} \Big/ \prod_{i=1}^{n+1} \mid R_{ii} \mid . \end{aligned}$$

Our problem of minimization is equivalent to minimizing

$$ln\left[\left(\sum_{i=1}^{n+1} \sigma_i \mid R_{ii} \mid^{1/2} \right)^2 \Big/ \prod_{i=1}^{n+1} \mid R_{ii} \mid^{1/n} \right]$$

with respect to r_{ij}, $i, j = 1, \cdots, n + 1$, subject to the restriction that $r_{ii} = p_i^{-2}$, $i = 1, \cdots, n + 1$ and $| R | = 0$ over the class of symmetric matrices R.

Let λ, $\mu_1, \cdots, \mu_{n+1}$ be Lagrange multipliers and we seek the extreme value of

$$L = ln\left(\sum_{i=1}^{n+1} \sigma_i \mid R_{ii} \mid^{1/2} \right)^2 - \frac{1}{n} \sum_{i=1}^{n+1} ln \mid R_{ii} \mid - \lambda \mid R \mid + \sum_{i=1}^{n+1} \mu_i (r_{ii} - p_i^{-2}) .$$

r_{ij} must satisfy:

$$\frac{\partial L}{\partial r_{ij}} = \rho^{-1} \sum_{\nu=1}^{n+1} \frac{\partial |R_{\nu\nu}|}{\partial r_{ij}} \frac{\sigma_\nu}{|R_{\nu\nu}|^{1/2}} - \frac{1}{n} \sum_{\nu=1}^{n+1} \frac{1}{R_{\nu\nu}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ij}} - \lambda \frac{\partial |R|}{\partial r_{ij}} = 0 , \quad i \neq j,\ i, j = 1, \cdots, n+1$$

and

$$\frac{\partial L}{\partial r_{ii}} = \rho^{-1} \sum_{\nu=1}^{n+1} \frac{\sigma_\nu}{|R_{\nu\nu}|^{1/2}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ii}} - \frac{1}{n} \sum_{\nu=1}^{n+1} \frac{1}{R_{\nu\nu}} \frac{\partial |R_{\nu\nu}|}{\partial r_{ii}} - \lambda \frac{\partial |R|}{\partial r_{ii}} + \mu_i = 0$$

where ρ is as defined in (3.9).

These equations reduce to

$$\frac{1}{2} \frac{\partial L}{\partial r_{ij}} = \sum_{\substack{\nu=1 \\ \nu \neq i,j}}^{n+1} (\rho^{-1}\sigma_\nu |R_{\nu\nu}|^{-1/2} - n^{-1}|R_{\nu\nu}|^{-1}) |R_{\nu\nu|ij}| - \lambda |R_{ij}| = 0 \quad \text{for } i \neq j;\ i, j = 1, \cdots, n+1$$

and

$$\frac{\partial L}{\partial r_{ii}} = \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n+1} (\rho^{-1}\sigma_\nu |R_{\nu\nu}|^{-1/2} - n^{-1}|R_{\nu\nu}|^{-1}) |R_{\nu\nu|ii}| - \lambda |R_{ii}| + \mu = 0$$

where $|R_{\nu\nu|ij}|$ is the co-factor of r_{ij} in $|R_{\nu\nu}|$.

Hence the minimizing values of r_{ij}, r_{ij}^*, say, must satisfy the equations in r_{ij}:

$$r_{ii} = p_i^{-2}$$

and

$$\text{(3.14)} \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n+1} r_{ij} \frac{1}{2} \frac{\partial L}{\partial r_{ij}} + r_{ii} \frac{\partial L}{\partial r_{ii}} = 0$$

and

$$\text{(3.15)} \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n+1} r_{kj} \frac{1}{2} \frac{\partial L}{\partial r_{ij}} + r_{kj} \frac{\partial L}{\partial r_{ii}} = 0 .$$

After obvious simplification (3.14) yields

$$\sum_{\substack{\nu=1 \\ \nu \neq i}}^{n+1} (\rho^{-1}\sigma_\nu |R_{\nu\nu}|^{-1/2} - n^{-1}|R_{\nu\nu}|^{-1}) |R_{\nu\nu}| + \mu_i p_i^{-2} = 0 ,$$

or

$$\text{(3.16)} \qquad \mu_i = p_i^2 \rho^{-1} \sigma_i R_{ii} .$$

From (3.15) we obtain for $k \neq i$,

$$(3.17)\quad \sum_{j=1}^{n+1}\sum_{\substack{\nu=1\\ \nu\neq i,j}}^{n+1} (\sigma_\nu |R_{\nu\nu}|^{-1/2}\rho^{-1} - n^{-1}|R_{\nu\nu}|^{-1}) r_{kj} |R_{\nu\nu|ij}| + \mu_i r_{ki} = 0 .$$

After some calculations we obtain

$$(3.18)\quad r_{ki} = \mu_i^{-1}(\sigma_k |R_{kk}|^{-1/2}\rho^{-1} - n^{-1}|R_{kk}|^{-1})|R_{ik}| .$$

It is easily seen from (3.11) that $|R_{ik}| = p_i p_k |M_{ik}|$ and

$$M_{ik} = |E_{i,n+1}||E_{k,n+1}|$$

and hence from (3.8),

$$|R_{ik}| = \sigma_{ik}|R_{ii}|^{1/2}|R_{kk}|^{1/2}$$

so that substituting for μ_i from (3.16) in (3.18) we obtain

$$(3.19)\quad p_i^2 r_{ki} = 1 - n^{-1}\rho\sigma_k |R_{kk}|^{-1/2} .$$

In obtaining (3.18) from (3.17), we illustrate the case for $i = 1$, $n + 1 = 4$ and $k = 2$, for the expression, for example:

$$\begin{aligned}
&\sum_{j=1}^{4}\sum_{\substack{\nu=1\\ \nu\neq 1,j}}^{4} \sigma_\nu |R_{\nu\nu}|^{-1/2} r_{kj} |R_{\nu\nu|ij}| \\
&= r_{21}(\sigma_2 |R_{22|11}||R_{22}|^{-1/2} + \sigma_3 |R_{33|11}||R_{33}|^{-1/2} + \sigma_4 |R_{44|11}||R_{44}|^{-1/2}) \\
&\quad + r_{22}(\sigma_3 |R_{33|12}||R_{33}|^{-1/2} + \sigma_4 |R_{44|12}||R_{44}|^{-1/2}) \\
&\quad + r_{23}(\sigma_2 |R_{22|13}||R_{22}|^{-1/2} + \sigma_4 |R_{44|13}||R_{44}|^{-1/2}) \\
&\quad + r_{24}(\sigma_2 |R_{22|14}||R_{22}|^{-1/2} + \sigma_3 |R_{33|14}||R_{33}|^{-1/2}) \\
&= \sigma_2 |R_{21}||R_{22}|^{-1/2} .
\end{aligned}$$

The last expression is obtained from the coefficients of $|R_{22}|^{-1/2}$; the coefficients of $|R_{33}|^{-1/2}$ or $|R_{11}|^{-1/2}$ are easily seen to vanish identically, since they represent expansion by alien co-factors.

In the summation appearing in (3.17) only the term with $\nu = k$ survives;

$$\sum_{\substack{j=1\\ j\neq k}}^{n+1} r_{kj}|R_{kk|ij}|$$

is the expansion of the determinant obtained by replacing the elements of the i-th row of $|R|$ by those of the k-th row of $|R|$ with the k-th row and k-th column deleted. Transferring the elements r_{ki} appearing in the i-th row to the k-th row, there results the minor of r_{ki} in $|R|$. Hence multiplying by $(-1)^{i-k}$ and $(-1)^{i+k}$ we obtain $|R_{ki}|$. It is thus seen that

$$\sum_{\substack{j=1\\ j\neq k}}^{n+1} r_{kj}|R_{kk|ij}| = |R_{ki}| = |R_{ik}| .$$

From (3.19) it is easily checked that we have

$$p_i^2 p_k^2 r_{ik} = p_j^2 p_k^2 r_{jk} , \tag{3.20}$$

for all $i, j = 1, \cdots, n+1$, with $i \neq k, j \neq k$.

Since the matrix

$$(p_i^2 r_{ij} p_j^2) = D^2 R D^2 = D^2 D^{-1} M D^{-1} D^2 = DMD = (p_i m_{ij} p_j)$$

is symmetric, and (3.20) implies that nondiagonal elements of each row or column of this matrix are equal we conclude, (in a manner analogous to (2.5)) that $r_{ii}^* = p_i^{-2}$, $i = 1, \cdots, n+1$ and

$$p_i^2 r_{ij}^* p_j^2 = p_i p_j m_{ij}^* = v ,$$

say, for $i \neq j; i, j = 1, \cdots, n+1$ so that

$$\begin{cases} m_{ii}^* = 1 & \text{for } i = 1, \cdots, n+1 \\ m_{ij}^* = \dfrac{v}{p_i p_j} & \text{for } i \neq j; i, j = 1, \cdots, n+1. \end{cases} \tag{3.21}$$

We obtain values of v by equating $|r_{ij}^*| = 0$ or equivalently by setting $|DMD| = |(p_i p_j m_{ij}^*)| = 0$, where $p_i p_j m_{ij}^* = v$, $i \neq j$ and $p_i^2 m_{ii}^* = p_i^2$, and it is seen from (2.7) that v must satisfy

$$D_{n+1}(v; p_1^2, \cdots, p_{n+1}^2) = 0 .$$

and hence

$$\left(1 + v \sum_{i=1}^{n+1} (p_i^2 - v)^{-1}\right) \prod_{i=1}^{n+1} (p_i^2 - v) = 0 . \tag{3.22}$$

We also note from (3.13), (3.8), (3.9) and (3.12) that

$$\begin{aligned} (n! C(S_n))^2 &= \rho^{2n} \cdot \prod_{i=1}^{n+1} (|R_{ii}|^{-1} \cdot p_i^2) \\ &= p_1^2 |R_{11}|^{-1} \cdot \prod_{i=2}^{n+1} (\rho |R_{ii}|^{-1/2})^2 . \end{aligned} \tag{3.23}$$

But from (3.19) we have

$$\rho \sigma_k |R_{kk}|^{-1/2} = n(1 - p_i^2 r_{ki}^*)$$

so that $\rho \sigma_k |R_{ii}|^{-1/2} = n(p_i^2 - v)/p_i$, from (3.21). Also from (3.21), since $r_{ij}^* = v/(p_i^2 p_j^2)$ and $r_{ii}^* = p_i^{-2}$ it is easily seen that

$$\begin{aligned} |R_{11}| \prod_{i=2}^{n+1} p_i^2 &= D_n(v; p_2^2, \cdots, p_{n+1}^2) \\ &= \left(1 + v \sum_{i=2}^{n+1} (p_i^2 - v)^{-1}\right) \prod_{i=2}^{n+1} (p_i^2 - v) \\ &= (p_1^2 - v)^{-1}\Big(-v(p_1^2 - v)^{-1} + \end{aligned}$$

$$+ 1 + v\sum_{i=1}^{n+1}(p_1^2 - v)^{-1}\Big)\prod_{i=1}^{n+1}(p_i^2 - v)$$
$$= -v(p_1^2 - v)^{-2}\prod_{i=1}^{n+1}(p_i^2 - v) \quad \text{from} \quad (3.22) .$$

Substituting in (3.23) we readily find that

$$(n!C(S_n))^2 = v^{-1}n^{2n}\prod_{i=1}^{n+1}(p_i^2 - v) . \tag{3.24}$$

Thus (1.6) is proved.

In order that S_n is nondegenerate $v \neq p_i^2$, $i = 1, \cdots, n + 1$. Hence from (3.22) v must satisfy

$$1 + v\sum_{i=1}^{n+1}(p_i^2 - v)^{-1} = 0 . \tag{3.25}$$

Thus we have exactly the same equation as (2.12) with d_i replaced by p_i and u replaced by v. By exactly the same argument that follows (2.12) we conclude that, when $p_1, \cdots, p_{n+1}$ are distinct, if the roots of (3.25) are so labelled that the unique negative root of (3.25) is v_1 and the smallest positive root for v is v_2 and if the p_i are labelled so that p_1 is the smallest and p_2 the second smallest p_i, $i = 1, \cdots, n + 1$, we have the two eligible roots of (3.26) as v_1 and v_2 satisfying

$$v_1 < 0 < p_1^2 < v_2 < p_2^2 . \tag{3.26}$$

It remains to prove that v_1 corresponds to the case when A is inside S_n while v_2 corresponds to the case when A is outside S_n.

We will prove that, for the extremal simplexes obtained above, the vertex B_i and the fixed point A lie on the same side of π_i if

$$p_i^2 - v > 0$$

while A and B_i lie on opposite sides if $p_i^2 - v < 0$.

Let

$$L_i(x_1, \cdots, x_n) = e_{i,1}x_1 + \cdots + e_{i,n}x_n - e_{i,n+1} .$$

Then $L_i(0, \cdots, 0) = -e_{i,n+1} = -p_i$, and

$$\begin{aligned} &L_i(x_{i1}, \cdots, x_{in}) \\ &\quad = -\sum_{j=1}^{n+1} e_{i,j}|E_{i,j}|/|E_{i,n+1}| && \text{(by virtue of (3.5))} \\ &\quad = -|E|/|E_{i,n+1}| \\ &\quad = -p_i\rho/\sigma_i|R_{ii}|^{1/2} && \text{(from (3.8) and (3.12))} \\ &\quad = -np_i(1 - p_k^2 r_{ki}^*) && \text{(from (3.19))} \\ &\quad = -np_i(1 - v/p_i^2) && \text{(from (3.21))} \end{aligned}$$

Hence $L_i(0, \cdots, 0) \cdot L_i(x_{i,1}, \cdots, x_{i,n}) = n(p_i^2 - v)$. Now the equation of π_i is $L_i(x_1, \cdots, x_n) = 0$. Hence $p_i^2 - v > 0$ implies that A and B_i lie on the same side of π_i while $p_i^2 - v < 0$ implies that A and B_i lie on opposite sides of π_i. Since $p_i^2 - v_1$ is positive for $i = 1, \cdots, n+1$ we conclude from (3.26) that corresponding to v_1, A is inside S_n. Also from (3.26) we find $p_1^2 - v_2$ is negative so that corresponding to v_2 the point A lies outside S_n. Hence it is readily checked that we have proved (1.5), (1.6), (1.7) and (1.8).

Finally, using an argument analogous to that used to obtain (2.15) we find that

$$-v_1^{-1} \prod_{i=1}^{n+1} (p_i^1 - v_1) > -v_2^{-1} \prod_{i=1}^{n+1} (p_i^2 - v_2)$$

so that from (3.24) we conclude that the content of S_n corresponding to v_1 is greater than the content of S_n corresponding to v_2.

Obvious modifications in the foregoing proofs are easily made when some or all the $p_1, \cdots, p_{n+1}$ are equal.

When $p_1 = p_2 = \cdots = p_{n+1}$, (3.25) has a unique negative solution for v and in this case A must lie inside the extremal simplex.

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